Unit 14

Factoring Polynomials

Topic A: Factoring

- Highest / greatest common factor
- Factoring polynomials by grouping
- Factoring difference of squares

Topic B: Factoring trinomials

- Factoring $x^2 + b\,x + c$
- Factoring $ax^2 + b\,x + c$
- More on factoring $ax^2 + b\,x + c$
- Factoring trinomials: AC method
- Factoring special products

Topic C: Application of factoring

- Quadratic equations
- Solving quadratic equations
- Application of quadratic equations

Unit 14 Summary

Unit 14 Self-test

Topic A: Factoring

Highest / Greatest Common Factor

Factoring whole numbers: write the number as a product (multiply) of its prime factors.

Prime factor: it is a prime number that has only two factors, 1 and itself.

Example: Factor 42.

$\mathbf{42} = \boxed{2 \cdot 3 \cdot 7}$ 2, 3 and 7 are prime factors.

Common factor: a number or an expression that is a factor of each term of a group of terms.

Greatest / highest common factor (GCF or HCF): the product of the common factors.

Examples:

Expression	Factors	Common factor	GCF or GCF
30 42	$\mathbf{2} \cdot \mathbf{3} \cdot 5$ $\mathbf{2} \cdot \mathbf{3} \cdot 7$	2, 3	6
$2xy^3$ $6xy^2$	$\mathbf{2} \cdot \boldsymbol{x} \cdot y \cdot \boldsymbol{y^2}$ $\mathbf{2} \cdot 3 \cdot \boldsymbol{x} \cdot \boldsymbol{y^2}$	2, x, y^2	$2xy^2$

$2 \cdot 3 = 6$

$2 \cdot x \cdot y^2 = 2xy^2$

Factoring a polynomial: express a polynomial as a product of other polynomials. (Factoring is the reverse of multiplication.)

Multiplying (or expanding) →

$$(a + b)\,c = ac + bc$$

← Factoring

Distributive property.

The common factor is c.

Example:

Multiplying	Factoring	GCF or HCF
$3xy\,(2x - 4xy + 3)$ $= 6x^2y - 12x^2y^2 + 9xy$	$6x^2y - 12x^2y^2 + 9xy$ $= \mathbf{3xy}\,(2x) - \mathbf{3xy}\,(4xy) + \mathbf{3xy} \cdot 3$ $= \mathbf{3xy}\,(2x - 4xy + 3)$	$3xy$

Examples

Expression	Factoring	GCF or HCF
$6a^2 - 9a$	$\mathbf{3a} \cdot 2a - \mathbf{3a} \cdot 3 = \mathbf{3a}\,(2a - 3)$	$3a$
$4x^4y + 12x^3y - 16xy$	$\mathbf{4xy} \cdot x^3 + \mathbf{4xy} \cdot 3x^2 - \mathbf{4xy} \cdot 4 = \mathbf{4xy}\,(x^3 + 3x^2 - 4)$	$4xy$
$13z^2(z + 2) - (3z + 6)$	$13z^2\,\mathbf{(z + 2)} - 3\,\mathbf{(z + 2)} = \mathbf{(z + 2)}\,(13z^2 - 3)$	$z + 2$
$\frac{2}{3}w^2 - \frac{4}{3}wz^2 + \frac{1}{3}w$	$\frac{1}{3}\mathbf{w} \cdot 2w - \frac{1}{3}\mathbf{w} \cdot 4z^2 + \frac{1}{3}\mathbf{w} \cdot 1 = \frac{1}{3}\mathbf{w}\,(2w - 4z^2 + 1)$	$\frac{1}{3}w$
$-5x^4 - 10x^2 + 15x$	$\mathbf{-5x} \cdot x^3 - \mathbf{5x} \cdot 2x + (\mathbf{-5x}) \cdot (-3) = \mathbf{-5x}\,(x^3 + 2x - 3)$	$-5x$

Tips:
- Factor each term and pull out the GCF.
- If the first term is negative, factor out a negative GCF to make the first term positive.

Factoring Polynomials by Grouping

Steps for factoring polynomials by grouping:

Steps	Example
	Factor $\mathbf{16x^2 - 4x + 28x - 7}$.
▪ Regroup terms with the GCF.	$16x^2 - 4x + 28x - 7 = (16x^2 - 4x) + (28x - 7)$
▪ Factor out the GCF from each group.	$= 4x\,\mathbf{(4x - 1)} + 7\mathbf{(4x - 1)}$
▪ Factor out the GCF again from last step.	$= \boxed{(4x - 1)(4x + 7)}$

Factoring completely: continue factoring until no further factors can be found.

Example: Factor the following completely.

1) $\mathbf{35xy^2 - 7x^2y + 5y - x} = (35xy^2 - 7x^2y) + (5y - x)$ Regroup terms with the GCF.

$= \mathbf{7xy}\,(5y - x) + (5y - x) \cdot 1$ Factor out $7xy$.

$= \boxed{(5y - x)(7xy + 1)}$ Factor out $(5y - x)$.

2) $\mathbf{3xy + yz - 5yz + 6xy} = (3\mathbf{xy} + 6\mathbf{xy}) + (\mathbf{yz} - 5\mathbf{yz})$ Regroup.

$= \mathbf{3xy}\,(1 + 2) + \mathbf{yz}\,(1 - 5)$ Factor out the GCF.

$= 3xy\,(3) + yz\,(-4)$ Simplify.

