Unit 13

More about Polynomials

Topic A: Adding and subtracting polynomials

- Polynomials review
- Adding and subtracting polynomials

Topic B: Multiplication of polynomials

- Multiplying polynomials
- Special binomial products

Topic C: Polynomial division

- Dividing polynomials
- Long division of polynomials
- Missing terms in long division

Unit 13 Summary

Unit 13 Self-test

Topic A: Adding and Subtracting Polynomials

Polynomials Review

Review of basic algebraic terms:

Algebraic term	Definition	Example
Algebraic expression	A mathematical phrase that contains numbers, variables, and arithmetic operations (+, – , ×, ÷, etc.).	$5x + 2$ $3a - (4b + 6)$ $\frac{2x}{3} + 4y - z^2 + 11$
Constant	A number.	$x + \mathbf{2}$ constant: 2
Variable	A letter that can be assigned different values.	$3 - \boldsymbol{x}$ variable: x
Coefficient	The number that is in front of a variable.	$\mathbf{-6}\,x$ coefficient: -6 xz^3 coefficient: 1
Term	A term can be a constant, a variable, or the product of a number and variable(s). (Terms are separated by a plus or minus sign.)	$3x - \frac{2}{5} + 13y^2 + 73xy$ Terms: $3x$, $-\frac{2}{5}$, $13y^2$, $73xy$
Like terms	The terms that have the same variables and exponents.	$2x - y^2 - \frac{2}{5} + 5x - 7 + 13y^2$ Like terms: $2x$ and $5x$ $-y^2$ and $13y^2$, $-\frac{2}{5}$ and -7
Factor	A number or variable that multiplies with another. A number or expression can have many factors.	$24 = 2 \cdot 3 \cdot 4$ factors: 2, 3, 4 $5xy = 5 \cdot x \cdot y$ factors: 5, x, y

Polynomial		Example	Coefficient
Monomial	(one term)	$7a$	7
Binomial	(two terms)	$3x - 5$	3
Trinomial	(three terms)	$-4x^2 + xy + 7$	-4, 1
Polynomial	(one or more terms)	$2pq + 4p^3 + p + 11$	2, 4, 1

The degree of a term with more variables: the sum of the exponents of its variables.

Example: $-3x^3y^5z^2$ degree: $3 + 5 + 2 = \boxed{10}$

The degree of a polynomial with more variables: the highest degree of any individual term.

Example: $\underset{4}{4ab^3} + \underset{7}{3a^2b^2c^3} - \underset{1}{5a} + 1$ degree: $\boxed{7}$ $2 + 2 + 3 = 7$

Additive (or negative) inverse or opposite: the opposite of a term (two terms whose sum is 0).

Example: **1)** The additive inverse of **5** is $\boxed{-5}$ $5 + (-5) = 0$

2) The additive inverse of $-\frac{3}{4}\boldsymbol{y}$ is $\boxed{\frac{3}{4}y}$ $-\frac{3}{4}y + \frac{3}{4}y = 0$

3) The additive inverse of $\mathbf{4ab^3 - 3a^2 + b^3}$ is $\boxed{-4ab^3 + 3a^2 - b^3}$

Adding and Subtracting Polynomials

Add or subtract polynomials:

Example: Add $4x^3 - 5x^2 - x + 3$ and $3x^3 + 3x^2 - 5x + 2$.

Steps	Solution
	$\mathbf{(4x^3 - 5x^2 - x + 3) + (3x^3 + 3x^2 - 5x + 2)}$
▪ Regroup like terms:	$= (4x^3 + 3x^3) + (-5x^2 + 3x^2) + (-x - 5x) + (3 + 2)$
▪ Combine like terms:	$= \boxed{7x^3 - 2x^2 - 6x + 5}$

Example: Subtract $6x^2 + 7x - 5$ and $3x^2 - 4x + 16$.

