College Mathematics for Everyday Life

A College Level Liberal Arts Mathematics Text

2nd Edition

By Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, and Kim Sonier

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This work is dedicated to our families. We deeply appreciate all of your support throughout the writing of this textbook.

Acknowledgements:

Many thanks to the following people for reviewing this textbook:

Albert Gossler

Donald Young

Chandler Jameson

Many thanks to Coconino Community College administrators for their support:

Leah Bornstein, President

Russ Rothamer, Vice President of Academic Affairs

Jami Van Ess, Vice President of Business and Administrative Services

Ingrid Lee, Dean of Arts and Sciences

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Chapter 1: Statistics: Part 1

Section 1.1: Statistical Basics

Data are all around us. Researchers collect data on the effectiveness of a medication for lowering cholesterol. Pollsters report on the percentage of Americans who support gun control. Economists report on the average salary of college graduates. There are many other areas where data are collected. In order to be able to understand data and how to summarize it, we need to understand statistics.

Suppose you want to know the average net worth of a current U.S. Senator. There are 100 Senators, so it is not that hard to collect all 100 values, and then summarize the data. If instead you want to find the average net worth of all current Senators and Representatives in the U.S. Congress, there are only 435 members of Congress. So even though it will be a little more work, it is not that difficult to find the average net worth of all members. Now suppose you want to find the average net worth of everyone in the United States. This would be very difficult, if not impossible. It would take a great deal of time and money to collect the information in a timely manner before all of the values have changed. So instead of getting the net worth of every American, we have to figure out an easier way to find this information. The net worth is what you want to measure, and is called a variable. The net worth of every American is called the population. What we need to do is collect a smaller part of the population, called a sample. In order to see how this works, let's formalize the definitions.

Variable: Any characteristic that is measured from an object or individual.
Population: A set of measurements or observations from all objects under study
Sample: A set of measurements or observations from some objects under study (a subset of a population)

Example 1.1.1: Stating Populations and Samples

Determine the population and sample for each situation.

a. A researcher wants to determine the length of the lifecycle of a bark beetle. In order to do this, he breeds 1000 bark beetles and measures the length of time from birth to death for each bark beetle.

   Population: The set of lengths of lifecycle of all bark beetles
   Sample: The set of lengths of lifecycle of 1000 bark beetles

b. The National Rifle Association wants to know what percent of Americans support the right to bear arms. They ask 2500 Americans whether they support the right to bear arms.

   Population: The set of responses from all Americans to the question, "Do you support the right to bear arms?"
   Sample: The set of responses from 2500 Americans to the question, "Do you support the right to bear arms?"

c. The Pew Research Center asked 1000 mothers in the U.S. what their highest attained education level was.

   Population: The set of highest education levels of all mothers in the U.S.
   Sample: The set of highest education level of 1000 mothers in the U.S.

It is very important that you understand what you are trying to measure before you actually measure it. Also, please note that the population is a set of measurements or observations, and not a set of people. If you say the population is all Americans, then you have only given part of the story. More important is what you are measuring from all Americans. The question is, do you want to measure their race, their eye color, their income, their education level, the number of children they have, or other variables? Therefore, it is very important to state what you measured or observed, and from whom or what the measurements or observations were taken. Once you know what you want to measure or observe, and the source from which you want to take measurements or observations, you need to collect the data.

A **data set** is a collection of values called data points or data values. **N** represents the number of data points in a population, while **n** represents the number of data points in a sample. A data value that is much higher or lower than all of the other data values is called an **outlier**. Sometimes outliers are just unusual data values that are very interesting and should be studied further, and sometimes they are mistakes. You will need to figure out which is which.

In order to collect the data, we have to understand the types of variables we can collect. There are actually two different types of variables. One is called qualitative and the other is called quantitative.

Qualitative (Categorical) Variable: A variable that represents a characteristic. Qualitative variables are not inherently numbers, and so they cannot be added, multiplied, or averaged, but they can be represented graphically with graphs such as a bar graph.

Examples: gender, hair color, race, nationality, religion, course grade, year in college, etc.

Quantitative (Numerical) Variable: A variable that represents a measurable quantity. Quantitative variables are inherently numbers, and so can they be added, multiplied, averaged, and displayed graphically.

Examples: Height, weight, number of cats owned, score of a football game, etc.

Quantitative variables can be further subdivided into other categories – continuous and discrete.

Continuous Variable: A variable that can take on an uncountable number of values in a range. In other words, the variable can be any number in a range of values. Continuous variables are usually things that are measured.

Examples: Height, weight, foot size, time to take a test, length, etc.

Discrete Variable: A variable that can take on only specific values in a range. Discrete variables are usually things that you count.

Examples: IQ, shoe size, family size, number of cats owned, score in a football game, etc.

Example 1.1.2: Determining Variable Types

Determine whether each variable is quantitative or qualitative. If it is quantitative, then also determine if it is continuous or discrete.

a. Length of race
   Quantitative and continuous, since this variable is a number and can take on any value in an interval.

b. Opinion of a person about the President
   Qualitative, since this variable is not a number.

c. House color in a neighborhood
   Qualitative, since this variable is not a number.

d. Number of houses that are in foreclosure in a state
   Quantitative and discrete, since this variable is a number but can only be certain values in an interval.

e. Weight of a baby at birth
Quantitative and continuous, since this variable is a number and can take on any value in an interval.

f. Highest education level of a mother
Qualitative, since the variable is not a number.

Section 1.2: Random Sampling

Now that you know that you have to take samples in order to gather data, the next question is how best to gather a sample? There are many ways to take samples. Not all of them will result in a representative sample. Also, just because a sample is large does not mean it is a good sample. As an example, you can take a sample involving one million people to find out if they feel there should be more gun control, but if you only ask members of the National Rifle Association (NRA) or the Coalition to Stop Gun Violence, then you may get biased results. You need to make sure that you ask a cross-section of individuals. Let's look at the types of samples that can be taken. Do realize that no sample is perfect, and may not result in a representation of the population.

Census: An attempt to gather measurements or observations from all of the objects in the entire population.

A true census is very difficult to do in many cases. However, for certain populations, like the net worth of the members of the U.S. Senate, it may be relatively easy to perform a census. We should be able to find out the net worth of each and every member of the Senate since there are only 100 members. But, when our government tries to conduct the national census every 10 years, you can believe that it is impossible for them to gather data on each and every American.

The best way to find a sample that is representative of the population is to use a random sample. There are several different types of random sampling. Though it depends on the task at hand, the best method is often simple random sampling which occurs when you randomly choose a subset from the entire population.

Simple Random Sample: Every sample of size n has the same chance of being chosen, and every individual in the population has the same chance of being in the sample.

An example of a simple random sample is to put all of the names of the students in your class into a hat, and then randomly select five names out of the hat.

Stratified Sampling: This is a method of sampling that divides a population into different groups, called strata, and then takes random samples inside each strata.

An example where stratified sampling is appropriate is if a university wants to find out how much time their students spend studying each week; but they also want to know if different majors spend more time studying than others. They could divide the student body into the different majors (strata), and then randomly pick a number of people in each major to ask them how much time they spend studying. The number of people asked in each major (strata) does not have to be the same.

Systematic Sampling: This method is where you pick every kth individual, where k is some whole number. This is used often in quality control on assembly lines.

For example, a car manufacturer needs to make sure that the cars coming off the assembly line are free of defects. They do not want to test every car, so they test every 100^{th} car. This way they can periodically see if there is a problem in the manufacturing process. This makes for an easier method to keep track of testing and is still a random sample.

Cluster Sampling: This method is like stratified sampling, but instead of dividing the individuals into strata, and then randomly picking individuals from each strata, a cluster sample separates the individuals into groups, randomly selects which groups they will use, and then takes a census of every individual in the chosen groups.

Cluster sampling is very useful in geographic studies such as the opinions of people in a state or measuring the diameter at breast height of trees in a national forest. In both situations, a cluster sample reduces the traveling distances that occur in a simple random sample. For example, suppose that the Gallup Poll needs to perform a public opinion poll of all registered voters in Colorado. In order to select a good sample using simple random sampling, the Gallup Poll would have to have all the names of all the registered voters in Colorado, and then randomly select a subset of these names. This may be very difficult to do. So, they will use a cluster sample instead. Start by dividing the state of Colorado up into categories or groups geographically. Randomly select some of these groups. Now ask all registered voters in each of the chosen groups. This makes the job of the pollsters much easier, because they will not have to travel over every inch of the state to get their sample but it is still a random sample.

Quota Sampling: This is when the researchers deliberately try to form a good sample by *creating* a cross-section of the population under study.

For an example, suppose that the population under study is the political affiliations of all the people in a small town. Now, suppose that the residents of the town are 70% Caucasian, 25% African American, and 5% Native American. Further, the residents of the town are 51% female and 49% male. Also, we know information about the religious affiliations of the townspeople. The residents of the town are 55% Protestant, 25% Catholic, 10% Jewish, and 10% Muslim. Now, if a researcher is going to poll the people of this town about their political affiliation, the researcher should gather a sample that is representative of the entire population. If the researcher uses quota sampling, then the researcher would try to artificially create a cross-section of the town by insisting that his sample should be 70% Caucasian, 25% African American, and 5% Native American. Also, the researcher would want his sample to be 51% female and 49% male. Also, the researcher would want his sample to be 55% Protestant, 25% Catholic, 10% Jewish, and 10% Muslim. This sounds like an admirable attempt to create a good sample, but this method has major problems with selection bias.

The main concern here is when does the researcher stop profiling the people that he will survey? So far, the researcher has cross-sectioned the residents of the town by race, gender, and religion, but are those the only differences between individuals? What about socioeconomic status, age, education, involvement in the community, etc.? These are all influences on the political affiliation of individuals. Thus, the problem with quota sampling is that to do it right, you have to take into account all the differences among the people in the town. If you cross-section the town down to every possible difference among people, you end up with single individuals, so you would have to survey the whole town to get an accurate result. The whole point of creating a sample is so that you do not have to survey the entire population, so what is the point of quota sampling?

Note: The Gallup Poll did use quota sampling in the past, but does not use it anymore.

Convenience Sampling: As the name of this sampling technique implies, the basis of convenience sampling is to use whatever method is easy and convenient for the investigator. This type of sampling technique creates a situation where a random sample is not achieved. Therefore, the sample will be biased since the sample is not representative of the entire population.

For example, if you stand outside the Democratic National Convention in order to survey people exiting the convention about their political views. This may be a convenient way to gather data, but the sample will not be representative of the entire population.
Of all of the sampling types, a random sample is the best type. Sometimes, it may be difficult to collect a perfect random sample since getting a list of all of the individuals to randomly choose from may be hard to do.

Example 1.2.1: Which Type of Sample?

Determine if the sample type is simple random sample, stratified sample, systematic sample, cluster sample, quota sample, or convenience sample.

a. A researcher wants to determine the different species of trees that are in the Coconino National Forest. She divides the forest using a grid system. She then randomly picks 20 different sections and records the species of every tree in each of the chosen sections.

   This is a cluster sample, since she randomly selected some of the groups, and all individuals in the chosen groups were surveyed.

b. A pollster stands in front of an organic foods grocery store and asks people leaving the store how concerned they are about pesticides in their food.

   This is a convenience sample, since the person is just standing out in front of one store. Most likely the people leaving an organic food grocery store are concerned about pesticides in their food, so the sample would be biased.

c. The Pew Research Center wants to determine the education level of mothers. They randomly ask mothers to say if they had some high school, graduated high school, some college, graduated from college, or advance degree.

   This is a simple random sample, since the individuals were picked randomly.

d. Penn State wants to determine the salaries of their graduates in the majors of agricultural sciences, business, engineering, and education. They randomly ask 50 graduates of agricultural sciences, 100 graduates of business, 200 graduates of engineering, and 75 graduates of education what their salaries are.

   This is a stratified sample, since all groups were used, and then random samples were taken inside each group.

e. In order for the Ford Motor Company to ensure quality of their cars, they test every 130^{th} car coming off the assembly line of their Ohio Assembly Plant in Avon Lake, OH.

   This is a systematic sample since they picked every 130^{th} car.

f. A town council wants to know the opinion of their residents on a new regional plan. The town is 45% Caucasian, 25% African American, 20% Asian, and 10% Native American. It also is 55% Christian, 25% Jewish, 12% Islamic, and 8% Atheist. In addition, 8% of the town did not graduate from high school, 12% have graduated from high school but never went to college, 16% have had some college, 45% have obtained bachelor's degree, and 19% have obtained a post-graduate degree. So the town council decides that the sample of residents will be taken so that it mirrors these breakdowns.