$= \boxed{9xy - 4yz}$

3) $\mathbf{t^3 - t^2w - tw^2 + w^3} = (\mathbf{t}^3 - \mathbf{t}^2w) - (t\mathbf{w}^2 - \mathbf{w}^3)$ Regroup.

$= \mathbf{t^2}\,(t - w) - \mathbf{w^2}\,(t - w)$ Factor out $(t - w)$.

$= \mathbf{(t - w)}\,(t^2 - w^2)$ Apply $a^2 - b^2 = (a + b)(a - b)$

$= \mathbf{(t - w)}\,(t + w)\,\mathbf{(t - w)}$

$= \boxed{(t - w)^2 (t + w)}$

Tip: Identify patterns of common factors such as $5y - x$, $t - w$ …

Factoring Difference of Squares

Factoring difference of squares:

Formula	Example
$\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$ or $\boldsymbol{a^2 - b^2 = (a - b)(a + b)}$	$x^2 - 49 = x^2 - 7^2 = (x + 7)(x - 7)$ $y^2 - 81 = y^2 - 9^2 = (y - 9)(y + 9)$

Note:
- $a^2 + b^2$ cannot be factored.
- Always factor out the greatest common factor (GCF) first.
- Determine the perfect square or the square root of each term.

Recall that **factoring is the reverse of multiplication**.

$$a^2 - b^2 \underset{\text{Multiplying}}{\overset{\text{Factoring}}{\rightleftarrows}} (a + b)(a - b)$$

Example: Factor the following completely.

1) $\mathbf{2x^2 - 18} = 2(x^2 - 9)$ — Factor out 2.

$= 2(x^2 - 3^2)$ — $9 = 3^2$ or $\sqrt{9} = 3$

$= \boxed{2(x + 3)(x - 3)}$ — $a^2 - b^2 = (a + b)(a - b)$: $a = x$, $b = 3$

2) $\mathbf{1 - 64u^2} = 1^2 - 8^2 u^2$ — $1 = 1^2$, $64 = 8^2$ or $\sqrt{64} = 8$

$= 1^2 - (8u)^2$ — $a^n b^n = (ab)^n$

$= \boxed{(1 + 8u)(1 - 8u)}$ — $a^2 - b^2 = (a + b)(a - b)$: $a = 1$, $b = 8u$

3) $\mathbf{100t^2 - 256} = 10^2 t^2 - 16^2$ — $256 = 16^2$ or $\sqrt{256} = 16$

$= (10t)^2 - 16^2$ — $a^n b^n = (ab)^n$

$= \boxed{(10t + 16)(10t - 16)}$ — $a^2 - b^2 = (a + b)(a - b)$: $a = 10t$, $b = 16$

4) $\mathbf{9x^2 - 16y^2} = 3^2x^2 - 4^2y^2 = (\underset{a}{\mathbf{3x}})^2 - (\underset{b}{\mathbf{4y}})^2$ — $a^n b^n = (ab)^n$

$= \boxed{(3x + 4y)(3x - 4y)}$ — $a^2 - b^2 = (a + b)(a - b)$: $a = 3x$, $b = 4y$

5) $\mathbf{36x^8 - 0.04} = 6^2 (x^4)^2 - 0.2^2$ — $0.04 = 0.2^2$ or $\sqrt{0.04} = 0.2$, $x^8 = (x^4)^2$

$= (6x^4)^2 - 0.2^2$ — $a^n b^n = (ab)^n$

$= \boxed{(6x^4 + 0.2)(6x^4 - 0.2)}$ — $a^2 - b^2 = (a + b)(a - b)$: $a = 6x^4$, $b = 0.2$

Topic B: Factoring Trinomials

Factoring $x^2 + bx + c$

Factoring $x^2 + bx + c$: cross-multiplication method

Steps	Standard form	Example
	$x^2 + bx + c$	$x^2 + 7x + 12$
▪ Setting up two sets of parenthesis.	$= (\quad)(\quad)$	$= (\quad)(\quad)$
	$x^2 + bx + c$	$x^2 + 7x + 12$
▪ Factor the first term x^2: $x^2 = x \cdot x$	$\begin{matrix} x & & c_1 \\ & \times & \\ x & & c_2 \end{matrix}$	$\begin{matrix} x & & 3 \\ & \times & \\ x & & 4 \end{matrix}$
▪ Factor the last term c (by trial and error): $c = c_1 \cdot c_2$	$x \cdot x = x^2$ $\quad c_1 \cdot c_2 = c$	$x \cdot x = x^2$ $\quad 3 \cdot 4 = 12$
▪ Cross multiply and then add up to the middle term.	$(c_1)(x) + (c_2)(x) = bx$	$3 \cdot x + 4 \cdot x = 7x$
▪ Complete the parenthesis with $x + c_1$ and $x + c_2$.	$x^2 + bx + c$	$x^2 + 3x + 2$
	$= \mathbf{(x + c_1)(x + c_2)}$	$= \boxed{(x + 3)(x + 4)}$

▪ Check using FOIL.

