# Unit 6

# Polynomials

**Topic A: Introduction to polynomials**

- Polynomials
- Degree of a polynomial
- Combine like terms
- Removing parentheses

**Topic B: Multiplying and dividing polynomials**

- Multiplying and dividing monomials
- Multiplying / dividing polynomials by monomials
- FOIL method to multiply binomials

**Unit 6 Summary**

**Unit R Self-test**

# Topic A: Introduction to Polynomials

## Polynomials

**Basic algebraic terms:**

| Algebraic term | Description | Example |
|---|---|---|
| **Algebraic expression** | A mathematical phrase that contains numbers, variables (letters), and arithmetic operations (+, – , × , ÷ , etc.). | $3x - 4$<br>$5a^2 - b + 3$<br>$12y^3 + 7y^2 - 5y + \frac{2}{3}$ |
| **Constant** | A number on its own. | $2y + 5$ constant: 5 |
| **Coefficient** | The number in front of a variable. | $-9x^2$ coefficient: -9<br>$x$ coefficient: 1 $(x = 1 \cdot x)$ |
| **Term** | A term can be a constant, a variable, or the product of a number and variable. (Terms are separated by a plus or minus sign.) | $2x^3 + 7x^2 - 9y - 8$<br>Terms: $2x^3$, $7x^2$, $-9y$, -8 |
| **Like terms** | The terms that have the same variables and exponents (differ only in their coefficients). | $2x$ and $-7x$<br>$-4y^2$ and $9y^2$<br>$0.5pq^2$ and $\frac{2}{3}pq^2$ |

**Polynomial**: an algebraic expression that contains one or more terms.

The prefix "poly-" means many.

**Example:** $7x$ , $5ax - 9b$ , $6x^2 - 5x + \frac{2}{3}$ , $7a^2 + 8b + ab - 5$

**There are special names for polynomials that have one, two, or three terms:**

- **Mono**mial: an algebraic expression that contains only one term.

  **Example**: $9x$ , $4xy^2$ , $0.8mn^2$ , $\frac{1}{3}a^2b$ The prefix "mono" means one.

- **Bino**mial: an algebraic expression that contains two terms. The prefix "bino-" means two.

  **Example:** $7x + 9$ , $9t^2 - 2t$ , $0.3y + \frac{1}{3}$

- **Tri**nomial: an algebraic expression that contains three terms.

  **Example:** $ax^2 + bx + c$ , $-4qp^2 + 3q + 5$ The prefix "tri-" means three.

**Polynomials in ascending or descending order:** a polynomial can be arranged in ascending or descending order.

- Descending order: the exponents of variables are arranged from largest to smallest number.

  **Example**: $5a^3 - 3a^2 + a + 1$ The exponents of $a$ decrease from left to right.

  $19y^4 + 31y^3 - y^2 + 2y - \frac{2}{3}$ The exponents of $y$ decrease from left to right.

- Ascending order: the exponents of variables are arranged from smallest to largest number.

  **Example**: $2 - 0.3x + 4.5x^2 - 7x^3$ The exponents of $x$ increase from left to right.

  $7 + \frac{3}{7}w + 4w^2 - 8w^3 + w^4$ The exponents of $w$ increase from left to right.

# Degree of a Polynomial

**Classification of polynomial:** polynomials are classified according to their number of terms and degrees.

**Degree of a term:**

- The degree of a term with one variable: the exponent of its variable.

  **Example:** $9x^3$ The degree of the term: 3

  $-7y^5$ The degree of the term: 5

- The degree of a term with more variables: the sum of the exponents of its variables.

  **Example:** $-8a^2 b^3 c^6$ The degree of the term: 11 $(2 + 3 + 6 = 11)$

- **More examples:**

| Monomial | Degree | Reason |
|---|---|---|
| $4x$ | 1 | $x = x^1$ ($x$ has an exponent of 1.) |
| $7xy^3$ | 4 | $1 + 3 = 4$ |
| $-\frac{3}{5}x^2y^4z$ | 7 | $2 + 4 + 1 = 7$ $(z = z^1)$ |
| 13 | 0 | $13 = 13 \cdot 1 = 13 \cdot x^0 = 13$ $(x^0 = 1)$ |