   This is a quota sample, since they tried to pick people who fit into these subcategories.

Section 1.3: Clinical Studies

Now you know how to collect a sample, next you need to learn how to conduct a study. We will discuss the basics of studies, both observational studies and experiments.

Observational Study: This is where data is collected from just observing what is happening. There is no treatment or activity being controlled in any way. Observational studies are commonly conducted using surveys, though you can also collect data by just watching what is happening such as observing the types of trees in a forest.

Survey: Surveys are used for gathering data to create a sample. There are many different kinds of surveys, but overall, a survey is a method used to ask people questions when interested in the responses. Examples of surveys are Internet and T.V. surveys, customer satisfaction surveys at stores or restaurants, new product surveys, phone surveys, and mail surveys. The majority of surveys are some type of public opinion poll.

Experiment: This is an activity where the researcher controls some aspect of the study and then records what happens. An example of this is giving a plant a new fertilizer, and then watching what happens to the plant. Another example is giving a cancer patient a new medication, and monitoring whether the medication stops the cancer from growing. There are many ways to do an experiment, but a clinical study is one of the more popular ways, so we will look at the aspects of this.

Clinical Study: This is a method of collecting data for a sample and then comparing that to data collected for another sample where one sample has been given some sort of treatment and the other sample has not been given that treatment (control). *Note: There are occasions when you can have two treatments, and no control. In this case you are trying to determine which treatment is better.*

Example 1.3.1: Clinical Study Examples

Here are examples of clinical studies.

a. A researcher may want to study whether or not smoking increases a person's chances of heart disease.
b. A researcher may want to study whether a new antidepressant drug will work better than an old antidepressant drug.
c. A researcher may want to study whether taking folic acid before pregnancy will decrease the risk of birth defects.

Clinical Study Terminology:

Treatment Group: This is the group of individuals who are given some sort of treatment. The word treatment here does not necessarily mean medical treatment. The treatment is the cause, which may produce an effect that the researcher is interested in.

Control Group: This is the group of individuals who are not given the treatment. Sometimes, they may be given some old treatment, or sometimes they will not be given anything at all. Other times, they may be given a placebo (see below).

Example 1.3.2: Treatment/Control Group Examples

Determine the treatment group, control group, treatment, and control for each clinical study in Example 1.3.1.

a. A researcher may want to study whether or not smoking increases a person's chances of heart disease.
   The treatment group is the people in the study who smoke and the treatment is smoking. The control group is the people in the study who do not smoke and the control is not smoking.

b. A researcher may want to study whether a new antidepressant drug will work better than an old antidepressant drug.

The treatment group is the people in the study who take the new antidepressant drug and the treatment is taking the new antidepressant drug. The control group is the people in the study who take the old antidepressant drug and the control is taking the old antidepressant drug. *Note: In this case the control group is given some treatment since you should not give a person with depression a non-treatment.*

c. A researcher may want to study whether taking folic acid before pregnancy will decrease the risk of birth defects.

The treatment group is the women who take folic acid before pregnancy and the treatment is taking folic acid. The control group is the women who do not take folic acid before pregnancy and the control is not taking the folic acid. *Note: In this case, you may choose to do an observational study of women who did or did not take folic acid during pregnancy so that you are not inducing women to avoid folic acid during pregnancy which could be harmful to their baby.*

Confounding Variables: These are other possible causes that may produce the effect of interest rather than the treatment under study. Researchers minimize the effect of confounding variables by comparing the results from the treatment group versus the control group.

Controlled Study: Any clinical study where the researchers compare the results of a treatment group versus a control group.

Placebo: A placebo is sometimes used on the control group in a study to mimic the treatment that the treatment group is receiving. The idea is that if a placebo is used, then the people in the control group and in the treatment group will all think that they are receiving the treatment. However, the control group is merely receiving something that looks like the treatment, but should have no effect on the outcome. An example of a placebo could be a sugar pill if the treatment is a drug in pill form.

Example 1.3.3: Placebo Examples

For each situation in Example 1.3.1, identify if a placebo is necessary to use.

a. A researcher may want to study whether or not smoking increases a person's chances of heart disease.

In this example, it is impossible to use a placebo. The treatment group is comprised of people who smoke and the control group is comprised of people who do not smoke. There is no way to get the control group to think that they are smoking as well as the treatment group.

b. A researcher may want to study whether a new antidepressant drug will work better than an old antidepressant drug.

   In this example, a placebo is not needed since we are comparing the results of two different antidepressant drugs.

c. A researcher may want to study whether taking folic acid before pregnancy will decrease the risk of birth defects.

   In this example, the control group could be given a sugar pill instead of folic acid. However, they may think that they are taking folic acid and so the psychological effect on a person's health can be being measured. This way, when we compare the results of taking folic acid versus taking a sugar pill, we can see if there were any dramatic differences in the results.

Blind Study: Usually, when a placebo is used in a study, the people in the study will not know if they received the treatment or the placebo until the study is completed. In other words, the people in the study do not know if they are in the treatment group or in the control group. This type of study is called a blind study. *Note: When researchers use a placebo in a blind study, the people in the study are told ahead of time that they may be getting the actual treatment, or they may be getting the placebo.*

Double-Blind Study: Sometimes when researchers are conducting a very extensive study using many healthcare workers, the researchers will not tell the people in the study or the healthcare workers which patients will receive the treatment and which patients will receive the placebo. In other words, the healthcare workers who are administering the treatment or placebo to the people in the study do not know which people are in the treatment group and which people are in the control group. This type of study is called a double-blind study.

Randomized Controlled Study: Any clinical study in which the treatment group and the control group are selected randomly from the population.

Whether you are doing an observational study or an experiment, you need to figure out what to do with the data. You will have many data values that you collected, and it sometimes helps to calculate numbers from these data values. Whether you are talking about the population or the sample, determines what we call these numbers.

Parameter: A numerical value calculated from a population
Statistic: A numerical value calculated from a sample, and used to estimate the parameter

Some examples of parameters that can be estimated from statistics are the percentage of people who strongly agree to a question and mean net worth of all Americans. The statistic would be the percentage of people asked who strongly agree to a question, and the mean net worth of a certain number of Americans.

Notation for Parameter and Statistics:
Parameters are usually denoted with Greek letters. This is not to make you learn a new alphabet. It is because there just are not enough letters in our alphabet. Also, if you see a letter you do not know, then you know that the letter represents a parameter. Examples of letters that are used are μ (mu), σ (sigma), ρ (rho), and p (yes this is our letter because there is not a good choice in the Greek alphabet).
Statistics are usually denoted with our alphabet, and in some cases we try to use a letter that would be equivalent to the Greek letter. Examples of letter that are used are $\bar{x}$ (x-bar), s, r, and $\hat{p}$ (p-hat, since we already used p for the parameter).
In addition, N is used to denote the size of the population and n is used to denote the size of the sample.

Sampling Error: This is the difference between a parameter and a statistic. There will always be some error between the two since a statistic is an estimate of a parameter. Sampling error is attributed to chance error and sample bias.

Chance Error: The error inherent in taking information from a sample instead of from the whole population. This comes from the fact that two different samples from the same population will likely give two different statistics.

Sample Bias: The error from using a sample that does not represent the population. To avoid this, use some sort of random sample.

Sampling Rate: The fraction of the total population that is in the sample. This can be denoted by n/N.

Section 1.4: Should You Believe a Statistical Study?

Now we have looked at the basics of a statistical study, but how do you make sure that you conduct a good statistical study? You need to use the following guidelines.

Guidelines for Conducting a Statistical Study:

1. State the goal of your study precisely. Make sure you understand what you actually want to know before you collect any data. Determine exactly what you would like to learn about.
2. State the population you want to study and state the population parameter(s) of interest.
3. Choose a sampling method. A simple random sample is the best type of sample, though sometimes a stratified or cluster sample may be better depending on the question you are asking.
4. Collect the data for the sample and summarize these data by finding sample statistics.
5. Use the sample statistics to make inferences about the population parameters.
6. Draw conclusions: Determine what you learned and whether you achieved your goal.

The mistake that most people make when doing a statistical study is to collect the data, and then look at the data to see what questions can be answered. This is actually backwards. So, make sure you know what question you want to answer before you collect any data.

Even if you do not conduct your own study, you will be looking at studies that other people have conducted. Every day you hear and see statistics on the news, in newspapers and magazines, on the Internet and other places. Some of these statistics may be legitimate and beneficial, but some may be inaccurate and misleading. Here are some steps to follow when evaluating whether or not a statistical study is believable.

Steps for Determining whether a Statistical Study is Believable:

1. Are the population, goal of the study, and type of study clearly stated?
You should be able to answer the following questions when reading about a statistical study:

- Does the study have a clear goal? What is it?
- Is the population defined clearly? What is it?
- Is the type of study used clear and appropriate?

2. Is the source of the study identified? Are there any concerns with the source?
A study may not have been conducted fairly if those who funded the study are biased.

Example 1.4.1: Source of Study 1

Suppose a study is conducted to find out the percentage of United States college professors that belong to the Libertarian party. If this study was paid for by the Libertarian party, or another political party, then there may have been bias involved with conducting the study. Usually an independent group is a good source for conducting political studies.

Example 1.4.2: Source of Study 2

There was once a full-page ad in many of the newspapers around the U.S. that said that global warming was not happening. The ad gave some reasons why it was not happening based on studies conducted. At the bottom of the page, in small print, were the words that the study and ad were paid for by the oil and gas industry. So, the study may have been a good study, but since it was funded by the industry that would benefit from the results, then you should question the validity of the results.

3. Are there any confounding variables that could skew the results of the study?
Confounding variables are other possible causes that may produce the effect of interest besides the variable under study. In a scientific experiment researchers may be able to minimize the effect of confounding variables by comparing the results from a treatment group versus a control group.

Example 1.4.3: Confounding Variable

A study was done to show that microwave ovens were dangerous. The study involved plants, where one plant was given tap water and one plant was given water that was boiled in the microwave oven. The plant given the water that was boiled died. So the conclusion was that microwaving water caused damage to the water and thus caused the plant to die. However, it could easily have been the fact that boiling water was poured onto the plant that caused the plant to die.

4. Could there be any bias from the sampling method that was used?
Sometimes researchers will take a sample from the population and the results may be biased.

Selection Bias: This occurs when the sample chosen from the population is not representative of the population.
Participation Bias (or Nonresponse Bias): This occurs when the intended objects in the sample do not respond for many different reasons. Those who feel strongly about an issue will be more likely to participate.

Example 1.4.4: Bias

The 1936 *Literary Digest* Poll. The *Literary Digest* was a magazine that was founded in 1890. Starting with the 1916 U.S. presidential election, the magazine had predicted the winner of the each election. In 1936, the *Literary Digest* predicted that Alfred Landon would win the election in a landslide over Franklin Delano Roosevelt with fifty-seven percent of the popular vote. The process for predicting the winner was that the magazine sent out ten million mock ballots to its subscribers and names of people who had automobiles and telephones. Two million mock ballots were sent back. In reality, Roosevelt won the election with 62% of the popular vote. ("Case Study 1: The 1936 *Literary Digest* Poll," n.d.)

A side note is that at the same time that the *Literary Digest* was publishing its prediction, a man by the name of George Gallup also conducted a poll to predict the winner of the election. Gallup only polled about fifty thousand voters using random sampling techniques, yet his prediction was that Roosevelt would win the election. His polling techniques were shown to be the more accurate method, and have been used to present-day.

Selection Bias: Because of the people whom the *Literary Digest* polled, they created something called a selection bias. The poll asked ten million people who owned cars, had telephones, and subscribed to the magazine. Today, you would probably think that this group of people would be representative of the entire U.S. However, in 1936 the country was in the midst of the Great Depression. So the people polled were mostly in the upper middle to upper class. They did not represent the entire country. It did not matter that the sample was very large. The most important part of a sample is that it is representative of the entire population. If the sample is not, then the results could be wrong, as demonstrated in this case. It is important to collect data so that it has the best chance of representing the entire population.