$$\overset{F\quad O\quad I\quad L}{(x + 3)(x + 4) = x^2 + 4x + 3x + 12}$$

$$(x + 3)(x + 4) = x^2 + 7x + 12 \quad \surd$$

Factoring $x^2 + bx + c$ using the cross-multiplication method	
In general	**Example**
$x^2 + bx + c = (\quad)(\quad)$	$x^2 - 8x + 15 = (\quad)(\quad)$
$\begin{matrix} x & & c_1 \\ & \times & \\ x & & c_2 \end{matrix}$	$\begin{matrix} x & & -5 \\ & \times & \\ x & & -3 \end{matrix}$
$x \cdot x = x^2$ $\quad c_1 \cdot c_2 = c$	$x \cdot x = x^2$ $\quad (-3)(-5) = 15$
$(c_1)(x) + (c_2)(x) \overset{?}{=} bx$	$(-5)x + (-3)x \overset{?}{=} -8x$ yes!
$x^2 + bx + c = \mathbf{(x + c_1)(x + c_2)}$	$x^2 - 8x + 15 = \boxed{(x - 5)(x - 3)}$

Check: $-5 + (-3) = -8$ $\surd$

Tips: - Cross multiply and then add up to the middle term.
- Write the factors with their appropriate signs (+ or –) to get the right middle term.

Summary: Factoring $x^2 + bx + c$	**Example: $x^2 - 8x + 15$**
$x^2 + (c_1 + c_2)x + c_1c_2 = (x + c_1)(x + c_2)$	$x^2 + [-5 + (-3)]x + 15 = \boxed{(x - 5)(x - 3)}$
$\begin{matrix} x & & c_1 \\ & \times & \\ x & & c_2 \end{matrix}$	$\begin{matrix} x & & -5 \\ & \times & \\ x & & -3 \end{matrix}$
Check: $c_1x + c_2x \overset{?}{=} bx$	Check: $-5x + (-3x) \overset{?}{=} -8x$ yes!

Example: Factor the following:

Trial and error process

1) $\mathbf{a^2 - 11a + 30} = (\quad)(\quad)$

$$\begin{matrix} a & & -5 \\ & \times & \\ a & & -6 \end{matrix}$$

$a \cdot a = a^2$ $\quad (-5)(-6) = 30$

$(-5)a + (-6)a \overset{?}{=} -11a$ yes! $\quad$ Check: $-5 + (-6) = -11$ $\surd$

Answer: $a^2 - 11a + 30 = \boxed{(a - 5)(a - 6)}$

Trial and error:

$a^2 - 11a + 30$	$a^2 - 11a + 30$
$\begin{matrix} a & & 3 \\ & \times & \\ a & & 10 \end{matrix}$	$\begin{matrix} a & & 6 \\ & \times & \\ a & & 5 \end{matrix}$
$3a + 10a \overset{?}{=} -11a$ no	$6a + 5a \overset{?}{=} -11a$ no

2) $\mathbf{3x^2 + 24x - 27 = 3(x^2 + 8x - 9)}$

$$\begin{matrix} x & & -1 \\ & \times & \\ x & & 9 \end{matrix}$$

$x \cdot x = x^2$ $\quad (-1)(9) = -9$

$(-1)x + 9x \overset{?}{=} 8x$ yes! $\quad$ Check: $-1 + 9 = 8$ $\surd$

Answer: $3(x^2 + 8x - 9) = \boxed{3(x - 1)(x + 9)}$

Trial and error:

$x^2 + 8x - 9$	$x^2 + 8x - 9$
$\begin{matrix} x & & 3 \\ & \times & \\ x & & -3 \end{matrix}$	$\begin{matrix} x & & -3 \\ & \times & \\ x & & 3 \end{matrix}$
$3x + (-3)x \overset{?}{=} 8x$ no	$(-3)x + 3x \overset{?}{=} 8x$ no

Note: Always factor out the greatest common factor (GCF) and rewrite in descending order or standard form ($ax^2 + bx + c$) first.

Factoring $ax^2 + bx + c$

Procedure for factoring $ax^2 + bx + c$ using the cross-multiplication method:

Steps	In general	Example
	$\mathbf{ax^2 + bx + c}$	$\mathbf{3x^2} - 2x - \mathbf{8}$
▪ Setting up two sets of parenthesis.	$= (\quad)(\quad)$	$= (\quad)(\quad)$
	$ax^2 + bx + c$	$3x^2 - 2x - 8$
▪ Factor the first term ax^2: $ax^2 = a_1x \cdot a_2x$	$a_1x \qquad c_1$	$x \qquad -2$
▪ Factor the last term c (by trial and error):	$a_2x \qquad c_2$	$3x \qquad 4$
$c = c_1 \cdot c_2$	$a_1x \cdot a_2x = ax^2 \qquad c_1 \cdot c_2 = c$	$3x^2 = x \cdot 3x \qquad -8 = -2 \cdot 4$
▪ Cross multiply and then add up to the middle term.	$c_1(a_2x) + c_2(a_1x) = bx$	$(-2)(3x) + 4(x) = -2x$
▪ Complete the parenthesis with $(a_1x + c_1)$ and $(a_2x + c_2)$.	$ax^2 + bx + c$	$3x^2 - 2x - 8$
	$= \mathbf{(a_1x + c_1)(a_2x + c_2)}$	$= \boxed{(x-2)(3x+4)}$

▪ Check using FOIL.

$$(x-2)(3x+4) = \overset{F}{3x^2} + \overset{O}{4x} - \overset{I}{6x} - \overset{L}{8}$$

$$(x-2)(3x+4) \overset{\surd}{=} 3x^2 - 2x - 8$$

(Original expression)

Tip: Write the factors with their appropriate signs (+ or –) to get the right middle term.

