Unit 11

Exponents, Roots and Scientific Notation

Topic A: Exponents

- Basic exponent properties review
- Degree of a polynomial

Topic B: Properties of exponents

- Properties of exponents
- Properties of exponents – examples
- Simplifying exponential expressions

Topic C: Scientific notation and square roots

- Scientific notation
- Square roots
- Simplifying square roots

Unit 11: Summary

Unit 11: Self-test

Topic A: Exponents

Basic Exponent Properties Review

Exponent review: a^n or $\text{Base}^{\text{Exponent}}$

Exponential notation	Example
Power **Exponent** $\boxed{a^n} = a \cdot a \cdot a \cdot a \ldots a$ Base Read "*a* to the *n*th" or "the *n*th power of *a*."	$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$ Read "2 to the 4th."

Exponents - basic properties:

Name	Property	
Zero Exponent a^0	$a^0 = 1$	(0^0 is undefined)
One Exponent a^1	$a^1 = a$	
	$1^n = 1$	

Example: Write the following exponential expressions in expanded form.

	Exponential expressions	Expanded form	
1)	$\mathbf{4^3}$	$\boxed{4 \cdot 4 \cdot 4}$	$a^n = a \cdot a \cdot a \ldots$
2)	$\mathbf{(-u)^3}$	$\boxed{(-u)\,(-u)\,(-u)}$	
3)	$\mathbf{-u^3}$	$\boxed{-u \cdot u \cdot u}$	
4)	$\mathbf{(2x^3y^0)^2}$	$\boxed{(2x^3y^0)\,(2x^3y^0)}$	
5)	$\left(\frac{-5}{7}w\right)^3$	$\boxed{\left(\frac{-5}{7}w\right)\left(\frac{-5}{7}w\right)\left(\frac{-5}{7}w\right)}$	

Example: Write each of the following in the exponential form.

	Expanded form	Exponential notation	
1)	**(0.3) (0.3) (0.3)**	$\boxed{(0.3)^3}$	
2)	$\mathbf{(4t)\,(4t)\,(4t)\,(4t)}$	$\boxed{(4t)^4}$	
3)	$\mathbf{(3x)\,(2y)\,(x)\,(2y)}$	$\boxed{12x^2\,y^2}$	$12(x \cdot x)\,(y \cdot y)$

Example: Evaluate.

1) $\mathbf{2x^3 + y}$, **for** $\mathbf{x = 2}$, $\mathbf{y = 3}$

$2x^3 + y = 2 \cdot 2^3 + 3$ — Substitute *x* for 2 and *y* for 3.

$= 2\,(8) + 3 = \boxed{19}$

2) $\mathbf{(2a)^4 - b}$, **for** $\mathbf{a = 1}$, $\mathbf{b = 4}$ — Substitute *a* for 1 and *b* for 4.

$(2a)^4 - b = (2 \cdot 1)^4 - 4$

$= 2^4 - 4 = \boxed{12}$

Degree of a Polynomial

The degree of a *term* with *one* variable: the exponent of its variable.

Example: $\mathbf{9x^3}$ degree: $\boxed{3}$

$\mathbf{-7u^5}$ degree: $\boxed{5}$

$\mathbf{2a}$ degree: $\boxed{1}$ $2a = 2a^1$, $a^1 = a$

The degree of a *term* with *more* variables: the sum of the exponents of its variables.

Example: $\mathbf{-8x^2 y^4 z^3}$ degree: $2 + 4 + 3 = \boxed{9}$

The degree of a *polynomial* with more variables: the highest degree of any individual term.