Nonresponse Bias: When looking at the number of ballots returned, two million appears to be a very large number. However, ten million ballots were sent out. So that means that only about one-fifth of all the ballots were actually returned. This is known as a nonresponse bias. The only people who probably took the time to fill out and return the ballot were those who felt strongly about the issue. So when you send out a survey, you have to pay attention to what percentage of surveys are actually returned. If at all possible, it is better to conduct the survey in person or through the telephone. Most credible polls conducted today, such as Gallup, are conducted either in person or over the telephone. Do be careful though, just because a polling group conducts the poll in person or on the telephone does not mean that it is necessarily credible.

5. Are there any problems with the setting of a survey?
The setting of the survey can create bias. So you want to make sure that the setting is as neutral as possible, so that someone does not answer based on where the survey is conducted or who is giving the survey.

Example 1.4.5: Setting Example

Suppose a survey is being conducted to learn more about illegal drug use among college students. If a uniformed police officer is conducting the survey, then the results will very likely be biased since the college students may feel uncomfortable telling the truth to the police officer.

6. Are there any problems with the wording of a survey?
How a question is worded can elicit a particular response. Also the order of the questions may affect a person's answers. So make sure that the questions are worded in a way that would not lead a person to a particular answer.

Example 1.4.6: Wording Example

A question regarding the environment may ask "Do you think that global warming is the most important world environmental issue, or pollution of the oceans?" Alternatively, the question may be worded "Do you think that pollution of the oceans is the most important world environmental issue, or global warming?" The answers to these two questions will vary greatly simply because of how they are worded. The best way to handle a question like this is to present it in multiple choice format as follows:
What do you think is the most important world environmental issue?
a. Global warming
b. Pollution of the oceans
c. Other

7. Are the results presented fairly?

Be sure that any concluding statements accurately represent the data and statistics that were calculated from the data. Many times people make conclusions that are beyond the scope of the study, or are beyond the results of the data.

Example 1.4.7: Wrong Conclusion

> Many studies have been done on cancer treatments using rats. The wrong conclusion is to say that because a treatment cured cancer in rats, then it will cure cancer in people. The fact that a treatment cured cancer in rats, means that there is a chance that it will cure cancer in people, but you would have to try it on people before making such claims. Rats and people have different physiology, so you cannot assume what works on one will work on the other.

8. Are there any misleading graphics?
Be sure that any graphics that are presented along with the results are not misleading. Some examples of misleading graphs are:

- The vertical axis does not start at zero. This means that any changes will look more dramatic than they really are.
- There is no title. This means that you do not know what the graph is actually portraying.
- There are missing labels or units. This means that you do not know what the variables are or what the units are.
- The wrong type of graph is used. Sometimes people use the wrong graph, like using a bar graph when a line graph would be more appropriate.

9. Final considerations
Ask yourself the following questions about the overall effectiveness of the statistical study.

- Do the conclusions of the study answer the initial goal of the study?
- Do the conclusions of the study follow from the data and statistics?
- Do the conclusions of the study indicate practical changes should be made?

Overall, you should follow these steps when analyzing the validity of any statistical study.

Section 1.5: Graphs

Once we have collected data, then we need to start analyzing the data. One way to analyze the data is using graphical techniques. The type of graph to use depends on the type of data you have. Qualitative data use graphs like bar graphs, pie graphs, and pictograms. Quantitative data use graphs such as histograms. In order to create any graphs, you must first create a summary of the data in the form of a frequency distribution. A frequency distribution is created by listing all of the data values (or grouping of data values) and how often the data value occurs.

Frequency: Number of times a data value occurs in a data set.

Frequency Distribution: A listing of each data value or grouping of data values (called classes) with their frequencies.

Relative Frequency: The frequency divided by n, the size of the sample. This gives the percent of the total for each data value or class of data values.

Relative Frequency Distribution: A listing of each data value or class of data values with their relative frequencies.

How to create a frequency distribution depends on whether you have qualitative or quantitative variable. We will now look at how to create each type of frequency distribution according to the type of variable, and the graphs that go with them.

Qualitative Variable:
First let's look at the types of graphs that are commonly created for qualitative variables. Remember, qualitative variables are words, and not numbers.

Bar graph: A graph where rectangles represent the frequency of each data value or class of data values. The bars can be drawn vertically or horizontally. *Note: The bars do not touch and they are the same width.*

Pie Chart: A graph where the "pie" represents the entire sample and the "slices" represent the categories or classes. To find the angle that each "slice" takes up, multiple the relative frequency of that slice by 360°. *Note: The percentages in each slice of a pie chart must all add up to 100%.*

Pictograms: A bar graph where the bars are made up of icons instead of rectangles.

Pictograms are overused in the media and they are the same as a regular bar graph except more eye-catching. To be more professional, bar graphs or pie charts are better.

Example 1.5.1: Qualitative Variable Frequency Distribution and Graphs

Suppose a class was asked what their favorite soft drink is and the following is the results:

Table 1.5.1: Favorite Soft Drink

Coke	Pepsi	Mt. Dew	Coke	Pepsi	Dr. Pepper	Sprite	Coke	Mt. Dew
Pepsi	Pepsi	Dr. Pepper	Coke	Sprite	Mt. Dew	Pepsi	Dr. Pepper	Coke
Pepsi	Mt. Dew	Coke	Pepsi	Pepsi	Dr. Pepper	Sprite	Pepsi	Coke
Dr. Pepper	Mt. Dew	Sprite	Coke	Coke	Pepsi			

a. Create a frequency distribution for the data.

To do this, just list each drink type, and then count how often each drink comes up in the list. Notice Coke comes up nine times in the data set. Pepsi comes up 10 times. And so forth.

Table 1.5.2: Frequency Distribution of Favorite Soft Drink

Drink	**Coke**	**Pepsi**	**Mt Dew**	**Dr. Pepper**	**Sprite**
Frequency	9	10	5	5	4

b. Create a relative frequency distribution for the data.

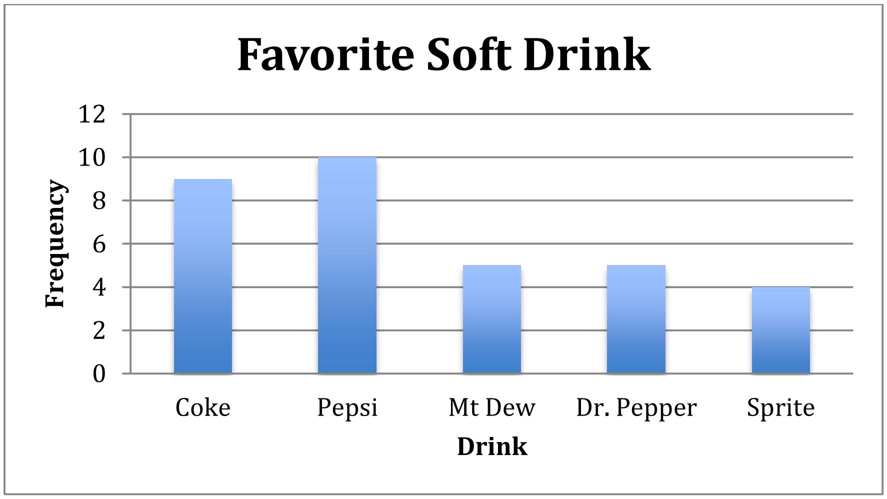
To do this, just divide each frequency by 33, which is the total number of data values. Round to three decimal places.

Table 1.5.3: Relative Frequency Distribution of Favorite Soft Drink

Drink	**Coke**	**Pepsi**	**Mt Dew**	**Dr. Pepper**	**Sprite**
Frequency	9	10	5	5	4
Relative Frequency	9/33 =0.273 =27.3%	10/33 =0.303 =30.3%	5/33 =0.152 =15.2%	5/33 =0.152 =15.2%	4/33 =0.121 =12.1%

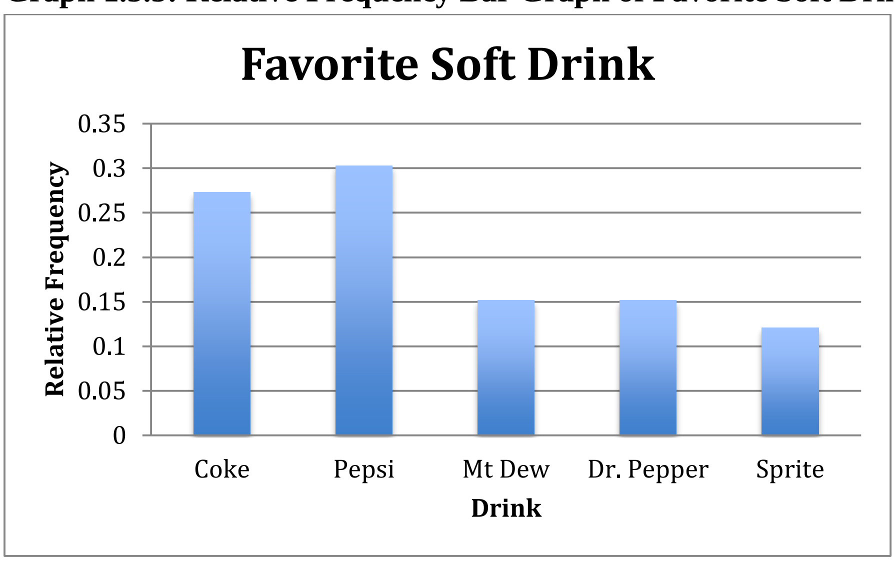
c. Draw a bar graph of the frequency distribution.
Along the horizontal axis you place the drink. Space these equally apart, and allow space to draw a rectangle above it. The vertical axis contains the frequencies. Make sure you create a scale along that axis in which all of the frequencies will fit. Notice that the highest frequency is 10, so you want to make sure the vertical axis goes to at least 10, and you may want to count by two for every tick mark. Using Excel, this is what your graph will look like.

Graph 1.5.4: Bar Graph of Favorite Soft Drink



d. Draw a bar graph of the relative frequency distribution.
This is similar to the bar graph for the frequency distribution, except that you use the relative frequencies instead. Notice that the graph does not actually change except the numbers on the vertical scale.

Graph 1.5.5: Relative Frequency Bar Graph of Favorite Soft Drink



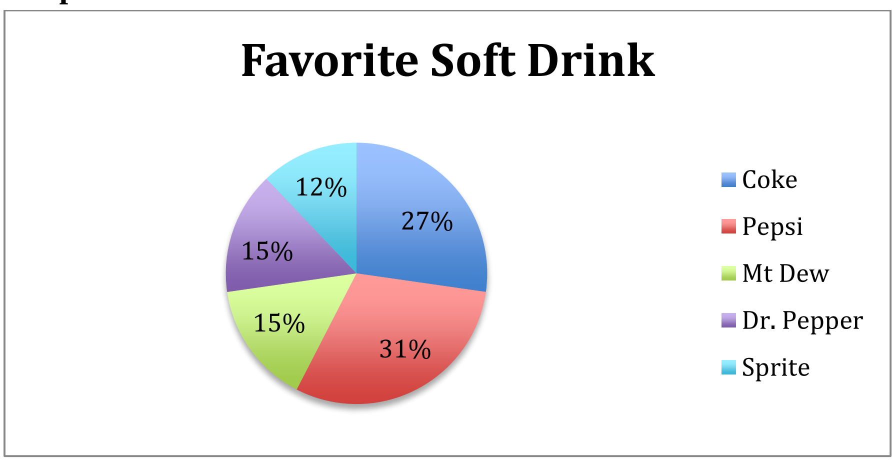
e. Draw a pie chart of the data.

To draw a pie chart, multiply the relative frequencies by 360°. Then use a protractor to draw the corresponding angle. Or, it is easier to use Excel, or some other spreadsheet program to draw the graph.

Table 1.5.6: Angles for the Pie Chart of Favorite Soft Drink

Drink	Coke	Pepsi	Mt Dew	Dr. Pepper	Sprite
Frequency	9	10	5	5	4
Relative Frequency	0.273	0.303	0.152	0.152	0.121
Angles	(9/33)*360 =98.2°	(10/33)*360 =109.1°	(5/33)*360 =54.5°	(5/33)*360 =54.5°	(4/33)*360 =43.6°

Graph 1.5.7: Pie Chart for Favorite Soft Drink



f. Draw a pictograph for the favorite soft drink data.