Factoring $ax^2 + bx + c$ using the cross-multiplication method	
In general	**Example**
$\mathbf{ax^2 + bx + c = (\quad)(\quad)}$	$\mathbf{4x^2 + 7x + 3 = (\quad)(\quad)}$
$a_1x \qquad c_1$	$4x \qquad 3$
$a_2x \qquad c_2$	$x \qquad 1$
$a_1x \cdot a_2x = ax^2 \qquad c = c_1 \cdot c_2$	$4x \cdot x = 4x^2 \qquad 3 \cdot 1 = 3$
$(c_1)(a_2x) + (c_2)(a_1x) \overset{?}{=} \mathbf{bx}$	$3 \cdot x + 4x \cdot 1 \overset{?}{=} \mathbf{7x}$ yes!
$ax^2 + bx + c = \mathbf{(a_1x + c_1)(a_2x + c_2)}$	$4x^2 + 7x + 3 = \boxed{(4x+3)(x+1)}$

Tip: Cross multiply and then add up to the middle term.

Summary: Factoring $ax^2 + bx + c$
$a_1a_2x^2 + (c_1a_2 + c_2a_1)x + c_1c_2 = (a_1x + c_1)(a_2x + c_2)$
$a_1x \qquad c_1$
$a_2x \qquad c_2$

Note: Always factor out the greatest common factor (GCF) and rewrite in descending order or standard form ($ax^2 + bx + c$) first.

More on Factoring $ax^2 + b\,x + c$

Example: Factor $\mathbf{6y^2 - 17y - 14}$.

$\mathbf{6y^2 - 17y - 14} = (\qquad)(\qquad)$

$3y \quad 2$
$2y \quad -7$

$3y \cdot 2y = 6y^2 \qquad 2(-7) = -14$

$(2)(2y) + (-7)(3y) \overset{?}{=} \mathbf{-17y}$ yes!

$6y^2 - 17y - 14 = \boxed{(3y + 2)(2y - 7)}$

Trial and error process

1) $6y^2 - 17y - 14$
$y \quad -7$
$6y \quad 2$
$(-7)(6y) + 2y \overset{?}{=} -17y$ no

2) $6y^2 - 17y - 14$
$3y \quad 7$
$2y \quad -2$
$7(2y) + (-2)(3y) \overset{?}{=} -17y$ no

3) $6y^2 - 17y - 14$
$6y \quad 2$
$y \quad -7$
$2y + (-7)(6y) \overset{?}{=} -17y$ no

Check: $(3y + 2)(2y - 7) = \underset{F}{6y^2} \underset{O}{- 21y} + \underset{I}{4y} \underset{L}{- 14}$

$(2y + 2)(2y - 7) \overset{\surd}{=} \boxed{6y^2 - 17y - 14}$ Correct!

Example: Factor the following completely.

1) $\mathbf{28x - 24 + 20x^2} = 20x^2 + 28x - 24$ Rewrite in descending order or standard form ($ax^2 + bx + c$).

$= 4(5x^2 + 7x - 6) = 4(\qquad)(\qquad)$ Factor out 4.

$x \quad 2$
$5x \quad -3$

$10x + (-3x) = 7x \quad \surd$

$4(5x^2 + 7x - 6) = \boxed{4(x + 2)(5x - 3)}$

Note: Always factor out the greatest common factor (GCF) and rewrite in descending order or standard form ($ax^2 + b\,x + c$) first.

2) $\mathbf{8a^2 - 6ab - 5b^2} = (\qquad)(\qquad)$

$2a \quad b$
$4a \quad -5b$

$-10ab + 4ab = -6ab \quad \surd$

$8a^2 - 6ab - 5b^2 = \boxed{(2a + b)(4a - 5b)}$

3) $\mathbf{2t^4 + 14t^2 + 20} = 2(t^4 + 7t^2 + 10) = 2(\qquad)(\qquad)$ Factor out 2.

$t^2 \quad 2$
$t^2 \quad 5$

$2t^4 + 14t^2 + 20 = \boxed{2(t^2 + 2)(t^2 + 5)}$

$5t^2 + 2t^2 = 7t^2 \quad \surd$

Factoring Trinomials: AC Method

AC method for factoring trinomials: $ax^2 + bx + c$

Factoring $ax^2 + bx + c = 0$ by Grouping	**Example**
Steps	**Solve** $14x + 6 = -8x^2$
▪ Convert to standard form (descending order) if necessary.	$8x^2 + 14x + 6 = 0$
▪ Factor out the greatest common factor (GCF).	$2\,(\mathbf{4}x^2 + 7x + \mathbf{3}) = 0$
▪ Multiply a and c in $ax^2 + bx + c$.	$ac = \mathbf{4 \cdot 3} = 12$
▪ Factor the product ac that sum to the middle coefficient b.	$4 \cdot 3 = \mathbf{12}, \quad 4 + 3 = \mathbf{7}$
▪ Rewrite the middle term as the sum using the factors found in last step.	$2\,(4x^2 + \mathbf{7}x + 3) = 0$ $2\,(4x^2 + \mathbf{4}x + \mathbf{3}x + 3) = 0$
▪ Factor by grouping.	$2\,[4x\,(\mathbf{x+1}) + \mathbf{3}\,(\mathbf{x+1})] = 0$ $\boxed{2\,(x+1)\,(4x+3) = 0}$ Factor out $(x + 1)$.