Steps	Solution
	$\mathbf{(6x^2 + 7x - 5) - (3x^2 - 4x + 16)}$
▪ Remove parenthesis:	$= 6x^2 + 7x - 5 \mathbf{-} 3x^2 \mathbf{+} 4x \mathbf{-} 16$ (Reverse each sign in second parenthesis.)
▪ Regroup like terms:	$= (6x^2 - 3x^2) + (7x + 4x) + (-5 - 16)$
▪ Combine like terms:	$= \boxed{3x^2 + 11x - 21}$

Add or subtract polynomials using the column method:

Example: Add $4x^3 - 3x^2 + 6x - 5$ and $3x^3 + 2x + 3$.

Steps	Solution
▪ Line up like terms in columns: ▪ Add: Leave spaces for missing terms.	$$\begin{array}{rrrrr} & \mathbf{4x^3} & \mathbf{- 3x^2} & \mathbf{+ 6x} & \mathbf{- 5} \\ +) & \mathbf{3x^3} & & \mathbf{+ 2x} & \mathbf{+ 3} \\ \hline & \boxed{7x^3 - 3x^2 + 8x - 2} \end{array}$$

Example: Subtract $(7x^2 - 3x + 4)$ and $(3x^2 - 1)$.

Steps	Solution
▪ Line up like terms in columns: ▪ Change signs in the minuend and **add**:	$$\begin{array}{rrrrl} & \mathbf{7x^2} & \mathbf{- 3x} & \mathbf{+ 4} & \longleftarrow \text{Subtrahend} \\ +) & \mathbf{-3x^2} & & \mathbf{+ 1} & \longleftarrow \text{Minuend} \\ \hline & \boxed{4x^2 - 3x + 5} & & & \longleftarrow \text{Difference} \end{array}$$ Or $(7x^2 - 3x + 4) - (3x^2 - 1) = 7x^2 - 3x + 4 - 3x^2 + 1$ $= 4x^2 - 3x + 5$

Topic B: Multiplication of Polynomials

Multiplying Polynomials

Multiplying monomials

Example: $(-4a^2 b^3)(5a^3 b^5) = (-4 \cdot 5)(a^2 \cdot a^3)(b^3 \cdot b^5)$ Multiply the coefficients and add the exponents.

$= \boxed{-20\, a^5 b^8}$ $a^m a^n = a^{m+n}$

Multiplying monomial and polynomial

Example: $5x^2 (4x^3 - 3x) = (5x^2)(4x^3) - (5x^2)(3x)$ Distributive property: $a(b + c) = ab + ac$

$= (5 \cdot 4)(x^{2+3}) - (5 \cdot 3)(x^{2+1})$ Multiply the coefficients and add the exponents.

$= \boxed{20x^5 - 15x^3}$ $a^m a^n = a^{m+n}$

Example: $3xy^3 (4xy^2 + x^3 y - y)$ Distribute

$= (3xy^3)(4xy^2) + (3xy^3)(x^3 y) + (3xy^3)(-y)$ Multiply the coefficients and add the exponents.

$= \boxed{12x^2y^5 + 3x^4y^4 - 3xy^4}$ $a^m a^n = a^{m+n}$

Multiplying binomials (2 terms × 2 terms)

Example: Find the following product.

$(4a - 5)(2a - 3) = \overset{F}{4a \cdot 2a} + \overset{O}{4a(-3)} - \overset{I}{5 \cdot 2a} - \overset{L}{5(-3)}$ FOIL

$= 8a^2 - 12a - 10a + 15$ $a^m a^n = a^{m+n}$

$= \boxed{8a^2 - 22a + 15}$ Combine like terms.

Multiplying binomial and polynomial

Example: Multiply: $2x - 3x^2$ and $x^2 + x - 4$

Steps	**Solution**
	$(2x - 3x^2)(x^2 + x - 4)$
▪ Use the distributive property:	$= 2x \cdot x^2 + 2x \cdot x + 2x(-4) - 3x^2 \cdot x^2 - 3x^2 \cdot x - 3x^2(-4)$
▪ Multiply coefficients and add exponents:	$= 2x^3 + 2x^2 - 8x - 3x^4 - 3x^3 + 12x^2$
▪ Combine like terms and write in descending order:	$= \boxed{-3x^4 - x^3 + 14x^2 - 8x}$

Multiplying polynomials mentally (no need to write out each step).