Here you can get creative. One thing to draw would be glasses. Now you would not want to draw 10 glasses. So what you can do is let each glass be worth a certain number of data values, let's say one glass = frequency of two. So this means that you will need to draw half of a glass for some of the frequencies. So for the first drink, with a frequency of nine, you need to draw four and a half glasses. For the second drink, with a frequency of 10, you need to draw five glasses. And so on.

Graph 1.5.8: Pictograph for Favorite Soft Drink

Pictographs are not really useful graphs. The makers of these graphs are trying to use graphics to catch a person's eye, but most of these graphs are missing labels, scaling, and titles. Additionally, it can sometimes be unclear what ½ or ¼ of an icon represents. It is better to just do a bar graph, and use color to catch a person's eye.

Quantitative Variable:
Quantitative variables are numbers, so the graph you create is different from the ones for qualitative data. First, the frequency distribution is created by dividing the interval containing the data values into equally spaced subintervals. Then you count how many data values fall into each subinterval. Since the subintervals do not overlap, but do touch, then the graph you create has the bars touching.

Histogram: A graph of a quantitative variable where rectangles are used for each subinterval, the height of the rectangle represents the frequency of the data values in the subinterval, and there are no gaps in between the rectangles. Sometimes the midpoint of each subinterval is graphed instead of the endpoints of the subinterval.

Example 1.5.2: Quantitative Variable Frequency Distribution and Graphs

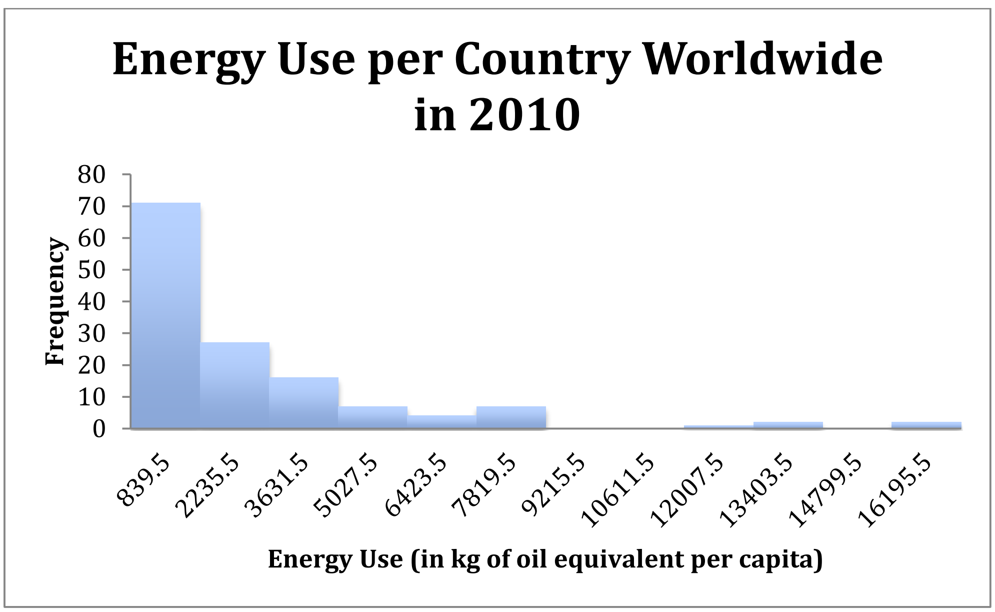
The energy used (in kg of oil equivalent per capita) in 2010 of 137 countries around the world is summarized in the following frequency distribution. Use this distribution draw a histogram. (World Bank, 2010).

This frequency distribution was created by dividing the range of the data into 12 equally spaced subintervals, sometimes called classes.

Table 1.5.9: Frequency Distribution for Energy Used

Lower limit	Upper limit	Midpoint	Frequency
142	1537	839.5	71
1538	2933	2235.5	27
2934	4329	3631.5	16
4330	5725	5027.5	7
5726	7121	6423.5	4
7122	8517	7819.5	7
8518	9913	9215.5	0
9914	11309	10611.5	0
11310	12705	12007.5	1
12706	14101	13403.5	2
14102	15497	14799.5	0
15498	16893	16195.5	2

Graph 1.5.10: Histogram for Energy Used in 2010 for 137 Countries in the World

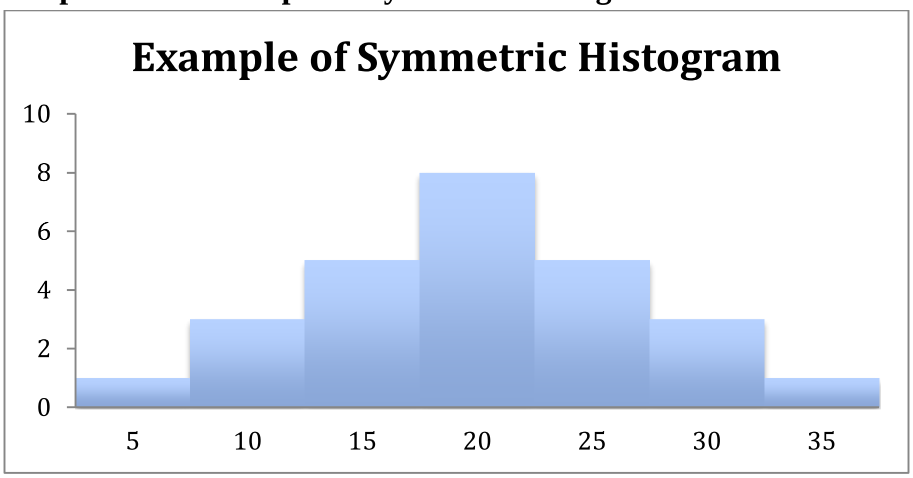


Notice that the vertical axes starts at 0, there is a title on the graph, the axes have labels, and the tick marks are labeled. This is a correct way to draw a graph, and allows people to know what the data represents.

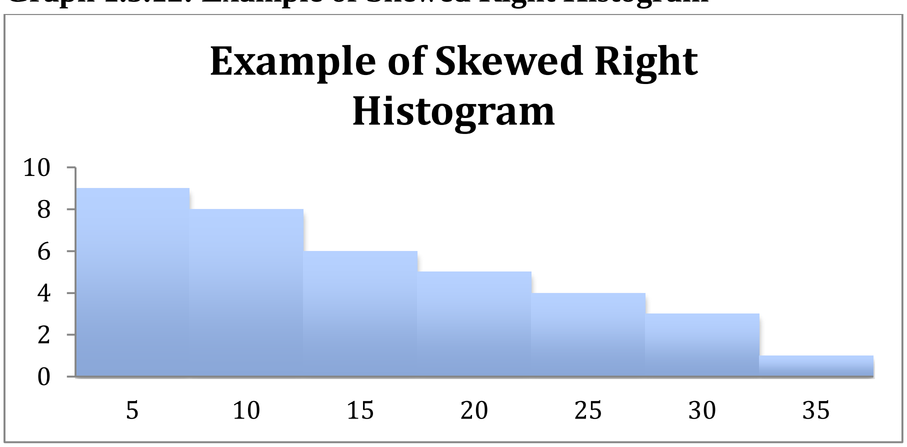
Interpreting graphs:

It is important to be able to interpret graphs. If you look at the graphs in Example 1.5.1, you can see that Pepsi is more popular than any of the other drinks. You can also see that Sprite is the least popular, and that Mt. Dew and Dr. Pepper are equally liked. If you look at the graph in Example 1.5.2, you can see that most countries use around 839.5 kg of energy per capita. You can also see that the graph is heavily weighted to the lower amounts of energy use, and that there is a gap between the bulk of the amounts and the higher ends. So there are very few countries that use over 9215.5 kg of energy per capita. Since the data is quantitative, we can talk about the shape of the distribution. This graph would be called skewed right, since the data on the right side of the graph is the unusual data, and if it was not there, then the graph may look more symmetric. Some basic shapes of histograms are shown below.

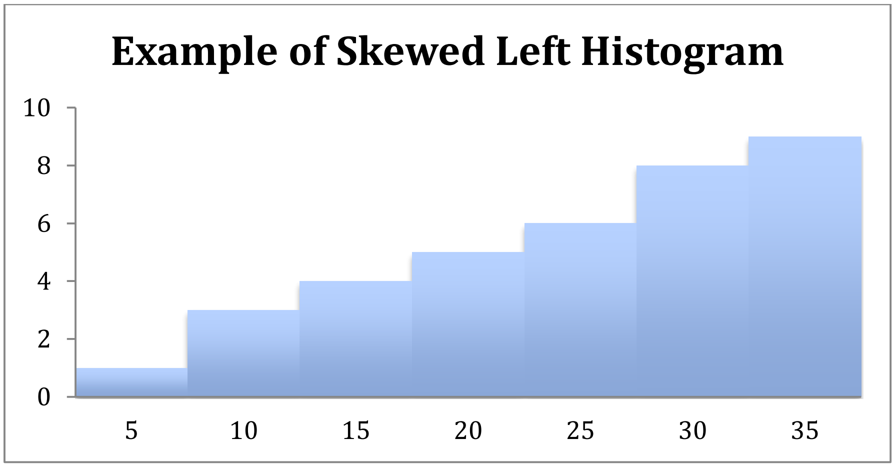
Graph 1.5.11: Example of Symmetric Histogram



Graph 1.5.12: Example of Skewed Right Histogram



Graph 1.5.13: Example of Skewed Left Histogram



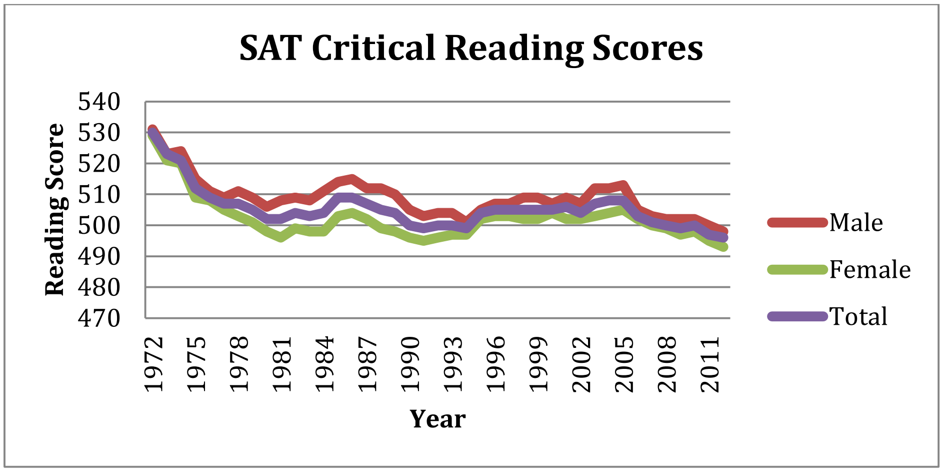
Section 1.6: Graphics in the Media

There are many other types of graphs you will encounter in the media.