Example: Factor $\mathbf{6x^2 - 16 = 4x}$ using ac method.

Steps	**Solution**
	$\mathbf{6x^2 - 16 = 4x}$
▪ Write in standard form:	$6x^2 - 4x - 16 = 0$
▪ Factor out the greatest common factor:	$2(\mathbf{3}x^2 - 2x - \mathbf{8}) = 0$
▪ Multiply a and c in $ax^2 + bx + c$:	$ac = \mathbf{3} \cdot (\mathbf{-8}) = -24$
▪ Factor the product ac that sum to the middle coefficient b. (There are different pairs to get the product of ac of **-24**. Try to find two numbers that multiply to ac and add to obtain b = **-2**.)	

Some factors of ac (-24)	Sum of factors (b = -2)
-3 & 8	-3 + 8 = 5
-4 & 6	-4 + 6 = 2
8 & -3	8 + (-3) = 5
4 & -6	4 + (- 6) = **-2** Correct!

The right choices are 4 an -6, since they both add up to b = -2. 4 (-6) = -24, 4 + (-6) = -2

▪ Rewrite the middle term as $\mathbf{4}x - \mathbf{6}x$.	$2(3x^2 - \mathbf{2}x - 8) = 0$ $2\,(3x^2 + \mathbf{4}x - \mathbf{6}x - 8) = 0$
▪ Factor by grouping.	$2\,[x\,(3\mathbf{x} + \mathbf{4}) - 2(3\mathbf{x} + \mathbf{4}) = 0$ Factor out $(3x + 4)$ $\boxed{2(3x+4)\,(x-2) = 0}$

Factoring Special Products

Recall that **factoring is the reverse of multiplication**.

$$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2} \quad \xrightarrow{\text{Factoring}} \quad \xleftarrow{\text{Multiplying}}$$

Recognize some polynomials as special products can factor more quickly.

Special products:

Name	Formula	Example
Square of sum (perfect square trinomial)	$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$	$x^2 + 10x + 25 = (x+5)^2$ $a = x, \quad b = 5$ $x \quad 5$ $x \quad 5$ Check: $(x+5)^2 = x^2 + 2 \cdot x \cdot 5 + 5^2 = x^2 + 10x + 25$ √
Square of difference (perfect square trinomial)	$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$	$9y^2 - 24y + 16 = (3y-4)^2$ $a = 3y, \quad b = 4$ $3y \quad -4$ $3y \quad -4$ Check: $(3y-4)^2 = (3y)^2 - 2(3y)(4) + 4^2 = 9y^2 - 24y + 16$ √

Note: The quickest way to factor an expression is to recognize it as a special product.

Memory aid: $(a^2 \pm 2ab + b^2) = (a \pm b)^2$ v Notice the reversed plus or minus sign in the second term.

To use perfect square trinomial formulas: use cross-multiplication method to factor a perfect square. Then use the square formula to check.

Example: Factor the following completely.

1) $\boldsymbol{28z + 49 + 4z^2} = 4z^2 + 28z + 49$ Rewrite in standard form: $ax^2 + bx + c$

$2z \quad 7$
$2z \quad 7$

$= (2z+7)(2z+7)$ $7(2z) + 7(2z) = 28x$

$= \boxed{(2z+7)^2}$

Check: $(2z+7)^2 = (2z)^2 + 2 \cdot 2z \cdot 7 + 7^2 = 4z^2 + 28z + 49$ √ $a^2 + 2ab + b^2 = (a+b)^2$: $a = 2z$, $b = 7$

2) $\boldsymbol{50p^2 - 40p + 8} = 2(25p^2 - 20p + 4)$ Factor out 2.

$2(25p^2 - 20p + 4) = \boxed{2(5p-2)^2}$

$5p \quad -2$
$5p \quad -2$ $-2(5p) + -2(5p) = -20p$

Check: $2(5p-2)^2 = 2[(5p)^2 - 2(5p)(2) + (2)^2] = 2(25p^2 - 20p + 4)$ √ $a^2 - 2ab + b^2 = (a-b)^2$: $a = 5p$, $b = 2$

3) $\boldsymbol{16n^{10} - 48n^5 + 36} = 4(4n^{10} - 12n^5 + 9)$ Factor out 4.

$2n^5 \quad -3$
$2n^5 \quad -3$ $(2n^5)(-3) + (2n^5)(-3) = -12n^5$

$= \boxed{4(2n^5-3)^2}$ $a^m a^n = a^{m+n}$

Check: $(2n^5-3)^2 = (2n^5)^2 - 2(2n^5)(3) + (3)^2 = 4n^{10} - 12n^5 + 9$ √ $a^2 - 2ab + b^2 = (a-b)^2$: $a = 2n^5$, $b = 3$

Topic C: Application of Factoring

Quadratic Equations

Quadratic equation: an equation that has a squared term, such as $7x^2 + 3x - 5 = 0$.

Quadratic equations in standard form	
$ax^2 + bx + c = 0$	$a \neq 0$

Incomplete quadratic equation

Incomplete quadratic equation		Example	a	b	c
$ax^2 + bx = 0$	$(c = 0)$	$4x^2 - 3x = 0$	4	-3	0
$ax^2 + c = 0$	$(b = 0)$	$8x^2 + 5 = 0$	8	0	5

Zero-product property:

Zero-product property
If $A \cdot B = 0$, then either $A = 0$ or $B = 0$ (or both) (A and B are algebraic expressions.)