Example: Multiply.

a) $2x^3 (3x^2 - 2) = \boxed{6x^5 - 4x^3}$ $a(b + c) = ab + ac$, $a^n a^m = a^{n+m}$

b) $(a - 3)(2a - 1) = \boxed{2a^2 - 7a + 3}$ FOIL

Special Binomial Products

Special binomial products – squaring binominals

Special products	Formula	Initial expansion	Example
Difference of squares	$(a+b)(a-b) = a^2 - b^2$ It does not matter if $(a-b)$ comes first	$(a+b)(a-b) = a^2 - \cancel{ab} + \cancel{ba} - b^2$ $= a^2 - b^2$	$(x+3)(x-3) = x^2 - 3^2 = x^2 - 9$ $(a = x,\ b = 3)$ or $(x-3)(x+3) = x^2 - 3^2 = x^2 - 9$
Square of sum	$(a+b)^2 = a^2 + 2ab + b^2$ A perfect square trinomial	$(a+b)^2 = (a+b)(a+b)$ $= a^2 + ab + ba + b^2$ $= a^2 + 2ab + b^2$	$(y+2)^2 = y^2 + 2 \cdot y \cdot 2 + 2^2$ $= y^2 + 4y + 4$
Square of difference	$(a-b)^2 = a^2 - 2ab + b^2$ A perfect square trinomial	$(a-b)^2 = (a-b)(a-b)$ $= a^2 - ab - ba + b^2$ $= a^2 - 2ab + b^2$	$(z-5)^2 = z^2 - 2 \cdot z \cdot 5 + 5^2$ $= z^2 - 10z + 25$

Special binomial products: special forms of binomial products that are worth memorizing.

Memory aid: $(a \pm b)^2 = (a^2 \pm 2ab + b^2)$ Notice the reversed plus or minus sign in the second term.

Example: Find the following products.

1) $(5x + 3)(5x - 3) = \overset{a}{\overset{\downarrow}{(5x)}}{}^2 - \overset{b}{\overset{\downarrow}{3}}{}^2$ $\qquad (a+b)(a-b) = a^2 - b^2$

$= \boxed{25x^2 - 9}$ $\qquad a = 5x \ , \ b = 3$

2) $(2t - 1)^2 = (2t)^2 - 2(2t) + 1^2$ $\qquad (a-b)^2 = a^2 - 2ab + b^2$

$= \boxed{4t^2 - 4t + 1}$ $\qquad a = 2t \ , \ b = 1$

3) $(3w + \frac{1}{3})^2 = (3w)^2 + 2(3w)(\frac{1}{3}) + (\frac{1}{3})^2$ $\qquad (a+b)^2 = a^2 + 2ab + b^2$

$= \boxed{9w^2 + 2w + \frac{1}{9}}$ $\qquad a = 3w \ , \ b = \frac{1}{3}$

4) $(5u - \frac{1}{2}v)^2 = (5u)^2 - 2(5u)(\frac{1}{2}v) + (\frac{1}{2}v)^2$ $\qquad (a-b)^2 = a^2 - 2ab + b^2$

$= \boxed{25u^2 - 5uv + \frac{1}{4}v^2}$ $\qquad a = 5u \ , \ b = \frac{1}{2}v$

5) $(\frac{1}{3}t - \frac{1}{2})(\frac{1}{3}t + \frac{1}{2}) = (\frac{1}{3}t)^2 - (\frac{1}{2})^2$ $\qquad (a+b)(a-b) = a^2 - b^2$

$= \boxed{\frac{1}{9}t^2 - \frac{1}{4}}$ $\qquad a = \frac{1}{3}t \ , \ b = \frac{1}{2}$

Topic C: Polynomial Division

Dividing Polynomials

Dividing a monomial by a monomial

Example: $\frac{\mathbf{14a^5}}{\mathbf{a^2}} = 14\,a^{5-2}$ Apply $\frac{a^m}{a^n} = a^{m-n}$