**Degree of a polynomial:** the highest degree of any individual term in it.

**Examples**:

| Polynomial | Degree | Reason |
|---|---|---|
| $7x^8 + 5x^5 + 8$ | 8 | The highest exponent of the term is $7x^8$. |
| $3a^2 + 4a^2b^3 + 7a^4b^5c^2$ <br> 2 $\quad$ 2 + 3 = 5 $\quad$ 4 + 5 + 2 = 11 | 11 | The highest degree of the term is $7a^4b^5c^2$. |

**Example**: Arrange polynomials in descending order and identify the degrees and coefficients.

**a)** $5 + 2a - 4a^2 + a^3$

Descending order: $a^3 - 4a^2 + 2a + 5$

Coefficients: 1 -4 2

Degree of the polynomial: 3

**b)** $-2xy + 9x^3 + 5x^5y + \frac{3}{4} + 7x^2 - \frac{1}{2}x^4$

Descending order: $5x^5y - \frac{1}{2}x^4 + 9x^3 + 7x^2 - 2xy + \frac{3}{4}$

Coefficients: 5 $-\frac{1}{2}$ 9 7 -2

Degree of the polynomial: 6 $y = y^1$

# Combine Like Terms

**Like terms:** terms that have the same variables and exponents (the coefficients can be different).

**Examples:**

| Example | Like or unlike terms |
|---|---|
| $7y$ and $-9y$ | Like terms |
| $6a^2$, $-32a^2$, and $-a^2$ | Like terms |
| $0.3\,x^2y$ and $-4.8\,x^2y$ | Like terms |
| $\frac{-2}{7}u^2v^3$ and $\frac{3}{5}u^2v^3$ | Like terms |
| $-8y$ and $78x$ | Unlike terms |
| $6m^3$ and $-9m^2$ | Unlike terms |
| $-9u^3w^2$ and $-9w^3u^2$ | Unlike terms |

**Combine like terms:** add or subtract their coefficients and keep the same variables and exponents.

**Note**: unlike terms cannot be combined.

**Example:** Combine like terms.

**a)** $3a + 7b - 9a + 15b = (3a - 9a) + (7b + 15b)$ Regroup like terms.

$= \boxed{-6a + 22b}$ Combine like terms.

**b)** $2y^2 - 4x + 3x - 5y^2 = (2y^2 - 5y^2) + (-4x + 3x)$ Regroup like terms.

$= -3y^2 - 1x$ Combine like terms.

$= \boxed{-3y^2 - x}$

**c)** $8xy^2 - x^2y + 4x^2y - 6xy^2$

$= \underline{8xy^2} - \mathbf{x^2y} + \mathbf{4x^2y} \underline{- 6xy^2}$ Or underline like terms and without regrouping.

$= \boxed{2xy^2 + 3x^2y}$ Combine like terms.

**d)** $2(2m + 3n) + 3(m - 4n) = \underline{4m} + \mathbf{6n} + \underline{3m} - \mathbf{12n}$ Distributive property.

$= \boxed{7m - 6n}$ Combine like terms.

**e)** $8v + 4(2v - u^2) + 3(u^2 + v) = \underline{8v} + \underline{8v} - \mathbf{4u^2} + \mathbf{3u^2} + \underline{3v}$ Distributive property.

$= \boxed{-u^2 + 19v}$ Combine like terms.

# Removing Parentheses

**If the sign preceding the parentheses is positive (+),** do not change the sign of terms inside the parentheses, just remove the parentheses.

**Example**: $(x - 5) = x - 5$

**If the sign preceding the parentheses is negative (-),** remove the parentheses and the negative sign (in front of parentheses), and change the sign of each term inside the parentheses.

**Example**: $-(x - 7) = -x + 7$

**Remove parentheses:**

| Algebraic expression | Remove parentheses | Example |
|---|---|---|
| $(ax + b)$ | $ax + b$ | $(5x + 2) = 5x + 2$ |
| $(ax - b)$ | $ax - b$ | $(9y - 4) = 9y - 4$ |
| $-(ax + b)$ | $-ax - b$ | $-(\frac{3}{4}x + 7) = -\frac{3}{4}x - 7$ |
| $-(ax - b)$ | $-ax + b$ | $-(0.5b - 2.4) = -0.5b + 2.4$ |

**Example:** Simplify.

**a)** $9x^2 + 7 - (2x^2 - 2) = \underline{9x^2} + \mathbf{7} - \underline{2x^2} + \mathbf{2}$ Remove parentheses.

$= \boxed{7x^2 + 9}$ Combine like terms.