Note: "or" means possibility of both.

Solving incomplete quadratic equations

Incomplete quadratic equation	Steps	Example
Use the zero-product property to solve $ax^2 + bx = 0$	- Express in $ax^2 + \boldsymbol{b}x = 0$ - Factor: $x(ax + b) = 0$ - Apply the zero-product property: $x = 0$ ⋮ $ax + b = 0$ - Solve for x: $x = 0$ ⋮ $x = -\frac{b}{a}$	**Solve $11x^2 = -6x$** $\mathbf{11}x^2 + \mathbf{6}x = 0$ Add $6x$. $x(11x + 6) = 0$ $x = 0$ ⋮ $11x + 6 = 0$ $x = 0$ ⋮ $x = -\frac{6}{11}$
Use the square root method to solve $ax^2 - c = 0$ (or $ax^2 = c$)	- Express in $\boldsymbol{a}x^2 = \boldsymbol{c}$ - Divide both sides by a: $x^2 = \frac{c}{a}$ - Take the square root of both sides: $x = \pm\sqrt{\frac{c}{a}}$	**Solve $64x^2 - 9 = 0$** $\mathbf{64}x^2 = \mathbf{9}$ $x^2 = \frac{9}{64}$ $x = \pm\sqrt{\frac{9}{64}} = \pm\frac{3}{8}$

Solving Quadratic Equations

Solve a quadratic equation: a quadratic equation $ax^2 + bx + c = 0$ can be written as:

$(x + a)(x + b) = 0$ — Factor.

Set each term equal to zero: $x + a = 0$ | $x + b = 0$ — Zero-product property.

Solutions: $x = -a$ | $x = -b$ — Solve for x.

Example: Solve for x. $\quad \boldsymbol{(x + 6)(x - 11) = 0}$

$x + 6 = 0$ | $x - 11 = 0$ — Zero-product property.

$\boldsymbol{x} = \boxed{-6}$ | $\boldsymbol{x} = \boxed{11}$ — Solve for x.

Example: Solve the quadratic equation $\boldsymbol{x^2 - x - 20 = 0}$.

1) $\boldsymbol{x^2 - x - 20 = 0}$

$$\begin{array}{cc} x & 4 \\ x & -5 \end{array}$$

Factor.
$4x + (-5)x = -x$

$(x + 4)(x - 5) = 0$

$x + 4 = 0$ | $x - 5 = 0$ — Zero-product property.

$\boldsymbol{x} = \boxed{-4}$ | $\boldsymbol{x} = \boxed{5}$

2) $\boldsymbol{6x^2 - 13x = 15}$ — Rewrite in standard form: $ax^2 + bx + c = 0$

$6x^2 - 13x - 15 = 0$ — Set the equation equal to 0.

$$\begin{array}{cc} 6x & 5 \\ x & -3 \end{array}$$

Factor.
$5x + (-3)(6x) = -13x$

$(6x + 5)(x - 3) = 0$

$6x + 5 = 0$ | $x - 3 = 0$ — Zero-product property.

$\boldsymbol{x} = \boxed{-\frac{5}{6}}$ | $\boldsymbol{x} = \boxed{3}$

3) $\boldsymbol{x^2 - \frac{2}{9} = \frac{1}{3}x}$ — Rewrite in standard form: $ax^2 + bx + c = 0$

$x^2 - \frac{1}{3}x - \frac{2}{9} = 0$ — Set the equation equal to 0.

$$\begin{array}{cc} x & \frac{1}{3} \\ x & -\frac{2}{3} \end{array}$$

Factor.
$\frac{1}{3}\left(-\frac{2}{3}\right) = -\frac{2}{9}$, $\quad \frac{1}{3}x + \left(-\frac{2}{3}x\right) = -\frac{1}{3}x$

$(x + \frac{1}{3})(x - \frac{2}{3}) = 0$

$x + \frac{1}{3} = 0$ | $y - \frac{2}{3} = 0$ — Zero-product property.

$\boldsymbol{x} = \boxed{-\frac{1}{3}}$ | $\boldsymbol{x} = \boxed{\frac{2}{3}}$

Application of Quadratic Equations

Review number problems - examples

English phrase	Algebraic expression/equation
6 more than the difference of the square of a number and 11 is 32.	$(x^2 - 11) + 6 = 32$
The quotient of 5 and the product of 9 and a number is 7 less than the number.	$\frac{5}{9x} = x - 7$
The product of 9 and the square of a number decreased by 13 is 21.	$9x^2 - 13 = 21$
15 more than the quotient of 4*x* by 7 is 5 times the square of a number.	$15 + \frac{4x}{7} = 5x^2$

Let x = a number; y = a number

Review consecutive integers

English phrase	Algebraic expression	Example
Three consecutive **odd** integers	x, $x + 2$, $x + 4$	If $x = \mathbf{1}$, $x + 2 = \mathbf{3}$, $x + 4 = \mathbf{5}$
Three consecutive **even** integers	x, $x + 2$, $x + 4$	If $x = \mathbf{2}$, $x + 2 = \mathbf{4}$, $x + 4 = \mathbf{6}$
The product of two consecutive odd integers is 35.	$x(x + 2) = 35$	
Three consecutive even integers whose sum is 12.	$x + (x + 2) + (x + 4) = 12$	

Examples:

1) The product of a number and 4 more than the square of the number is 21. Find the number(s).