Example: $\underbrace{\mathbf{9t^2u}}_{3} + \underbrace{\mathbf{4t^3u^2v^5}}_{10} - \underbrace{\mathbf{6t}}_{1} + \mathbf{5}$ degree: $\boxed{10}$ $3 + 2 + 5 = 10$, $a^1 = a$

Examples of degree of a polynomial:

Polynomial	$\mathbf{5x^3 - x^2 + 21}$	$\mathbf{2x^2y - 5z + 7x^4y^2z}$
Term	$5x^3$, $-x^2$, 21	$2x^2y$, $-5z$, $7x^4y^2z$
Degree of the term	3 , 2, 0	3, 1 , 7
Degree of the polynomial	3	7

Example: What is the degree of the following term / polynomial?

1) $\mathbf{3xy^3}$ degree: $\boxed{4}$

2) $\underbrace{\boldsymbol{2bc^3d^5}}_{9} + \underbrace{\boldsymbol{5e^2}}_{2} - \underbrace{\boldsymbol{fg^2}}_{3} + \underbrace{\boldsymbol{2e^0}}_{0}$ degree: $\boxed{9}$

Descending order: the exponent of a variable ***decreases*** for each succeeding term.

Example: $9\boldsymbol{x^4} - 7\boldsymbol{x^3} + \boldsymbol{x^2} - \boldsymbol{x} + 2$ $a^1 = a$

$21\boldsymbol{uv^3} - uv^2 + 4\boldsymbol{v} - 67$ The descending order of exponent v.

Ascending order: the exponent of a variable ***increases*** for each succeeding term.

Example: $13 - 8\boldsymbol{a} + 34\boldsymbol{a^2} - 12\boldsymbol{a^3}$ $a^1 = a$

$-7 + \frac{3}{5}w\boldsymbol{z} + 3.5w^2\boldsymbol{z^2} - 5\boldsymbol{z^3} + \boldsymbol{z^4}$ The ascending order of power z.

Topic B: Properties of Exponents

Properties of Exponents

Properties of exponents:

Name	Rule	Example
Product rule	$a^m a^n = a^{m+n}$	$2^3\, 2^2 = 2^{3+2} = 2^5 = 32$
Quotient rule (the same base)	$\frac{a^m}{a^n} = a^{m-n}$ $(a \neq 0)$	$\frac{y^4}{y^2} = y^{4-2} = y^2$
Power of a power	$(a^m)^n = a^{mn}$	$(x^3)^2 = x^{3 \cdot 2} = x^6$
Power of a product (different bases)	$(a \cdot b)^n = a^n b^n$	$(2 \cdot 3)^2 = 2^2\, 3^2 = 4 \cdot 9 = 36$
	$(a^m \cdot b^n)^p = a^{mp} b^{np}$	$(t^3 \cdot s^4)^2 = t^{3 \cdot 2} s^{4 \cdot 2} = t^6 s^8$
Power of a quotient (different bases)	$(\frac{a}{b})^n = \frac{a^n}{b^n}$ $(b \neq 0)$	$(\frac{2}{3})^2 = \frac{2^2}{3^2} = \frac{4}{9}$
	$(\frac{a^m}{b^n})^p = \frac{a^{mp}}{b^{np}}$ $(b \neq 0)$	$(\frac{q^2}{p^4})^3 = \frac{q^{2 \cdot 3}}{p^{4 \cdot 3}} = \frac{q^6}{p^{12}}$
Negative exponent a^{-n}	$a^{-n} = \frac{1}{a^n}$ $(a \neq 0)$	$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$
	$\frac{1}{a^{-n}} = a^n$ $(a \neq 0)$	$\frac{1}{4^{-2}} = 4^2 = 16$
Zero exponent a^0	$a^0 = 1$	$15^0 = 1$
One exponent a^1	$a^1 = a$ (But $1^n = 1$)	$7^1 = 7$, $1^{13} = 1$

Properties of exponents explained:

- **Product rule** (multiplying the same base): when multiplying two powers with the same base, keep the base and add the exponents. $\boxed{a^m a^n = a^{m+n}}$ a^n or $\text{Base}^{\text{Exponent}}$

  Example: $\mathbf{2^3\, 2^2} = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2^5 = \boxed{32}$