$= \boxed{14\,a^3}$

Example: $\frac{-28u^6v^2}{7u^4v^5}$

Steps	**Solution**
Divide the coefficients:	$\frac{\mathbf{-28u^6v^2}}{\mathbf{7u^4v^5}} = (\frac{-28}{7})\,(\frac{u^6v^2}{u^4v^5})$
Divide like variables (apply $\frac{a^m}{a^n} = a^{m-n}$):	$= -4\,(\frac{u^6}{u^4})(\frac{v^2}{v^5})$ $\quad \frac{v^2}{v^5} = v^{2-5} = v^{-3}$
	$= \boxed{-4\left(\frac{u^2}{v^3}\right)}$ $\quad a^{-m} = \frac{1}{a^m}$

Dividing a polynomial by a monomial

Example: $\frac{15a^2+5a-4}{5a}$

Steps	**Solution**
Split the polynomial into three parts:	$\frac{\mathbf{15a^2+5a-4}}{\mathbf{5a}} = \frac{15a^2}{5a} + \frac{5a}{5a} - \frac{4}{5a}$
Divide a monomial by a monomial:	$= \boxed{3a + 1 - \frac{4}{5a}}$

Example: $\frac{4x^2+8x+2x+4}{x+2}$

Steps	**Solution**
Group:	$\frac{\mathbf{4x^2+8x+2x+4}}{\mathbf{x+2}} = \frac{(4x^2+8x)+(2x+4)}{x+2}$
Factor out the greatest common factor (GCF):	$= \frac{4x(x+2)+2(x+2)}{x+2}$
Split the polynomial into two parts:	$= \frac{4x\cancel{(x+2)}}{\cancel{x+2}} + \frac{2\cancel{(x+2)}}{\cancel{x+2}}$
Divide a monomial by a monomial:	$= \boxed{4x + 2 = 2\,(2x + 1)}$

Long Division of Polynomials

Long division for numbers:

Example:

$$\begin{array}{r} \underline{\text{Quotient}} \\ \text{Divisor})\,\text{Dividend} \\ -\underline{\phantom{\text{Dividend}}} \\ \text{Remainder} \end{array} \qquad\qquad \begin{array}{r} \underline{7} \\ 3\,)22 \\ -\ \underline{21} \\ 1 \end{array}$$

Polynomial long division: a method used for dividing a polynomial by another polynomial of the same or lower degree (it is very similar to long division for numbers).

Example: $\frac{4x^2+8x+1}{2x}$

Steps	Solution	Long division for numbers
Write in ***divisor) Dividend*** form:	$\mathbf{2x\,)\,\overline{4x^2+8x+1}}$	$2)\,\overline{481}$
Divide the first term:	$\begin{array}{rl} & 2x \\ 2x\,) & \overline{4x^2+8x+1} \\ - & \underline{4x^2 \qquad (2x)(2x)=4x^2} \end{array}$	$\begin{array}{rl} & 2 \\ 2) & \overline{481} \\ - & \underline{4} \end{array}$ $2\cdot 2=4$
Divide the second term:	$\begin{array}{rl} & 2x + 4 \\ 2x\,) & \overline{4x^2+8x+1} \\ & \underline{4x^2} \\ \text{Bring } 8x \text{ down} & 8x \\ (2x)(4)=8x & -\underline{\ 8x} \\ & 1 \end{array}$	$\begin{array}{rl} & 240 \\ 2) & \overline{481} \\ & \underline{4} \\ & 8 \quad \text{Bring 8 down} \\ & \underline{-8} \quad 2\cdot 4=8 \\ & 1 \end{array}$

Remainder

- Quotient = $\boxed{2x+4}$, remainder = $\boxed{1}$
- Tip: continue until the degree of the remainder is less than the degree of the divisor.
  (i.e. $1 = 1\cdot x^{\mathbf{0}}$ and $2x = 2x^{\mathbf{1}}$, $0 < 1$)

$$\begin{array}{r} \underline{\text{Quotient}} \\ \text{Divisor})\,\text{Dividend} \\ -\underline{\phantom{\text{Dividend}}} \\ \text{Remainder} \end{array}$$

- Check: $\boxed{\text{Dividend} = \text{Quotient}\cdot\text{Divisor} + \text{Remainder}}$

$$4x^2+8x+1 \overset{?}{=} (2x+4)\,(2x)+1$$

Distribute

$$4x^2+8x+1 \overset{\surd}{=} 4x^2+8x+1$$

Correct!