- Let x = the number
- Equation $\mathbf{x^2 + 4x = 21}$
- Solve for x: $x^2 + 4x - 21 = 0$ Rewrite in standard form.

$x \quad 7$
$x \quad -3$

Factor.
$7x + (-3)x = 4x$

$(x + 7)(x - 3) = 0$

$x + 7 = 0 \quad | \quad x - 3 = 0$ Zero-product property.

$\mathbf{x} = \boxed{-7} \quad | \quad \mathbf{x} = \boxed{3}$

2) The product of two consecutive even integers is 48. Find the integers.

- Let x = the first even integer
- Equation $\mathbf{x(x + 2) = 48}$ The 2nd integer is $x + 2$.
- Solve for x: $x^2 + 2x - 48 = 0$ Rewrite in standard form.

$x \quad -6$
$x \quad 8$

Factor.
$-6x + 8x = 2x$

$(x - 6)(x + 8) = 0$

$x - 6 = 0 \quad | \quad x + 8 = 0$ Zero-product property.

$\mathbf{x} = \boxed{6} \quad | \quad \mathbf{x} = \boxed{-8}$

If $x = 6$, $x + 2 = 8$ | If $x = -8$, $x + 2 = -6$

Dimension (length and width) problems:

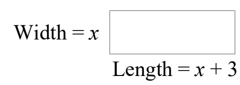
Example: Robert is going to replace the old carpet in his bedroom, which is a rectangle and has a ***length 3 meters greater than its width***. If the ***area*** of his bedroom is ***54*** square meters ($\boldsymbol{m^2}$), what will be the dimensions of the carpet?

Steps	Solution

- Organize the facts.

Facts	Area $A = 54\text{m}^2$ Length = Width + 3m
Unknowns	Width = x , Length = x + 3m

- Draw a diagram.



- Equation: $\boldsymbol{x\,(x + 3) = 54}$ Area: $A = w\,l$
- Solve the equation. $x^2 + 3x = 54$ Distribute.
  - Standard form: $x^2 + 3x - 54 = 0$ $ax^2 + bx + c = 0$
  - Factor:

    x 9
    x -6 $9x + (-6)\,x = 3x$

    $(x + 9)\,(x - 6) = 0$
  - Zero-product property: $x + 9 = 0$ | $x - 6 = 0$
  - Solutions: $x = -9$ | $x = 6$

  Since the width of a rectangle cannot be negative, eliminate x = -9.

- Answer (the size of the carpet):

  Width = $\boldsymbol{x}$ = $\boxed{6\text{ m}}$

  Length = $\boldsymbol{x + 3}$ = 6 + 3 = $\boxed{9\text{ m}}$

Triangle problem:

Example: A triangle is 1 meter wider than it is tall. The area is 36 m^2. Find the base and the height.

- Organize the facts.

Facts	Area $A = 36\text{m}^2$ Base = Height + 1m
Unknowns	Height = x , Base = x + 1m

- Equation. $\frac{1}{2}(x + 1)\,x = 36$ Area: $A = \frac{1}{2}bh$
- Solve the equation. $x^2 + x = 72$ Multiply both sides by 2.
  - Standard form: $x^2 + x - 72 = 0$ $ax^2 + bx + c = 0$
  - Factor:

    x 9
    x -8 $9x + (-8)\,x = x$

    $(x + 9)\,(x - 8) = 0$
  - Zero-product property: $x + 9 = 0$ | $x - 8 = 0$
  - Solutions: $x = -9$ | $x = 8$
- Answer:

  Height = $\boldsymbol{x}$ = $\boxed{8\text{ m}}$

  (Since the height of a triangle cannot be negative, eliminate x = -9.)

  Base = $\boldsymbol{x + 1}$ = 8 + 1 = $\boxed{9\text{m}}$

Unit 14: Summary

Factoring Polynomials

Factoring whole numbers: write the number as a product of its prime factors.

Common factor: a number or an expression that is a factor of each term of a group of terms.

Greatest / highest common factor (GCF or HCF): the product of the common factors.

Factoring a polynomial: express a polynomial as a product of other polynomials. It is the reverse of multiplication.

Steps for factoring polynomials by grouping:

- Regroup terms with the GCF.
- Factor out the GCF from each group.
- Factor out the GCF again from last step.

Special products:

Name	Formula
Difference of squares	$a^2 - b^2 = (a + b)\,(a - b)$
Square of sum	$a^2 + 2ab + b^2 = (a + b)^2$
Square of difference	$a^2 - 2ab + b^2 = (a - b)^2$

Memory aid: $(a^2 \pm ab + b^2) = (a \pm b)^2$

Cross-multiplication method:

Factoring $x^2 + bx + c$ using the cross-multiplication method	
In general	**Example**
$x^2 + bx + c = (\quad)(\quad)$	$x^2 - 8x + 15 = (\quad)(\quad)$
$x \quad c_1$ $x \quad c_2$	$x \quad -5$ $x \quad -3$
$x \cdot x = x^2 \qquad c_1 \cdot c_2 = c$	$x \cdot x = x^2 \qquad (-3)(-5) = 15$
$(c_1)(x) + (c_2)(x) \stackrel{?}{=} bx$	$(-5)x + (-3)x \stackrel{?}{=} -8x$ yes!
$x^2 + bx + c = (x + c_1)(x + c_2)$	$x^2 - 8x + 15 = \boxed{(x - 5)(x - 3)}$

Tips: - Cross multiply and then add up to the middle term.
- Write the factors with their appropriate signs (+ or –) to get the right middle term.