Multiple Line Graphs:

A line graph is useful for seeing trends over time. Multiple line graphs are useful for seeing trends over time, and also comparing two or more data sets. As an example, suppose you want to examine the average SAT critical reading score over time for Arizona students, but you further want to compare the averages overall and between genders. So, a multiple line graph like the following may be used. As you can see, the average score for SAT critical reading has been going down over the years. You can also see that the average score for male students is higher than the average score for female students for all of the years. The other interesting aspect that you can see is that the average scores for male and female used to be closer to each other, then they separated fairly far, and look to be getting closer to each other again. One last comment is that even though there is a difference between male and female average score, the average scores seem to follow each other. In other words, when the male score was going down, so was the female score. Do be careful. Do not try to make up a reason for the scores to go down. You cannot say why the scores have decreased, since you did not run an experiment. The scores could have decreased because our education system is not teaching as well, funding has decreased for education, or the intelligence of Arizona students has decreased. Or it could be that the percentage of the overall student body that takes the SAT has increased over the years, meaning that more than just the highest ranked students have been taking the SAT in later years, which could lower the averages. Any one of these reasons, or other reasons, could be the right one, and you cannot determine which it is. Do not make unsubstantiated claims. *Note: A next step in analyzing this data could be to compare the AZ trends to national trends.*

Graph 1.6.1: Multiple Line Graph for SAT Critical Reading Scores in AZ



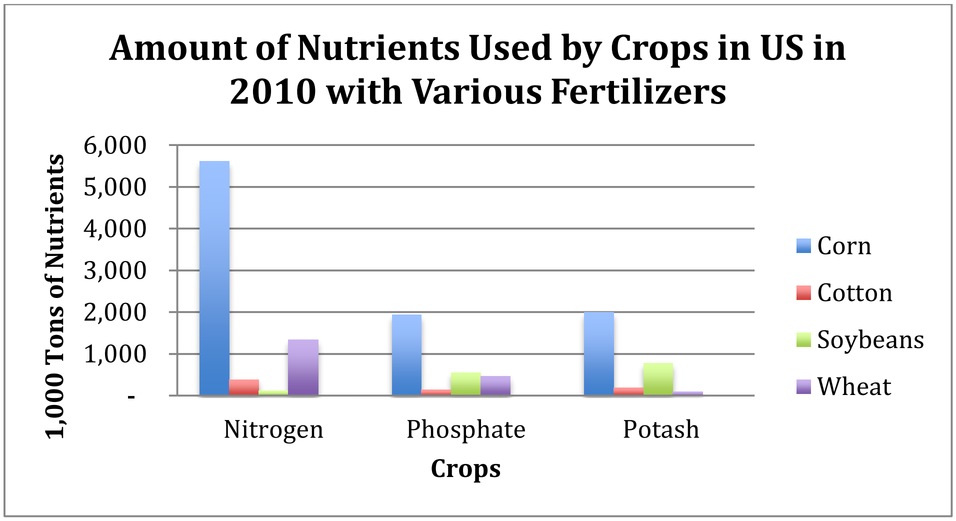
(College Board: Arizona, 2012)

Note: In this case the vertical axis did not start at zero, because if it did start at zero, the different lines would be very close together and difficult to see. When at all possible, the scale on the vertical axis should start at zero. Always carefully consider that if a vertical scale of a graph does not start at zero, the researchers could be attempting to exaggerate insignificant differences in the data.

Multiple Bar Graphs:

Sometimes you have information for multiple variables and instead of putting the information on different bar graphs, you can put them all on one so that you can compare the variables. The following is an example of where you might use this. The data is the amount of nutrients used by crops with various fertilizers. By creating a graph that has all of the fertilizers and all of the crops, you can see that corn with nitrogen uses the most nutrients, and soybeans with nitrogen uses the least amount of nutrients. You can also see that phosphate seems to use low amounts of nutrients for all of its crops. So, a multiple bar graph is useful to make all of these observations.

Graph 1.6.2: Multiple Bar Graph for the Amount of Nutrients Used by Crops: 2010

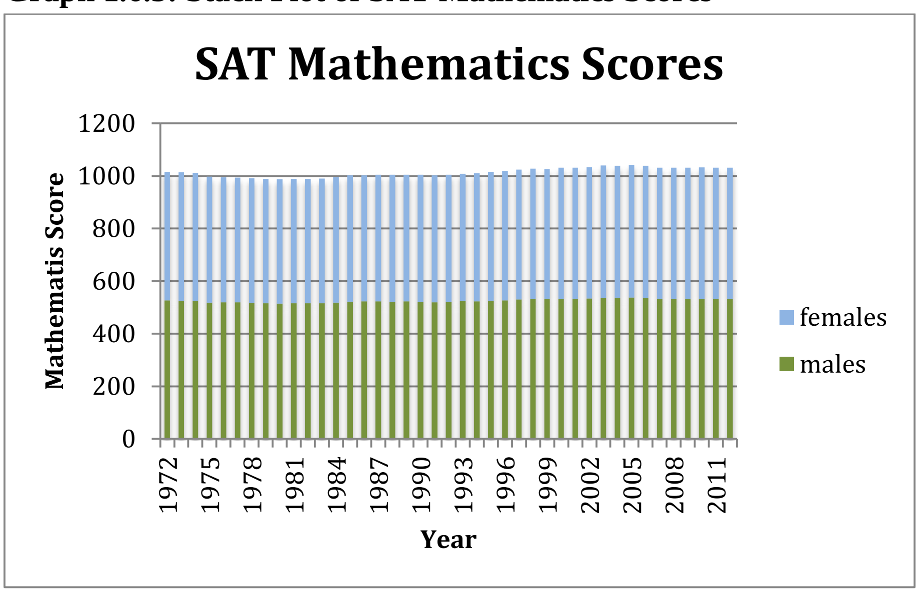


(United States Department of Agriculture [USDA], 2010)

Stack Plots:

A stack plot is basically a multiple line graph, but with the lines separated (or stacked) on top of each other instead of overlapping. This can be useful when it is difficult to interpret a multiple line graph since the lines are so close to one another. To read a given line on a stack plot, you must subtract that line from the line below it. In the example of a stack plot below, you can see that the SAT Mathematics score for males in 1972 is about 530 while the SAT Mathematics score for females in 1972 is about 1010 – 530 = 480.

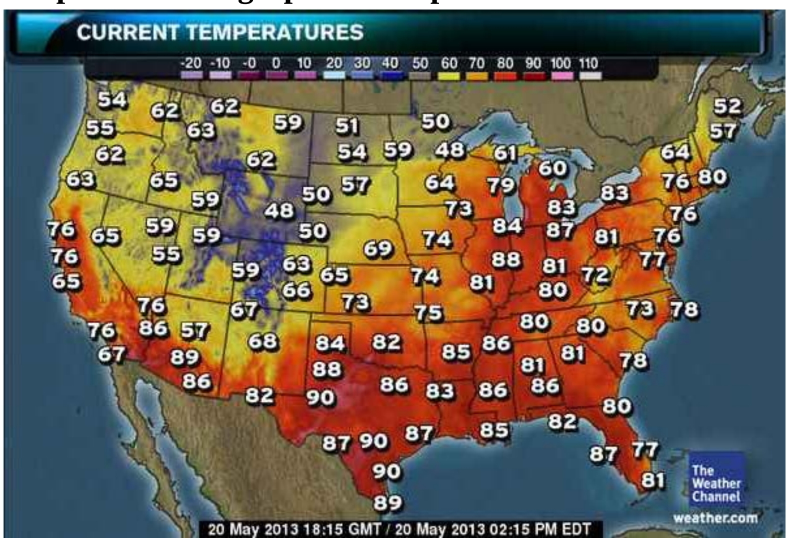
Graph 1.6.3: Stack Plot of SAT Mathematics Scores



(College Board: Arizona, 2012)

Geographical Graphs:
Weather maps, topographic maps, population distribution maps, gravity maps, and vegetation maps are examples of geographical graphs. They allow you to see a trend of information over a geographic area. The following is an example of a weather map showing temperatures. As you can see, the different colors represent certain temperature ranges. From this graph, you can see that on this date, the red in the south means that the temperature was in the 80s and 90s there, and the blue in the Rockies area means that the temperature was in the 40s there.

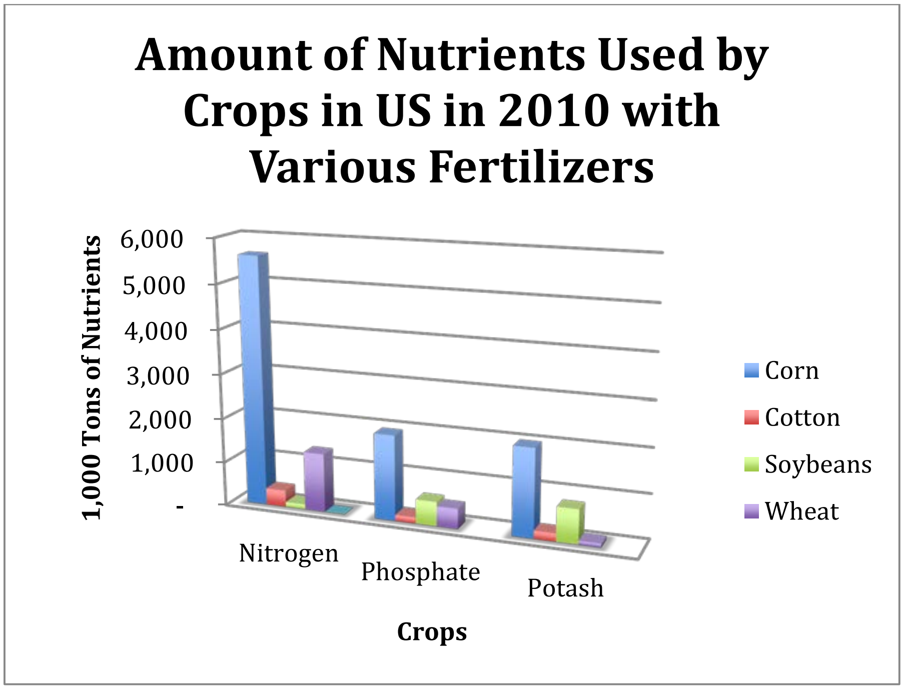
Graph 1.6.4: Geographical Graph



(Weather Channel, 2013)

Three-Dimensional Graphics
Some people like to show a bar graph in three-dimensions. Occasionally, a three-dimensional graph is used to graph three variables together on three axes, but this type of graph may be difficult to read. The following graph just represents two variables and so it is basically the same as a standard bar graph, but the three-dimensional look may add a bit more style.

Graph 1.6.5: Three-Dimensional Graph

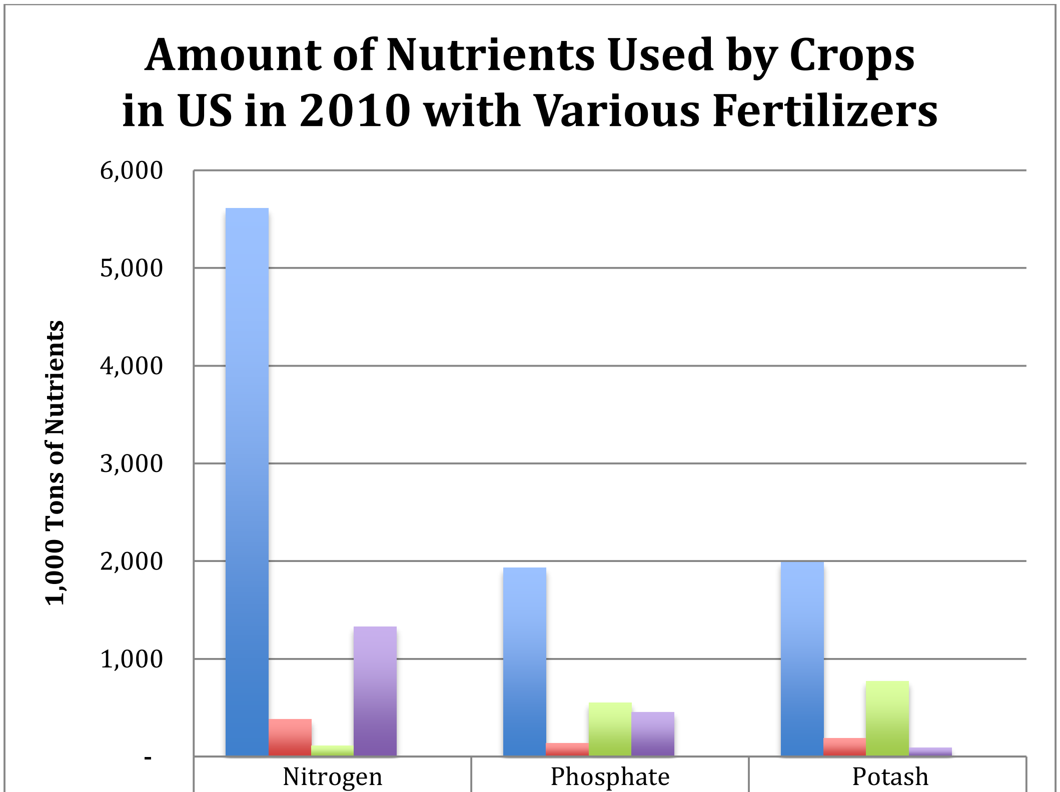


(USDA, 2010)

Combination Graphics

Some graphs are created so that they combine the data table and a graph or combine two types of graphs in one. The advantage is that you can see a graphical representation of the data, and still have the data to find exact values. The disadvantage is that they are busier, and usually people show graphical representation of data because people do not like looking at the data.