  Or $\mathbf{2^3\, 2^2} = 2^{3+2} = 2^5 = \boxed{32}$ A short cut, $a^m a^n = a^{m+n}$

- **Quotient rule** (dividing the same base): when dividing two powers with the same base, keep the base and subtract the exponents. $\boxed{\frac{a^m}{a^n} = a^{m-n}}$

  Example: $\frac{\mathbf{2^4}}{\mathbf{2^2}} = \frac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = 2^2 = \boxed{4}$

  Or $\frac{\mathbf{2^4}}{\mathbf{2^2}} = 2^{4-2} = 2^2 = \boxed{4}$ A short cut, $\frac{a^m}{a^n} = a^{m-n}$

  This law can also show that why $a^0 = 1$ (zero exponent a^0): $\frac{a^2}{a^2} = a^{2-2} = a^0 = 1$

- **Power rule:**
  - Power of a power: when raise a power to a power, just multiply the exponents.

$$\boxed{(a^m)^n = a^{mn}}$$

Example: $\mathbf{(4^3)^2} = (4^3)\,(4^3) = (4 \cdot 4 \cdot 4)\,(4 \cdot 4 \cdot 4) = 4^6 = \boxed{4096}$

Or $\mathbf{(4^3)^2} = 4^{3 \cdot 2} = 4^6 = \boxed{4096}$ A short cut, $(a^m)^n = a^{mn}$

  - Power of a product: when raise a power to different bases, distribute the exponent to each base. $\boxed{(a \cdot b)^n = a^n\, b^n}$

Example: $\mathbf{(2 \cdot 3)^2} = (2 \cdot 3)\,(2 \cdot 3) = 6 \cdot 6 = \boxed{36}$

Or $\mathbf{(2 \cdot 3)^2} = 2^2\, 3^2 = 4 \cdot 9 = \boxed{36}$ A short cut, $(a \cdot b)^n = a^n\, b^n$

  - Power of a product (different bases): when raise a power to a power with different bases, multiply each exponent inside the parentheses by the power outside the parentheses. $\boxed{(a^m \cdot b^n)^p = a^{mp}\, b^{np}}$

Example: $\mathbf{(2^2 \cdot 3^2)^2} = (2^2 \cdot 3^2)\,(2^2 \cdot 3^2) = (2^2 \cdot 2^2)\,(3^2 \cdot 3^2) = 16 \cdot 81 = \boxed{1296}$

Or $\mathbf{(2^2 \cdot 3^2)^2} = 2^{2 \cdot 2}\, 3^{2 \cdot 2} = 2^4\, 3^4 = 16 \cdot 81 = \boxed{1296}$ A short cut, $(a \cdot b)^n = a^n\, b^n$

- **Power of a quotient** (different bases):
  - When raise a fraction to a power, distribute the exponent to the numerator and denominator of the fraction. $\boxed{(\frac{a}{b})^n = \frac{a^n}{b^n}}$

Example: $(\mathbf{\frac{2}{3}})^{\mathbf{3}} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3} = \frac{2^3}{3^3} = \boxed{\frac{8}{27}}$

Or $(\mathbf{\frac{2}{3}})^{\mathbf{3}} = \frac{2^3}{3^3} = \boxed{\frac{8}{27}}$ A short cut, $(\frac{a}{b})^n = \frac{a^n}{b^n}$

  - When raise a fraction with powers to a power, multiply each exponent in the numerator and denominator by the power outside the parentheses. $\boxed{(\frac{a^m}{b^n})^p = \frac{a^{mp}}{b^{np}}}$

Example: $(\mathbf{\frac{2^2}{3^4}})^{\mathbf{3}} = \left(\frac{2^2}{3^4}\right)\left(\frac{2^2}{3^4}\right)\left(\frac{2^2}{3^4}\right) = \frac{4 \cdot 4 \cdot 4}{81 \cdot 81 \cdot 81} = \boxed{\frac{64}{531441}}$