Missing Terms in Long Division

If there is a missing consecutive power term in a polynomial (i.e. if there are x^3 and x but not x^2), add in the missing term with a coefficient of 0.

Example: $\frac{7-4x^2+x^3}{1+x}$

Steps | **Solution**

- Rewrite both polynomials in descending order:
  Descending order: $Ax^3 + Bx^2 + Cx + D$, $Ax + B$

$$\frac{x^3-4x^2+7}{x+1}$$

- Write in **divisor) $\overline{\textbf{Dividend}}$** form and insert a 0 coefficient for the missing power term.

$$x+1\,)\,\overline{x^3 - 4x^2 + \mathbf{0x} + 7}$$

Missing power

- Divide as usual:

$$\begin{array}{rlr}
 & \;x^2 - 5x + 5 & \\
\mathbf{x+1\,)} & \overline{\mathbf{x^3 - 4x^2 + 0x + 7}} & (x^2)(x) = x^3 \\
-) & \underline{x^3 + x^2} & (x^2)(1) = x^2 \\
 & \quad -5x^2 + 0x & (-5x)(x) = -5x^2 \\
-) & \quad \underline{-5x^2 - 5x} & (-5x)(1) = -5x \\
 & \qquad\quad 5x + 7 & (5)(x) = 5x \\
-) & \qquad\quad \underline{5x + 5} & (5)(1) = 5 \\
 & \qquad\qquad\quad 2 &
\end{array}$$

- Quotient = $\boxed{x^2 - 5x + 5}$, remainder = $\boxed{2}$
- Check: Dividend = Quotient · Divisor + Remainder

$$\begin{array}{c} \underline{\text{Quotient}} \\ \text{Divisor) Dividend} \\ -\underline{\qquad\qquad} \\ \text{Remainder} \end{array}$$

$$7 - 4x^2 + x^3 = (x^2 - 5x + 5)(x + 1) + 2$$

Distribute

$$7 - 4x^2 + x^3 \overset{?}{=} (x^3 + x^2 - 5x^2 - 5x + 5x + 5) + 2$$

Combine like terms.

$$7 - 4x^2 + x^3 \overset{\surd}{=} x^3 - 4x^2 + 7$$

Correct!

Unit 13: Summary

More about Polynomials

Basic algebraic terms:

Algebraic term	Definition
Algebraic expression	A mathematical phrase that contains numbers, variables, and arithmetic operations.
Constant	A number.
Variable	A letter that can be assigned different values.
Coefficient	The number that is in front of a variable.
Term	A term can be a constant, a variable, or the product of a number and variable(s). (Terms are separated by a plus or minus sign.)
Like terms	The terms that have the same variables and exponents.
Factor	A number or variable that multiplies with another.

Polynomial	**Description**
Monomial	One term.
Binomial	Two terms.
Trinomial	Three terms.
Polynomial	One or more terms.

The degree of a term with more variables: the sum of the exponents of its variables.

The degree of a polynomial with more variables: the highest degree of any individual term.

Additive (or negative) inverse or opposite: the opposite of a term.

Add or subtract polynomials:

- Regroup like terms.
- Combine like terms.

Add polynomials using the column method:

- Line up like terms in columns.
- Add.

Subtract polynomials using the column method:

- Line up like terms in columns.
- Change signs in minuend and add.

Multiplying binomial and polynomial:

- Use the distributive property. $a\,(b + c) = ab + ac$
- Multiply coefficients and add exponents. Apply $\frac{a^m}{a^n} = a^{m-n}$
- Combine like terms and write in descending order.

Special binomial products – squaring binominals

Special products	Formula
Difference of squares	$(a + b)(a - b) = a^2 - b^2$
Square of sum	$(a + b)^2 = a^2 + 2ab + b^2$
Square of difference	$(a - b)^2 = a^2 - 2ab + b^2$

Memory aid: $(a \pm b)^2 = (a^2 \pm 2ab + b^2)$

Dividing a monomial by a monomial

- Divide coefficients.
- Divide like variables (apply $\frac{a^m}{a^n} = a^{m-n}$).