**b)** $(-8y + 5z) - 4(y - 7z) = \underline{-8y} + \mathbf{5z} - \underline{4y} + \mathbf{28z}$ Remove parentheses.

$= \boxed{-12y + 33z}$ Combine like terms.

**c)** $-(3a^2 + 4a - 4) + 3(4a^2 - 6a + 7)$ Remove parentheses.

$= \mathbf{-3a^2} \underline{- 4a} + 4 + \mathbf{12a^2} \underline{- 18a} + 21$ Distributive property.

$= \boxed{9a^2 - 22a + 25}$ Combine like terms.

**d)** $-5(u^2 - 3u) + 3(2u - 4) - (5 - 3u + 4u^2)$ Distributive property.

$= \mathbf{-5u^2} + 15\underline{u} + \underline{6u} - 12 - 5 \underline{+ 3u} - \mathbf{4u^2}$ Remove parentheses.

$= \boxed{-9u^2 + 24u - 17}$ Combine like terms.

**e)** $8(pq - 4cd) - 3(-pq + 5cd) = \underline{8pq} - \mathbf{32cd} \underline{+ 3pq} - \mathbf{15cd}$ Distributive property.

$= \boxed{11pq - 47cd}$ Combine like terms.

# Topic B: Multiplying and Dividing Polynomials

## Multiplying and Dividing Monomials

**Basic rules of exponents:**

| Name | Rule | Example |
|---|---|---|
| Product of like bases (The same base) | $a^m\, a^n = a^{m+n}$ | $2^3\, 2^2 = 2^{3+2} = 2^5$ Since $2^3\, 2^2 = (2 \cdot 2 \cdot 2)\,(2 \cdot 2) = 2^5$ |
| Quotient of like bases (The same base) | $\frac{a^m}{a^n} = a^{m-n}$ | $\frac{x^5}{x^3} = x^{5-3} = x^2$ Since $\frac{x^5}{x^3} = \frac{x\,x\,x\,x\,x}{x\,x\,x} = x^2$ |
| Negative exponent $a^{-n}$ | $a^{-n} = \frac{1}{a^n}$ | $3^{-2} = \frac{1}{3^2} = \frac{1}{9} = 0.11$ Since $3^{-1} = \frac{1}{3} = 1 \div 3$, $3^{-2} = \frac{1}{3 \cdot 3} = \frac{1}{9} \approx 0.11$ |

**Example**: Simplify the following.

**a)** $x^4\, x^3 = x^{4+3} = \boxed{x^7}$ $\quad a^m\, a^n = a^{m+n}$

**b)** $\frac{y^{-6}}{y^3} = y^{-6-3} = y^{-9} = \boxed{\frac{1}{y^9}}$ $\quad \frac{a^m}{a^n} = a^{m-n}$, $a^{-n} = \frac{1}{a^n}$

**Multiplying monomials (one term):**

- Regroup coefficients and variables.
- Multiply coefficients (the numbers in front of the variable).
- Multiply variables (add exponents with the same base, apply $a^m\, a^n = a^{m+n}$).

**Example**: **1)** $(-4x^4\, y^3)\,(7x^3\, y^2) = (-4 \cdot 7)\,(x^4 \cdot x^3)\,(y^3 \cdot y^2)$ Regroup the coefficients & the variables.

$= -28\, x^{4+3}\, y^{3+2} = \boxed{-28\, x^7 y^5}$ Multiply the coefficients & add the exponents. $a^m\, a^n = a^{m+n}$

**2)** $\left(\frac{3}{4}a^2b^3c^2\right)\left(\frac{4}{6}ab^2c^2\right) = \left(\frac{3}{4} \cdot \frac{4}{6}\right)(a^2a)\,(b^3b^2)(c^2c^2)$ Regroup.

$= \boxed{\frac{1}{2}a^3b^5c^4}$ $\quad a = a^1$, $\quad a^m\, a^n = a^{m+n}$

**Dividing monomials:**

- Divide coefficients.
- Divide variables (subtract exponents with the same base, apply $\frac{a^m}{a^n} = a^{m-n}$).