Or $(\mathbf{\frac{2^2}{3^4}})^{\mathbf{3}} = \frac{2^{2 \cdot 3}}{3^{4 \cdot 3}} = \frac{2^6}{3^{12}} = \boxed{\frac{64}{531441}}$ A short cut, $(\frac{a^m}{b^n})^p = \frac{a^{mp}}{b^{np}}$

- **Negative exponent:** a negative exponent is the reciprocal of the number with a positive exponent. $\boxed{a^{-n} = \frac{1}{a^n}}$, $\boxed{\frac{1}{a^{-n}} = a^n}$ a^{-n} is the reciprocal of a^n.

Example: $\mathbf{3^{-4}} = \frac{1}{3^4} = \boxed{\frac{1}{81}}$ $a^{-n} = \frac{1}{a^n}$

Example: $\mathbf{\frac{1}{3^{-4}}} = 3^4 = \boxed{81}$ $\frac{1}{a^{-n}} = a^n$

Properties of Exponents − Examples

Example: Simplify (do not leave negative exponents in the answer).

1) $(-4)^1 = \boxed{-4}$ $\qquad a^1 = a$

2) $(-2345)^0 = \boxed{1}$ $\qquad a^0 = 1$

3) $(-0.3)^3 = \boxed{-0.027}$ $\qquad a^n = a \cdot a \cdot a \ldots$

4) $-5^2 = -(5^2) = \boxed{-25}$

5) $x^2\, x^3 = x^{2+3} = \boxed{x^5}$ $\qquad a^m\, a^n = a^{m+n}$

6) $\frac{y^6}{y^4} = y^{6-4} = \boxed{y^2}$ $\qquad \frac{a^m}{a^n} = a^{m-n}$

7) $(x^4)^{-3} = x^{4\,(-3)} = x^{-12} = \boxed{\frac{1}{x^{12}}}$ $\qquad (a^n)^m = a^{n\,m}$, $\frac{1}{a^{-n}} = a^n$

8) $7b^{-1} = 7 \cdot \frac{1}{b^1} = \boxed{\frac{7}{b}}$ $\qquad a^{-n} = \frac{1}{a^n}$, $a^1 = a$

9) $[(-4) \cdot (0.7)]^2 = (-4)^2 \cdot 0.7^2 = (16)\,(0.49) = \boxed{7.84}$ $\qquad (a \cdot b)^n = a^n\, b^n$

10) $(2t^3 \cdot w^2)^4 = 2^4\, t^{3\cdot 4} \cdot w^{2\cdot 4} = \boxed{16\, t^{12}\, w^8}$ $\qquad (a^m \cdot b^n)^p = a^{mp}\, b^{np}$

11) $\frac{1}{3^{-2}} = 3^2 = \boxed{9}$ $\qquad \frac{1}{a^{-n}} = a^n$

12) $(\frac{u}{z})^{-2} = \frac{u^{-2}}{z^{-2}} = \boxed{\frac{z^2}{u^2}}$ $\qquad (\frac{a}{b})^n = \frac{a^n}{b^n}$, $a^{-n} = \frac{1}{a^n}$, $\frac{1}{a^{-n}} = a^n$

13) $(\frac{x^4}{y^{-3}})^2 = \frac{x^{4\cdot 2}}{y^{(-3)(2)}} = \frac{x^8}{y^{-6}} = \boxed{x^8 y^6}$ $\qquad (\frac{a^m}{b^n})^p = \frac{a^{mp}}{b^{np}}$, $\frac{1}{a^{-n}} = a^n$

14) $(2^{-3})(2^3) = \frac{1}{2^3} \cdot 2^3 = \boxed{1}$ $\qquad \frac{1}{a^{-n}} = a^n$