Dividing a polynomial by a monomial

- Split the polynomial into parts.
- Divide a monomial by a monomial.

Polynomial long division: a method used for dividing a polynomial by another polynomial of the same or lower degree (it is very similar to long division for numbers).

Example: $\frac{8-3x+x^3}{2+x}$

Steps | **Solution**

- Rewrite both polynomials in descending order:

  Descending order: $Ax^3 + Bx^2 + Cx + D$, $Ax + B$

$$\frac{x^3-3x+8}{x+2}$$

- Write in **divisor) $\overline{\textbf{Dividend}}$** form and insert a 0 coefficient for the missing power term.

$$x+2 \,\big)\, \overline{x^3 + \mathbf{0}x^2 - 3\boldsymbol{x} + 8}$$

Missing power

- Divide as usual:

$$\begin{array}{rl l}
 & \;x^2 - 2x + 1 & \\
x+2 & \big)\,\overline{x^3 + \mathbf{0}x^2 - 3\boldsymbol{x} + 8} & (x^2)(x) = x^3 \\
-) & \;\underline{x^3 + 2x^2} & (x^2)(2) = 2x^2 \\
 & \qquad -2x^2 - 3x & (-2x)(x) = -2x^2 \\
-) & \qquad \underline{-2x^2 - 4x} & (-2x)(2) = -4x \\
 & \qquad\qquad\quad x + 8 & (1)(x) = x \\
-) & \qquad\qquad\quad \underline{x + 2} & (1)(2) = 2 \\
 & \qquad\qquad\qquad 6 &
\end{array}$$

- Quotient = $x^2 - 2x + 1$, remainder = 6
- Tip: continue until the degree of the remainder is less than the degree of the divisor.
- Check: Dividend = Quotient · Divisor + Remainder

$$\begin{array}{r}
\underline{\text{Quotient}} \\
\text{Divisor) Dividend} \\
\underline{-\qquad\qquad} \\
\text{Remainder}
\end{array}$$

Unit 13: Self-Test

More about Polynomials

Topic A

1. Determine the degree of the following.

a) $-8x^4 y^3 z^5$

b) $21x^5y + 32x^2y^3z + 6x^3y^4z^2$

c) $3.5a^4b + 6.1a^4b^3c - 7.3a + 5.4$

2. Determine the additive inverse.

a) $8y$

b) $-\frac{5}{8}x$

c) $9xy^2 - 4x^2 + y^3$

3. Add $5x^4 - 3x^3 - x + 7$ and $4x^4 + 2x^3 - 7x + 3$.

4. Subtract $8x^2 + 5x - 4$ and $4x^2 - 2x + 14$.

5. Add or subtract polynomials using the column method:

a) Add $7a^3 - 4a^2 + 3a - 6$ and $4a^3 + 6a + 8$.

b) Subtract $(9x^2 - 4x + 8)$ and $(4x^2 - 3)$.

Topic B

6. Multiply.

a) $(-6x^3 y^2)(4x^4 y^3)$

b) $4a^2(3a^4 - 6a)$

c) $7xy^2(2xy^4 + x^3y - 3y)$

d) $(3x - 4)(4x - 5)$

e) $(3a - 2a^2)(a^2 + a - 5)$

7. Find the following product.

a) $4t^4(2t^3 - 5)$

b) $(x - 5)(3x - 2)$

c) $(6a + 5)(6a - 5)$

d) $(3w - 1)^2$

e) $(5u + \frac{1}{2})^2$

f) $(6x - \frac{1}{3}y)^2$

g) $(\frac{1}{5}z - \frac{1}{4})(\frac{1}{5}z + \frac{1}{4})$

Topic C

8. Divide the following.

a) $\frac{56x^6}{x^3}$

b) $\frac{-81a^5b^3}{9a^3b^6}$

c) $\frac{28y^2+7y-3}{7y}$

d) $\frac{6a^2+18a+3a+9}{a+3}$

9. Use long division to divide the following.

a) $\frac{9x^2+6x+2}{3x}$

b) $\frac{30-3x^2+2x^3}{2+x}$