**Example**: **1)** $\frac{4a^5}{16a^2} = \left(\frac{4}{16}\right)\left(\frac{a^5}{a^2}\right)$ Regroup the coefficients & the variables.

$= \frac{1}{4}\, a^{5-2} = \boxed{\frac{1}{4}a^3}$ Divide the coefficients & subtract the exponents.

**2)** $\frac{t^2}{t^7} = t^{2-7} = t^{-5} = \boxed{\frac{1}{t^5}}$ $\quad \frac{a^m}{a^n} = a^{m-n}$, $a^{-n} = \frac{1}{a^n}$

**3)** $\frac{-12x^2y^5}{4x^3y^5} = \left(\frac{-12}{4}\right)\left(\frac{x^2}{x^3}\right)\left(\frac{y^5}{y^5}\right)$ Regroup.

$= -3x^{2-3}y^{5-5}$ $\quad \frac{a^m}{a^n} = a^{m-n}$

$= -3x^{-1}y^0 = \boxed{\frac{-3}{x}}$ $\quad x^{-1} = \frac{1}{x^1} = \frac{1}{x}$, $\quad y^0 = 1$

# Multiplying / Dividing Polynomials by Monomials

## Multiplying a monomial and a polynomial:

- Use the distributive property: $a\,(b + c) = ab + ac$
- Multiply coefficients and add exponents with the same base. Apply $a^m\,a^n = a^{m+n}$

**Examples**:

**1)** $3x^3\,(5x^2 - 2x) = (3x^3)\,(5x^2) - (3x^3)\,(2x)$ Distributive property: $a\,(b + c) = ab + ac$

$= (3\cdot 5)\,(x^3\,x^2) - (3\cdot 2)\,(x^3\,x^1)$ Regroup $x = x^1$

$= 15\,(x^{3+2}) - 6\,(x^{3+1})$ Multiply the coefficients & add the exponents.

$= \boxed{15x^5 - 6x^4}$ $a^m \cdot a^n = a^{m+n}$

**2)** $5ab^2\,(2a^2b + ab^2 - a)$ Distribute.

$= (5ab^2)\,(2a^2b) + (5ab^2)\,(ab^2) + (5ab^2)\,(-a)$ Multiply the coefficients and add exponents.

$= (5\cdot 2)\,(a^{1+2}\,b^{2+1}) + (5a^{1+1}\,b^{2+2}) - (5a^{1+1}b^2)$ $b = b^1$ , $a = a^1$

$= \boxed{10a^3b^3 + 5a^2b^4 - 5a^2b^2}$ $a^m \cdot a^n = a^{m+n}$

## Dividing a polynomial by a monomial

- Split the polynomial into several parts.
- Divide a monomial by a monomial. Apply $\frac{a^m}{a^n} = a^{m-n}$.

**Example**: $\frac{12x^2+4x-2}{4x}$

| **Steps** | **Solution** |
|---|---|
| ▪ Split the polynomial into three parts: | $\frac{12x^2+4x-2}{4x} = \frac{12x^2}{4x} + \frac{4x}{4x} - \frac{2}{4x}$ |
| ▪ Divide a monomial by a monomial: | $= \boxed{3x + 1 - \frac{1}{2x}}$ $\frac{a^m}{a^n} = a^{m-n}$ |

# FOIL Method to Multiply Binomials

**The FOIL method:** an easy way to find the product of two binomials (two terms).

| $(a + b)(c + d) = ac + ad + bc + bd$ <br> **F O I L** | | | **Example** |
|---|---|---|---|
| **F** - First terms | first term × first term | $(a + b)(c + d)$ | $(x + 5)(x + 4)$ |
| **O** - Outer terms | outside term × outside term | $(a + b)(c + d)$ | $(x + 5)(x + 4)$ |
| **I** - Inner terms | inside term × inside term | $(a + b)(c + d)$ | $(x + 5)(x + 4)$ |
| **L** - Last terms | last term × last term | $(a + b)(c + d)$ | $(x + 5)(x + 4)$ |

| FOIL method | Example |
|---|---|
| $(a + b)(c + d) = ac + ad + bc + bd$ <br> **F O I L** | $(x + 5)(x + 4) = x \cdot x + x \cdot 4 + 5x + 5 \cdot 4 = x^2 + 9x + 20$ <br> **F O I L** |

## Multiplying binomials (2 terms × 2 terms)

**Example**: Multiply.

**1)** $(2x + 3)(5x - 6) = 2x \cdot 5x + 2x(-6) + 3 \cdot 5x + 3(-6)$ <br> **F O I L** — The FOIL method.

$= 10x^2 - 12x + 15x - 18$ — $a^n a^m = a^{n+m}$

$= \boxed{10x^2 + 3x - 18}$ — Combine like terms.