Graph 1.6.6: Combination Graph



	Nitrogen	Phosphate	Potash
Corn	5,610	1,933	1,991
Cotton	380	139	188
Soybeans	111	552	774
Wheat	1,331	458	90
Fertilizer	1		

(USDA, 2010)

These are just a few of the different types of graphs that exist in the world. There are many other ones. A quick Google search on statistical graphs will show you many more. Just open up a newspaper, magazine, or website and you are likely to see others. The most important thing to remember is that you need to look at the graph objectively, and interpret for yourself what it says. Also, do not read any cause and effect into what you see.

Chapter 1 Homework

1. A study was conducted of schools across the U.S. about whether they require school uniforms. Two hundred ninety-six schools gave their response to the question, "Does your school require school uniforms?" State the population and sample.

2. The U.S. Department of Labor collects information on the average hourly earnings of professional and business services positions. Suppose that you look at 20 years, and collect data on the average hourly earnings in those years. State the population and sample.

3. A person collects the gas prices at 25 gas stations in Phoenix, AZ. State the population and sample.

4. The Center for Disease Control collects data on the number of children with autism. They collect data on 32,601 children in the state of Arizona, and then look to see how many have autism. State the population and sample.

5. Determine if the variable is qualitative or quantitative. If quantitative, then also state if the variable is discrete or continuous.
   a. Height of buildings in a town
   b. Eye color of all students at a college
   c. Weight of cars
   d. Number of dogs in a household
   e. Religion of people in a town
   f. Number of fish caught daily

6. Determine if the variable is qualitative or quantitative. If quantitative, then also state if the variable is discrete or continuous.
   a. Letter grades of students in a class
   b. Distance a person runs every day
   c. Number of prairie dogs on a parcel of land
   d. Time that a task takes to complete
   e. Gender of a person
   f. Number of cars at a dealership on a given day

7. A study to estimate the average salary of workers at a university was conducted using the following designs. Categorize the sampling method as a simple random sample, stratified sample, cluster sample, systematic sample, or convenience sample.
   a. Each person who is employed by the university is first divided into groups of administrative professional, classified, faculty, and part-time. Then each person in those groupings is given a number and then random samples are taken inside each grouping.
   b. Each researcher asks the first 40 people he or she encounters on campus what their salary is.
   c. The researchers number all employees, and then start with the 34^{th} person. Then they record the salary of every 10^{th} person after that.
   d. The researchers number all employees, and then use a random number generator to determine which employees they will use.
   e. Each college on campus is given a number. Then five colleges are chosen at random and all employees' salaries in each college is recorded.

8. A study to determine the opinion of Americans about the use of marijuana for medical purposes is being conducted using the following designs. Categorize the sampling method as a simple random sample, stratified sample, cluster sample, systematic sample, or convenience sample.
   a. The researchers attend a festival in a town in Kansas and ask all the people they can what their opinions are.
   b. The researchers divide Americans into groups based on the person's race, and then take random samples from each group.
   c. The researchers number all Americans and call the 50^{th} person on the list. Then they call every $10{,}000^{th}$ person after the 50^{th} person.
   d. The researchers call every person in each of 10 area codes that were randomly chosen.
   e. The researchers number every American, and then call all randomly selected Americans.

9. A biologist is looking to see the effect of microgravity on plant growth. The researcher sends some seeds to the International Space Station, and has the height of the plants measured on specific days. The researcher also plants the same number of seeds in a laboratory on Earth, using the same lighting, soil, and water conditions, and measures the height of the plants on specific days. Describe the treatment group and the control group.

10. To see if a new blood pressure medication works or not, volunteers are divided into two groups. One group is given the new medication and the other group is given an older medication. Describe the treatment group and the control group.

11. In each situation state if a placebo is needed or not and explain your reasoning.
    a. A new medication for treating heartworm in dogs is being investigated to see if it works.
    b. A new training method for managers is being investigated to see if it improves morale.
    c. A new medication is being tested to see if it reduces itching due to eczema.

12. In each situation state if a placebo is needed or not and explain your reasoning.
    a. A new drug is being developed to treat high blood pressure and needs to be tested to see if it works better than a previous type of drug.
    b. A new medication is being developed to treat headaches.
    c. A new drink flavor is being developed to see if it is marketable.

13. To see if a new medication works to reduce fevers, volunteers are divided into two groups. One group is given the new medication, and the other group is given a placebo. The volunteers do not know which group they are in, but the researchers do. Is this a blind or double blind study?

14. To see if a new blood pressure medication works or not, volunteers are divided into two groups. One group is given the new medication and the other group is given the old medication. The volunteers do not know which group they are in, and neither do the researchers. Is this a blind or double blind study?

15. In each situation, identify a potential source of bias and explain your reasoning.
    a. A study on teenage boys shows that a new drug works on curing acne. The company then concludes that the new drug will work on everyone.
    b. A radio station asks listeners to phone in their choice in a daily poll.
    c. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
    d. A poll asks, “Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?”

e. A study is conducted on whether artificial light is better for plants. A certain type of plant is grown under artificial light and its height is recorded. A different type of plant is grown under natural light and its height is recorded. Both plant groups get the same amount of water and soil.

16. In each situation, identify a potential source of bias and explain your reasoning.
    a. A study is conducted to see if grades improve if students are given tutoring. Students in a calculus class are given tutoring and compared to students in a statistics class who are not given tutoring.
    b. An organic grower conducts a study and shows that pesticides are harmful to people who eat food that used the pesticide.
    c. A poll asks, "If 53% of all people will need assisted living in the future, should you worry about needing assisted living insurance?"
    d. A website asks people to say if they support the President or not.
    e. A study of behavior modification therapy is tried on a specific breed of dogs. The study concludes that the behavior modification therapy works on all breeds of dogs.

17. The average SAT test scores for Michigan students in reading for the years 1971 to 2012 are given in the table below (College Board: Michigan, 2012). Create a table showing frequencies and relative frequencies for the data using the classes of 490 to 496, 497 to 503, 504 to 510, 511 to 517, 518 to 524, and 525 to 531.

496	500	502	504	505	507	509
497	500	503	504	505	507	512
499	500	503	505	505	508	521
499	500	504	505	506	508	523
499	501	504	505	507	509	530
500	502	504	505	507	509	509

18. The average hourly earnings of all employees in the U.S. in professional and business services for each month in the time period from March 1, 2006 to June 1, 2013 is given in the table below (Federal Reserve Bank of St. Louis, 2013). Create a table showing frequencies and relative frequencies for the data using the classes of 23 to 23.9, 24 to 24.9, 25 to 25.9, 26 to 26.9, 27 to 27.9, and 28 to 28.9.

23.40	23.81	24.85	25.72	26.73	26.84	27.19	27.23	27.74
23.87	24.22	24.44	25.46	26.74	26.98	27.14	27.39	27.48
23.23	24.42	24.91	25.47	26.89	26.98	27.43	28.19	27.67
23.14	24.56	24.65	25.72	27.37	26.88	26.96	27.81	28.09
23.68	24.45	24.80	25.59	27.32	27.27	27.06	27.55	27.79
23.31	24.84	25.33	25.66	27.02	27.19	27.31	27.57	27.97
23.64	24.46	25.33	25.89	26.84	27.32	27.18	27.84	28.55
23.97	24.45	25.48	26.05	26.74	27.49	27.21	27.45	28.17
28.03	28.30	27.85	27.76	28.24	27.68	28.27	27.90	28.04
28.58	28.56	28.61	28.43	28.38	28.24	28.57		

19. Create histogram of the data in problem 17. State the overall shape of this histogram.

20. Create a relative frequency histogram of the data in problem 17. State the overall shape of this histogram.

21. Create histogram of the data in problem 18. State the overall shape of this histogram.

22. Create a relative frequency histogram of the data in problem 18. State the overall shape of this histogram.

23. In Kenya, the government is interested in the number of health care facilities in each of the provinces. Below is a table showing this data in 2013 (Kenya Open Data, 2013).

Province: * The province's name was missing from the original data.	**Number of facilities**
RIFT VALLEY	1645
EASTERN	1093
NYANZA	962
NAIROBI	878
COAST	765
CENTRAL	745
WESTERN	541
N. EASTERN	131
*	9
PROV	3

a. Draw a bar graph of the data.
b. Draw a pie chart of the data.

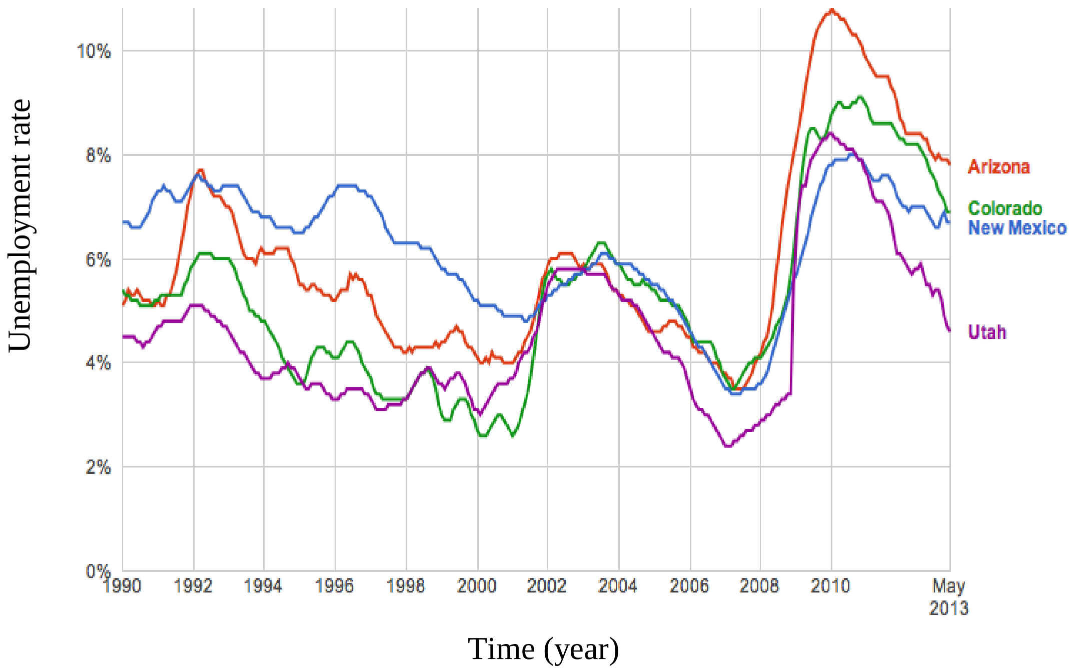
24. According to the Pew Research Center, the percentage of people who own a particular smartphone is given in the table (Smith, 2011).

Platform	**Percentage**
Android	35%
iPhone	24%
Blackberry	24%
Palm device	6%
Windows phone	4%
Unspecified	7%

a. Draw a bar graph of the data.
b. Draw a pie chart of the data.

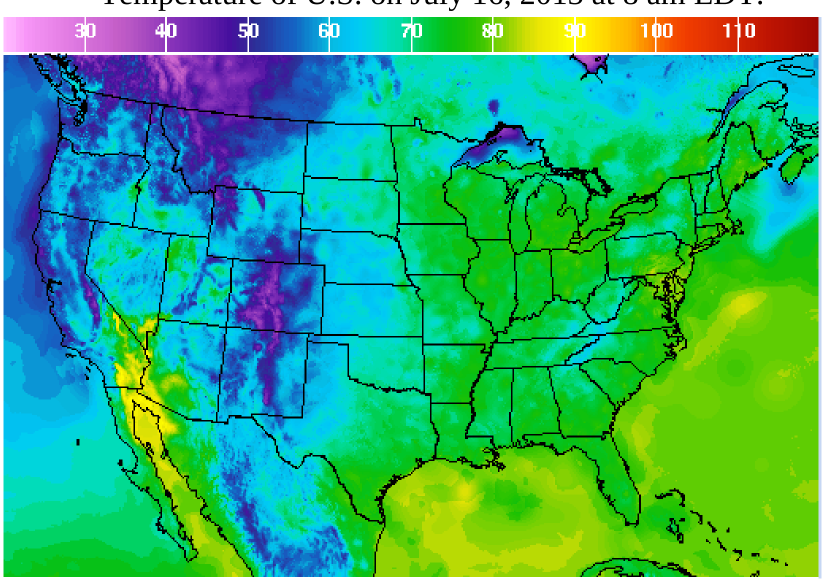
25. The following is a multiple line graph of unemployment percentages for the four corner states (Arizona, Colorado, New Mexico, and Utah). ("Unemployment Rate," n.d.) Compare and contrast the data for the four states, describing in detail at least three observations you identify from the graph.