15) $\frac{7x^4y^{-5}}{9^0 \cdot x^2 y^3} = \frac{7x^{4-2}y^{-5-3}}{1} = 7x^2y^{-8} = \boxed{\frac{7x^2}{y^8}}$ $\qquad a^0 = 1$, $\frac{a^m}{a^n} = a^{m-n}$, $a^{-n} = \frac{1}{a^n}$

16) $(\frac{e^{-3}f^2}{g^{-2}})^{-2} = \frac{e^{(-3)\,(-2)}\, f^{2(-2)}}{g^{(-2)(-2)}} = \frac{e^6\, f^{-4}}{g^4} = \boxed{\frac{e^6}{g^4\, f^4}}$ $\qquad (\frac{a^m}{b^n})^p = \frac{a^{mp}}{b^{np}}$, $\frac{1}{a^{-n}} = a^n$

Using a calculator: $4^2 = ?$ $\quad$ 4 $\boxed{x^2}$ $\boxed{=}$ (The display reads 16)

$3^4 = ?$ $\quad$ 3 $\boxed{x^y}$ 4 $\boxed{=}$ (The display reads 81)

(Or $\boxed{y^x}$ or $\boxed{\wedge}$ on some calculators.)

Simplifying Exponential Expressions

Steps for simplifying exponential expressions:

- Remove parentheses using "power rule" if necessary. $(a^m \cdot b^n)^p = a^{mp}\, b^{np}$
- Regroup coefficients and variables.
- Use "product rule" and "quotient rule". $a^m\, a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$
- Simplify.
- Use "negative exponent" rule to make all exponents positive if necessary.

Example: Simplify.

1) $(3x^3y^2)^2(2x^{-3}y^{-1})^3(-248z^{-19})^0$

$= 3^2x^{3\cdot 2}y^{2\cdot 2} \cdot 2^3x^{-3\cdot 3} \cdot y^{-1\cdot 3} \cdot 1$ Remove brackets. $(a^m \cdot b^n)^p = a^{mp}\, b^{np}$, $a^0 = 1$

$= (3^2 \cdot 2^3)(x^6\, x^{-9})(y^4\, y^{-3})$ Regroup coefficients and variables.

$= 72\, x^{-3}\, y^1$ Simplify. $a^m\, a^n = a^{m+n}$

$= \boxed{\frac{72y}{x^3}}$ Make exponent positive. $a^{-n} = \frac{1}{a^n}$, $a^1 = a$

2) $\left(\frac{(2x^4)(y^5)}{3x^3y^2}\right)^2 = \frac{(2x^4)^2(y^5)^2}{(3x^3y^2)^2}$ $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$

$= \frac{2^2x^{4\cdot 2}y^{5\cdot 2}}{3^2x^{3\cdot 2}y^{2\cdot 2}}$ Remove brackets. $(a \cdot b)^n = a^n\, b^n$

$= \frac{4}{9} \cdot \frac{x^8}{x^6} \cdot \frac{y^{10}}{y^4}$ Regroup coefficients and variables.

$= \boxed{\frac{4}{9}x^2y^6}$ Simplify. $\frac{a^m}{a^n} = a^{m-n}$

Example: Evaluate for $a = 2$, $b = 1$, $c = -1$.

1) $(-29a^{-5}b^4c^{-7})^0 = \boxed{1}$ $a^0 = 1$

2) $\left(\frac{a}{b}\right)^{-4} = \left(\frac{2}{1}\right)^{-4}$ Substitute 2 for *a* and 1 for *b*,

$= \frac{2^{-4}}{1^{-4}} = \frac{1^4}{2^4} = \boxed{\frac{1}{16}}$ $\frac{a^m}{a^n} = a^{m-n}$, $a^{-n} = \frac{1}{a^n}$, $\frac{1}{a^{-n}} = a^n$

3) $(a + b - c)^a = [2 + 1 - (-1)]^2 = 4^2 = \boxed{16}$ Substitute 2 for *a*, 1 for *b*, and -1 for *c*.