**2)** $(3r - t)(5r + t^2) = 3r \cdot 5r + 3r \cdot t^2 - t \cdot 5r - t \cdot t^2$ — FOIL

$= \boxed{15r^2 + 3rt^2 - 5rt - t^3}$ — $a^n a^m = a^{n+m}$

**3)** $(xy^2 + y)(2x^2y + x) = xy^2 \cdot 2x^2y + xy^2 \cdot x + y \cdot 2x^2 y + y\,x$ — FOIL

$= 2x^3 y^3 + x^2y^2 + 2x^2 y^2 + x\,y$ — $a^n a^m = a^{n+m}$

$= \boxed{2x^3 y^3 + 3x^2y^2 + x\,y}$ — Combine like terms.

**4)** $(a - \frac{1}{3})(a - \frac{1}{3}) = a^2 - \frac{1}{3}a - \frac{1}{3}a + (-\frac{1}{3})(-\frac{1}{3})$ — FOIL

$= \boxed{a^2 - \frac{2}{3}a + \frac{1}{9}}$ . — Combine like terms.

# Unit 6: Summary

## Polynomials

**Basic algebraic terms:**

| Algebraic term | Description | Example |
|---|---|---|
| **Algebraic expression** | A mathematical phrase that contains numbers, variables (letters), and arithmetic operations (+, –, ×, ÷, etc.). | $3x - 4$, $5a^2 - b + 3$ |
| **Constant** | A number on its own. | $2y + 5$ constant: 5 |
| **Coefficient** | The number in front of a variable. | $-9x^2$ coefficient: -9<br>$x$ coefficient: 1 |
| **Term** | A term can be a constant, a variable, or the product of a number and variable.<br>(Terms are separated by a plus or minus sign.) | $7a^2 - 6b + 8$<br>Terms: $7a^2$, $-6b$, 8 |
| **Like terms** | The terms that have the same variables and exponents (differ only in their coefficients). | $2x$ and $-7x$<br>$-4y^2$ and $9y^2$ |

| Polynomial | Example |
|---|---|
| **Mo**nomial **(one term)** | $0.67x$ |
| **Bi**nomial **(two terms)** | $4x - \frac{2}{3}$ |
| **Tri**nomial **(three terms)** | $2a^2 - ab + 5$ |
| Polynomial (one or more terms) | $2xy$, $4x^3 + 11$, $-\frac{2x}{3} + x - 5y + 4$ |

**Descending order:** the exponents of variables are arranged from largest to smallest number.

**Ascending order:** the exponents of variables are arranged from smallest to largest number.

**Degree of a term/polynomial:**

- The degree of a term with one variable: the exponent of its variable.
- The degree of a term with more variables: the sum of the exponents of its variables.
- Degree of a polynomial: the highest degree of any individual term in it.

**Like terms:** terms that have the same variables and exponents (the coefficients can be different.)

**Combine like terms:** add or subtract their numerical coefficients and keep the same variables and exponents.

**Remove parentheses:**

- If the sign preceding the parentheses is positive (+), do not change the sign of terms inside the parentheses, just remove the parentheses.

- If the sign preceding the parentheses is negative (-), remove the parentheses and the negative sign (in front of parentheses), and change the sign of terms inside the parentheses.

**Basic rules of exponents:**

| Name | Rule | | Example |
|---|---|---|---|
| Product of like bases (The same base) | $a^m\, a^n = a^{m+n}$ | $(a \neq 0)$ | $2^3\, 2^2 = 2^{3+2} = 2^5 = 32$ |
| Quotient of like bases (The same base) | $\frac{a^m}{a^n} = a^{m-n}$ | $(a \neq 0)$ | $\frac{y^3}{y^2} = y^{3-2} = y^1 = y$ |
| Negative exponent $a^{-n}$ | $a^{-n} = \frac{1}{a^n}$ | $(a \neq 0)$ | $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$ |

**Multiply monomials (one term):**

- Multiply coefficients.
- Multiply variables (add exponents with the same base, apply $a^m\, a^n = a^{m+n}$).