Unemployment Rate from 1990 to 2013 – Seasonally Adjusted



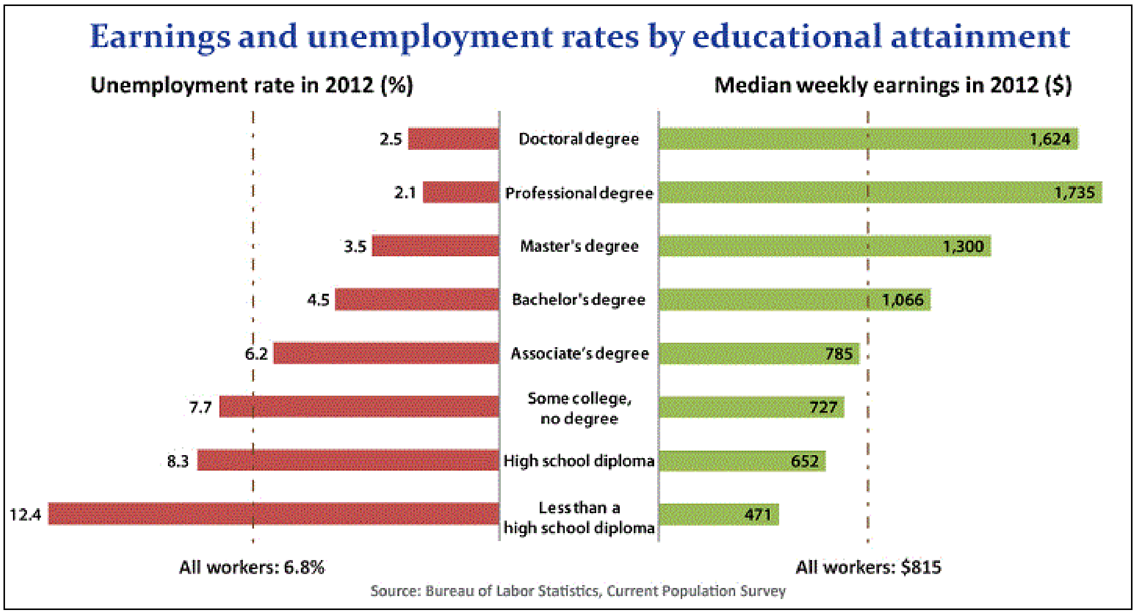
26. Below are the temperatures in the United States on July 16, 2013 at 8 am EDT (National Weather Service, 2013).

Temperature of U.S. on July 16, 2013 at 8 am EDT.



a. What temperature range did Arizona have on that day?
b. What temperature range did Montana have on that day?
c. What temperature range did Texas have on that day?

27. The following graph is the salaries and unemployment rate for different levels of education. ("Employment Projections," n.d.) Describe in detail at least three observations you identify from the graph.



Chapter 2: Statistics: Part 2

Graphical descriptions of data are important. However, many times we want to have a number to help describe a data set. As an example, in baseball a pitcher is considered good if he has a low number of earned runs per nine innings. A baseball hitter is considered good if he has a high batting average. These numbers tell us a great deal about a player. There are similar numbers in other sports such as percentage of field goals made in basketball. There are also similar numbers in other aspects of life. If you want to know how much money you will make when you graduate from college and are employed in your chosen field, you could look at the average salary that someone with your degree earns. If you want to know if you can afford to purchase a home, you could look at the median price of homes in the area. To understand how to find this information, we need to look at the different numerical descriptive statistics that exist out there.

Numerical Descriptive Statistics: These are numbers that are calculated from the sample and are used to describe or estimate the population parameter.

Statistics that we can calculate are proportion, location of center (average), measures of spread (variability), and percentiles. There are other numbers, but these are the ones that we will concentrate on in this book.

Section 2.1: Proportion

Proportions are usually calculated when dealing with qualitative variables. Suppose that you want to know the proportion of time that a basketball player will make a free throw. You could look at how often the player tries to make the free throw, and how often they do make a free throw. Then you could divide the number made by the number attempted. This is how we find proportion. This is a sample statistic, since we cannot look at all of the attempts, because the player could attempt more in the future. If the player retires, and never wants to play basketball ever again, then we could find the population parameter for that player. Since there are rare cases where you can find this, then we will define both the population parameter and the sample statistic. Remember though, usually we use the sample statistic to estimate the population parameter.

Population Proportion:

$$p = \frac{r}{N}$$

where r = number of successes observed
N = number of times the activity could be tried

Sample Proportion:

$$\hat{p} = \frac{r}{n}$$

where r = number of successes observed
n = number of times the activity was tried

Example 2.1.1: Finding Proportion

Suppose that you ask 140 people if they prefer vanilla ice cream to other flavors, and 86 say yes. What is the proportion of people who prefer vanilla ice cream?

Since you only asked 140 people, and there are many more than 140 people in the world, then this is a sample and we use the sample proportion formula.

$$\hat{p} = \frac{r}{n}$$

$r = 86$

$n = 140$

$$\hat{p} = \frac{86}{140} \approx 0.614 = 61.4\%$$

So 61.4% of the people in the sample like vanilla ice cream. This could mean that 61.4% of all people in the world like vanilla ice cream. We do not know for sure, but this is a good guess for the true proportion, p, as long as our sample was representative of the population. If you own an ice cream shop, then you probably want to make sure you order more vanilla ice cream than other flavors.

Section 2.2: Location of Center

The center of a population is very important. This describes where you expect to find values. If you know that you expect to make $50,000 annually when you graduate from college and are employed in your field of study, then that is the location of the center. It does not mean everyone will make that amount. It just means that you will make around that amount. The location of center is also known as the average. There are three types of averages—mean, median, and mode.

Mean: The mean is the type of average that most people commonly call "the average." You take all of the data values, find their sum, and then divide by the number of data values. Again, you will be using the sample statistic to estimate the population parameter, so we need formulas and symbols for each of these.

Population Mean:

$$\mu = \frac{x_1 + x_2 + \cdots + x_N}{N} = \frac{\sum x}{N}$$

where N = size of the population

$x_1,\ x_2,\ \cdots,\ x_N$ are data values

Note: $\sum \boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_{\boldsymbol{N}}$ *is a short cut way to write adding a bunch of numbers together*

Sample Mean:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{n} = \frac{\sum x}{n}$$

where n = size of the sample

$x_1,\ x_2,\ \cdots,\ x_n$ are data values

Note: $\sum \boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_{\boldsymbol{n}}$ *is a short cut way to write adding a bunch of numbers together*

Median: This is the value that is found in the middle of the ordered data set.

Most books give a long explanation of how to find the median. The easiest thing to do is to put the numbers in order and then count from both sides in, one data value at a time, until you get to the middle. If there is one middle data value, then that is the median. If there are two middle data values, then the median is the mean of those two data values. If you have a really large data set, then you will be using technology to find the value. There is no symbol or formula for median, neither population nor sample.

Mode: This is the data value that occurs most often.

The mode is the only average that can be found on qualitative variables, since you are just looking for the data value with the highest frequency. The mode is not used very often otherwise. There is no symbol or formula for mode, neither population nor sample. Unlike the other two averages, there can be more than one mode or there could be no mode. If you have two modes, it is called bimodal. If there are three modes, then it is called trimodal. If you have more than three modes, then there is no mode. You can also have a data set where no values occur most often, in which case there is no mode.

Example 2.2.1: Finding the Mean, Median, and Mode (Odd Number of Data Values)

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F)

Table 2.2.1: Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

(Weather Underground, n.d.)

Find the mean, median, and mode for the high temperature

Since there are only 11 days, then this is a sample.

Mean:

$$\bar{x} = \frac{71+59+69+68+63+57+57+57+57+65+67}{11}$$
$$= \frac{690}{11}$$
$$\approx 62.7^\circ F$$

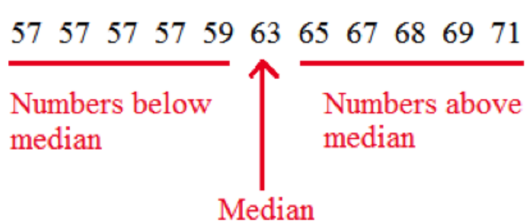
Median:

First put the data in order from smallest to largest.

57, 57, 57, 57, 59, 63, 65, 67, 68, 69, 71

Now work from the outside in, until you get to the middle number.

Figure 2.2.2: Finding the Median.



So the median is 63°F

Mode:

From the ordered list it is easy to see that 57 occurs four times and no other data values occur that often. So the mode is 57°F.

We can now say that the expected high temperature in early May in Flagstaff, Arizona is around 63°F.

Example 2.2.2: Finding the Mean, Median, and Mode (Even Number of Data Values)

Now let's look at the first 12 days of May 2013 in Flagstaff, AZ. The following is the high temperatures (in °F)

Table 2.2.3: Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	73

(Weather Underground, n.d.)

Find the mean, median, and mode for the high temperature
Since there are only 12 days, then this is a sample.

Mean:

$$\bar{x} = \frac{71+59+69+68+63+57+57+57+57+65+67+73}{12}$$

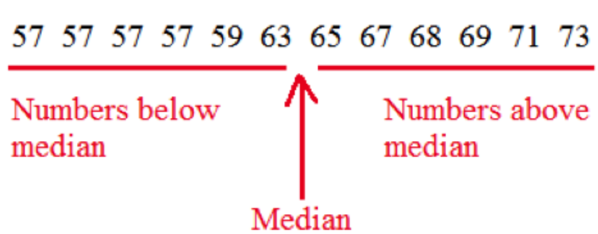
$$= \frac{763}{12}$$

$$\approx 63.6°F$$

Median:

First put the data in order from smallest to largest.
57, 57, 57, 57, 59, 63, 65, 67, 68, 69, 71, 73
Now work from the outside in, until you get to the middle number.

Figure 2.2.4: Finding the Median



This time there are two numbers that are in the middle. So the median is

$$\text{median} = \frac{63+65}{2} = 64°F.$$

Mode:

From the ordered list it is easy to see that 57 occurs 4 times and no other data values occurs that often. So the mode is 57°F.

Example 2.2.3: Effect of Extreme Values on the Mean and Median

A random sample of unemployment rates for 10 countries in the European Union (EU) for March 2013 is given:

Table 2.2.5: Unemployment Rates for EU Countries

11.0	7.2	13.1	26.7	5.7	9.9	11.5	8.1	4.7	14.5

(Eurostat, n.d.)

Find the mean, median, and mode.
Since the problem says that it is a random sample, we know this is a sample. Also, there are more than 10 countries in the EU.

Mean:

$$\begin{aligned}\bar{x} &= \frac{11.0+7.2+13.1+26.7+\cdots+14.5}{10}\\ &= \frac{112.4}{10}\\ &= 11.24\end{aligned}$$

The mean is 11.24%.

Median:
4.7, 5.7, 7.2, 8.1, 9.9, 11.0, 11.5, 13.1, 14.5, 26.7
Both 9.9 and 11.0 are the middle numbers, so the median is

$$\text{median} = \frac{9.9+11.0}{2} = 10.45$$

The median is 10.45%.

Note: This data set has no mode since there is no number that occurs most often.

Now suppose that you remove the 26.7 from your sample since it is such a large number (an outlier). Find the mean, median, and mode.

Table 2.2.6: Unemployment Rates for EU Countries

11.0	7.2	13.1	5.7	9.9	11.5	8.1	4.7	14.5

$$\begin{aligned}\bar{x} &= \frac{11.0+7.2+13.1+5.7+\cdots+14.5}{9}\\ &= \frac{85.7}{9}\\ &\approx 9.52\end{aligned}$$

The mean is 9.52%
The median is 9.9%.
There is still no mode.

Notice that the mean and median with the 26.7 were a bit different from each other. When the 26.7 value was removed, the mean dropped significantly, while the median dropped, but not as much. This is because the mean is affected by extreme values called outliers, but the median is not affected by outliers as much.

In section 1.5, there was a discussion on histogram shapes. If you look back at Graphs 1.5.11, 1.5.12, and 1.5.13, you will see examples of symmetric, skewed right, and skewed left graphs. Since symmetric graphs have their extremes equally on both sides, then the mean would not be pulled in any direction, so the mean and the median are essentially the same value. With a skewed right graph, there are extreme values on the right, and they will pull the mean up, but not affect the median much. So the mean will be higher than the median in skewed right graphs. Skewed left graphs have their extremes on the left, so the mean will be lower than the median in skewed left graphs.