Factoring $ax^2 + bx + c$ using the cross-multiplication method	
In general	**Example**
$ax^2 + bx + c = (\quad)(\quad)$	$4x^2 + 7x + 3 = (\quad)(\quad)$
$a_1x \quad c_1$ $a_2x \quad c_2$	$4x \quad 3$ $x \quad 1$
$a_1x \cdot a_2x = ax^2 \qquad c = c_1 \cdot c_2$	$4x \cdot x = 4x^2 \qquad 3 \cdot 1 = 3$
$(c_1)(a_2x) + (c_2)(a_1x) \stackrel{?}{=} bx$	$3 \cdot x + 4x \cdot 1 \stackrel{?}{=} 7x$ yes!
$ax^2 + bx + c = (a_1x + c_1)(a_2x + c_2)$	$4x^2 + 7x + 3 = \boxed{(4x + 3)(x + 1)}$

Tips: - Cross multiply and then add up to the middle term.
- Always factor out the greatest common factor (GCF) and rewrite in descending order or standard form ($ax^2 + bx + c$) first.

Factoring polynomials:

Polynomial	Method
Two terms (binomial)	- If it is a perfect square: $a^2 - b^2 = (a+b)(a-b)$ - If not, use the distributive property: $ac + bc = c\,(a+b)$
Three terms (trinomial)	$ax^2 + b\,x + c$: use the cross-multiplication or AC methods.
Four terms	Factor by grouping

Quadratic equation: an equation that has a squared term.

Quadratic equations in standard form	
$ax^2 + b\,x + c = 0$	$a \neq 0$

Incomplete quadratic equation

Incomplete quadratic equation	
$ax^2 + b\,x = 0$	$(c = 0)$
$ax^2 + c = 0$	$(b = 0)$

Zero-product property:

Zero-product property
If $A \cdot B = 0$, then either $A = 0$ or $B = 0$ (or both) (A and B are algebraic expressions.)

Solving incomplete quadratic equations

Incomplete quadratic equation	Steps
Use the zero-product property to solve $ax^2 + bx = 0$	- Express in $ax^2 + \boldsymbol{b}\,x = 0$ - Factor: $x\,(ax + b) = 0$ - Apply the zero-product property: $x = 0$ | $ax + b = 0$ - Solve for x: $x = 0$ | $x = -\frac{b}{a}$
Use the square root method to solve $ax^2 - c = 0$	- Express in $\boldsymbol{a}x^2 = \boldsymbol{c}$ - Divide both sides by a: $x^2 = \frac{c}{a}$ - Take the square root of both sides: $x = \pm\sqrt{\frac{c}{a}}$

Unit 14: Self-Test

Factoring Polynomials

Topic A

1. Factor 60.

2. Find the greatest common factor (GCF) for the following.

a) $5x^2 - 20x$

b) $3a^3b + 15a^4b - 21ab$

c) $17y^2(y + 4) - (2y + 8)$

d) $\frac{1}{4}x^3 - \frac{3}{4}xy^2 + \frac{5}{4}x$

e) $-4y^3 - 8y^2 + 20y$

3. Factor the following completely.

a) $25x^2 - 5x + 20x - 4$

b) $48ab^2 - 8a^2b + 6b - a$

c) $4uv + vw - 7vw + 21uv$

d) $x^3 - x^2y - xy^2 + y^3$

e) $5y^2 - 20$

f) $1 - 49w^2$

g) $81u^2 - 121$

h) $25a^2 - 36b^2$

i) $4y^6 - 0.09$

Topic B

4. Factor the following:

a) $x^2 + 9x + 20$

b) $x^2 - 10x + 24$

c) $x^2 - 3x - 18$

d) $2x^2 + 10x - 28$

e) $4x^2 - 7x - 15$

f) $5y^2 + 9y - 18$

g) $24ab^2 - 4a^2b + 6b - a$

h) $6uv + vs - 7vs + 11uv$

5. Factor the following using the *ac* method.

a) $6x^2 - 60 = 9x$

b) $6x^2 + 4x - 16$

6. Factor the following completely.

(Use the cross-multiplication method to factor a perfect square. Then use the square formula to check.)

a) $9x^2 + 30x + 25$

b) $27 + 12y^2 - 36y$

c) $18t^8 - 24t^4 + 8$

Topic C

7. Solve for x.

a) $23x^2 = -7x$

b) $81x^2 - 49 = 0$

c) $(x + 9)(x - 17) = 0$

8. Solve the following quadratic equations.

a) $x^2 - x - 42 = 0$

b) $7x^2 - 31x = 20$

c) $x^2 + \frac{3}{16} = x$

9. The product of a number and 5 more than the square of the number is 36. Find the number(s).

10. The product of two consecutive even integers is 24. Find the integers.

11. Lisa is going to replace old carpet in her living room, which is a rectangle and has a length 2 meters greater than its width. If the area of her living room is 63 square meters (m^2), what will be the dimensions of the carpet?

12. A triangle is 2 meters wider than it is tall. The area is $24m^2$. Find the base and the height.