**Dividing monomials:**

- Divide coefficients.
- Divide variables (subtract exponents with the same base, apply $\frac{a^m}{a^n} = a^{m-n}$).

**Multiplying a monomial and a polynomial:**

- Use the distributive property: $a\,(b + c) = ab + ac$
- Multiply coefficients and add exponents with the same base. Apply $a^m\, a^n = a^{m+n}$

**Dividing a polynomial by a monomial**

- Split the polynomial into several parts.
- Divide a monomial by a monomial. Apply $\frac{a^m}{a^n} = a^{m-n}$

**The FOIL method:**

| $(a + b)\,(c + d) = ac + ad + bc + bd$ <br> F O I L | | |
|---|---|---|
| **F** - First terms | first term × first term | $(a + b)\,(c + d)$ |
| **O** - Outer terms | outside term × outside term | $(a + b)\,(c + d)$ |
| **I** - Inner terms | inside term × inside term | $(a + b)\,(c + d)$ |
| **L** - Last terms | last term × last term | $(a + b)\,(c + d)$ |

# Unit 6: Self-Test

## Polynomials

### Topic A

**1.** Identify the terms of each polynomial.

**a)** $5x^3 - 8x^2 + 2x$

**b)** $-\frac{2}{3}y^4 + 9a^2 + a - 1$

**2.** Identify the coefficients and the degree of the polynomials.

**a)** $2a^3 - 7a^2b^3 + 9b + 11$

**b)** $-8xy^5 - \frac{2}{3}y^4 + 11x^2y^3 + 4y^2 - 23y + \frac{5}{6}$

**3.** Identify each polynomial as a monomial, binomial, or trinomial.

**a)** $3x^2 - 7x$

**b)** $-29xy^3$

**c)** $8mn^2 + 7m - 45$

**4.** Arrange polynomials in descending order.

**a)** $3 + 8x - 23x^2 + 15x^3$

**b)** $-3y^3 - 45y^2 + 4y + \frac{2}{3}y^4$

**5.** Combine like terms.

**a)** $7x + 10y - 8x + 9y$

**b)** $12a^2 - 33b + 2b - 6a^2$

**c)** $12uv^2 - 5u^2v + 15u^2v - 8uv^2$

**d)** $5(4t - 6r) + 3(t + 7r)$

**e)** $13n + 5(6n - m^2) + 7(2m^2 + 3n)$

**6.** Simplify.

**a)** $15a^2 + 9 - (5a^2 - 4)$

**b)** $(-13x + 9y) - 6(x - 5y)$

**c)** $-(7z^2 + 6z - 15) + 2(7z^2 - 5z + 8)$

**d)** $-11(y^2 - 3y) + 4(2y - 5) - (13 - 6y + 9y^2)$

**e)** $5(ab - 2xy) - 6(-2ab + 3xy)$

## Topic B

**7.** Simplify the following.

**a)** $a^3\, a^6$

**b)** $\frac{x^{-4}}{x^7}$

**c)** $\frac{t^3}{t^9}$

**d)** $(-6a^3\, b^5)\,(7a^4\, b^6)$

**e)** $\left(\frac{5}{6}x^3y^4z^5\right)\left(\frac{3}{10}xy^3z^4\right)$

**f)** $\frac{6y^8}{36y^3}$

**g)** $\frac{-81m^3n^9}{9m^4n^9}$

**8.** Perform the indicated operation.

**a)** $-4x^3\,(3x^4 - 7x)$

**b)** $9a^3b\,(3ab^2 + 2a^2\,b^2 - a)$

**c)** $\frac{35a^2 + 5a - 4}{5a}$

**d)** $(5y - 7)\,(8y + 9)$

**e)** $(7r - 2t)\,(3r + 4t^2)$

**f)** $(2ab^2 + 3b)\,(5a^2b + 3a)$

**g)** $\left(x - \frac{1}{3}\right)\left(x - \frac{2}{3}\right)$