Example 2.2.4: Finding the Average of a Qualitative Variable

Suppose a class was asked what their favorite soft drink is and the following is the results:

Table 2.2.7: Favorite Soft Drink

Coke	Pepsi	Mt. Dew	Coke	Pepsi	Dr. Pepper	Sprite	Coke	Mt. Dew
Pepsi	Pepsi	Dr. Pepper	Coke	Sprite	Mt. Dew	Pepsi	Dr. Pepper	Coke
Pepsi	Mt. Dew	Coke	Pepsi	Pepsi	Dr. Pepper	Sprite	Pepsi	Coke
Dr. Pepper	Mt. Dew	Sprite	Coke	Coke	Pepsi			

Find the average.

Remember, mean, median, and mode are all examples of averages. However since the data is qualitative, you cannot find the mean and the median. The only average you can find is the mode. Notice, Coke was preferred by 9 people, Pepsi was preferred by 10 people, Mt Dew was preferred by 5 people, Dr. Pepper was preferred by 5 people, and Sprite was preferred by 4 people. So Pepsi has the highest frequency, so Pepsi is the mode. If one more person came in the room and said that they preferred Coke, then Pepsi and Coke would both have a frequency of 10. So both Pepsi and Coke would be the modes, and we would call this bimodal.

Section 2.3: Measures of Spread

The location of the center of a data set is important, but also important is how much variability or spread there is in the data. If a teacher gives an exam and tells you that the mean score was 75% that might make you happy. But then if the teacher says that the spread was only 2%, then that means that most people had grades around 75%. So most likely you have a C on the exam. If instead you are told that the spread was 15%, then there is a chance that you have an A on the exam. Of course, there is also a chance that you have an F on the exam. So the higher spread may be good and it may be bad. However, without that information you only have part of the picture of the exam scores. So figuring out the spread or variability is useful.

Measures of Spread or Variability: These values describe how spread out a data set is.

There are different ways to calculate a measure of spread. One is called the range and another is called the standard deviation. Let’s look at the range first.

Range: To find the range, subtract the minimum data value from the maximum data value. Some people give the range by just listing the minimum data value and the maximum data value. However, to statisticians the range is a single number. So you want to actually calculate the difference.

$$\text{Range} = \text{maximum} - \text{minimum}$$

The range is relatively easy to calculate, which is good. However, because of this simplicity it does not tell the entire story. Two data sets can have the same range, but one can have much more variability in the data while the other has much less.

Example 2.3.1: Finding the Range

Find the range for each data set.

a. 10, 20, 30, 40, 50

$$\text{Range} = 50 - 10 = 40$$

b. 10, 35, 36, 37, 50

$$\text{Range} = 50 - 10 = 40$$

Notice both data sets from Example 2.3.1 have the same range. However, the one in part b seems to have most of the data closer together, except for the extremes. There seems to be less variability in the data set in part b than in the data set in part a. So we need a better way to quantify the spread.

Instead of looking at the difference between highest and lowest, let's look at the difference between each data value and the center. The center we will use is the mean. The difference between the data value and the mean is called the deviation.

Deviation from the Mean: $\text{data value} - \text{mean} = x - \bar{x}$

To see how this works, let's use the data set from Example 2.2.1. The mean was about 62.7°F

Table 2.3.1: Finding the Deviations

x	$x-\bar{x}$
71	8.3
59	-3.7
69	6.3
68	5.3
63	0.3
57	-5.7
57	-5.7
57	-5.7
57	-5.7
65	2.3
67	4.3
Sum	**0.3**

Notice that the sum of the deviations is around zero. If there is no rounding of the mean, then this should add up to exactly zero. So what does that mean? Does this imply that on average the data values are zero distance from the mean? No. It just means that some of the data values are above the mean and some are below the mean. The negative deviations are for data values that are below the mean and the positive deviations are for data values that are above the mean. So we need to get rid of the sign (positive or negative). How do we get rid of a negative sign? Squaring a number is a widely accepted way to make all of the numbers positive. So let's square all of the deviations.

Squared Deviations from the Mean: To find these values, square the deviations from the mean. Also, you can think of this as being the squared distance from the mean.

So for the data set, let's find the squared deviations.

Table 2.3.2: Finding the Squared Deviations.

x	$x-\overline{x}$	$\left(x-\overline{x}\right)^2$
71	8.3	68.89
59	-3.7	13.69
69	6.3	39.69
68	5.3	28.09
63	0.3	0.09
57	-5.7	32.49
57	-5.7	32.49
57	-5.7	32.49
57	-5.7	32.49
65	2.3	5.29
67	4.3	18.49
Sum	**0.3**	**304.19**

Now that we have the sum of the squared deviations, we should find the mean of these values. However, since this is a sample, the normal way to find the mean, summing and dividing by n, does not estimate the true population value correctly. It would underestimate the true value. So, to calculate a better estimate, we will divide by a slightly smaller number, $n-1$. This strange average is known as the sample variance.

Sample Variance: This is the sum of the squared deviations from the mean divided by $n-1$. The symbol for sample variance is s^2 and the formula for the sample variance is:

$$s^2 = \frac{\sum\left(x-\overline{x}\right)^2}{n-1}$$

For this data set, the sample variance is

$$s^2 = \frac{304.19}{11-1} = \frac{304.19}{10} = 30.419$$

The variance measures the average squared distance from the mean. Since we want to know the average distance from the mean, we will need to take the square root at this point.

Sample Standard Deviation: This is the square root of the variance. The standard deviation is a measure of the average distance the data values are from the mean. The symbol for sample standard deviation is s and the formula for the sample standard deviation is

$$s = \sqrt{s^2} = \sqrt{\frac{\sum\left(x - \bar{x}\right)^2}{n-1}}$$

Thus, for this data set, the sample standard deviation is $s = \sqrt{30.419} \approx 5.52^{\circ}F$.
Note: The units are the same as the original data.

Since the sample variance and the sample standard deviation are used to estimate the population variance and population standard deviation, we should define the symbols and formulas for those as well.

Population Variance: $\sigma^2 = \frac{\sum\left(x - \mu\right)^2}{N}$

Population Standard Deviation: $\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum\left(x - \mu\right)^2}{N}}$

Example 2.3.2: Finding the Range, Variance, and Standard Deviation

A random sample of unemployment rates for 10 counties in the EU for March 2013 is given

Table 2.3.3: Unemployment Rates for EU Countries

11.0	7.2	13.1	26.7	5.7	9.9	11.5	8.1	4.7	14.5

(Eurostat, n.d.)
Find the range, variance, and standard deviation.

Since this is a sample, then we will use the sample statistics formulas.
In Example 2.2.3, we calculated the mean to be 11.24%. The maximum value is 26.7% and the minimum value is 4.7%. So the range is:

range = 26.7 – 4.7 = 22.0%

To find the variance and the standard deviation, it is easier to use a table then the formula. The table follows the formula though, so they are the same thing.

Table 2.3.4: Finding the Variance and Standard Deviation

x	$x-\bar{x}$	$(x-\bar{x})^2$
11.0	-0.24	0.0576
7.2	-4.04	16.3216
13.1	1.86	3.4596
26.7	15.46	239.0116
5.7	-5.54	30.6916
9.9	-1.34	1.7956
11.5	0.26	0.0676
8.1	-3.14	9.8596
4.7	-6.54	42.7716
14.5	3.26	10.6276
Sum	**0**	**354.664**

Sample variance:

$$s^2 = \frac{354.664}{10-1} = \frac{354.664}{9} \approx 39.40711111$$

Sample standard deviation:

$$s = \sqrt{39.4071111} \approx 6.28\%$$

So, the unemployment rates for countries in the EU are approximately 11.24% with an average spread of about 6.28%. Since the sample standard deviation is fairly high compared to the mean, then there is a great deal of variability in unemployment rates for countries in the EU. This means that countries in the EU have rates that are much lower than the mean and some that have rates much higher than the mean.

Percentiles

There are other calculations that we can do to look at spread. One of those is called percentile. This looks at what data value has a certain percent of the data at or below it.

Percentiles: A value with k-percent of the data at or below this value.

For example, if a data value is in the 80th percentile, then 80% of the data values fall at or below this value.

We see percentiles in many places in our lives. If you take any standardized tests, your score is given as a percentile. If you take your child to the doctor, their height and weight are given as percentiles. If your child is tested for gifted or behavior problems, the score is given as a percentile. If your child has a score on a gifted test that is in the 92nd percentile, then that means that 92% of all of the children who took the same gifted test scored the same or lower than your child. That also means that 8% scored the same or higher than your child. This may mean that your child is gifted.

Example 2.3.3: Interpreting Percentiles

Suppose you took the SAT mathematics test and received your score as a percentile.

a. What does a score in the 90th percentile mean?
   90 percent of the scores were at or below your score (You did the same as or better than 90% of the test takers.)

b. What does a score in the 70th percentile mean?
   70% of the scores were at or below your score.

c. If the test was out of 800 points and you scored in the 80th percentile, what was your score on the test?
   You do not know! All you know is that you scored the same as or better than 80% of the people who took the test. If all the scores were really low, you could have still failed the test. On the other hand, if many of the scores were high you could have gotten a 95% on the test.

d. If your score was in the 95th percentile, does that mean you passed the test?
   No, it just means you did the same as or better than 95% of the other people who took the test. You could have failed the test, but still did the same as or better than 95% of the rest of the people.

There are three percentiles that are commonly used. They are the first, second, and third quartiles, where the quartiles divide the data into 25% sections.

First Quartile (Q_1): 25th percentile (25% of the data falls at or below this value.)
Second Quartile (Q_2 or M): 50th percentile, also known as the median (50% of the data falls at or below this value.).
Third Quartile (Q_3): 75th percentile (75% of the data falls at or below this value.)

To find the quartiles of a data set:
Step 1: Sort the data set from the smallest value to the largest value.
Step 2: Find the median (M or Q_2).
Step 3: Find the median of the lower 50% of the data values. This is the first quartile (Q_1).

Step 4: Find the median of the upper 50% of the data values. This is the third quartile (Q_3).

If we put the three quartiles together with the maximum and minimum values, then we have five numbers that describe the data set. This is called the five-number summary.

Five-Number Summary: Lowest data value known as the minimum (Min), the first quartile (Q_1), the median (M or Q_2), the third quartile (Q_3), and the highest data value known as the maximum (Max).

Also, since we have the quartiles, we can talk about how much spread there is between the 1st and 3rd quartiles. This is known as the interquartile range.

Interquartile Range (IQR): $IQR = Q_3 - Q_1$

There are times when we want to look at the five-number summary in a graphical representation. This is known as a box-and-whiskers plot or a box plot.

Box Plot: Plot of the five-number summary

A box plot is created by first setting a scale (number line) as a guideline for the box plot. Then, draw a rectangle that spans from Q_1 to Q_3 above the number line. Mark the median with a vertical line through the rectangle. Next, draw dots for the minimum and maximum points to the sides of the rectangle. Finally, draw lines from the sides of the rectangle out to the dots.

Example 2.3.4: Find the Five-Number Summary and IQR and Draw a Box Plot (Odd Number of Data Points)

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F):

Table 2.3.5: Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	

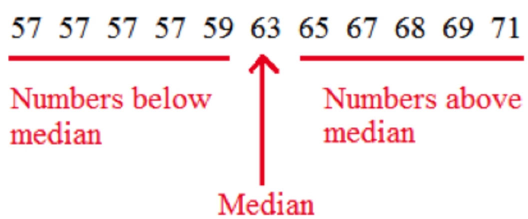
(Weather Underground, n.d.)

Find the five-number summary and IQR and draw a box plot.

To find the five-number summary, you must first put the numbers in order from smallest to largest.
57, 57, 57, 57, 59, 63, 65, 67, 68, 69, 71

Then find the median. The number 63 is in the middle of the data set, so the median is 63°F. To find Q_1, look at the numbers below the median. Since 63 is the median, you do not include that in the listing of the numbers below the median. To find Q_3, look at the numbers above the median. Since 63 is the median, you do not include that in the listing of the numbers above the median.

Figure 2.3.6: Finding the median, Q_1, and Q_3

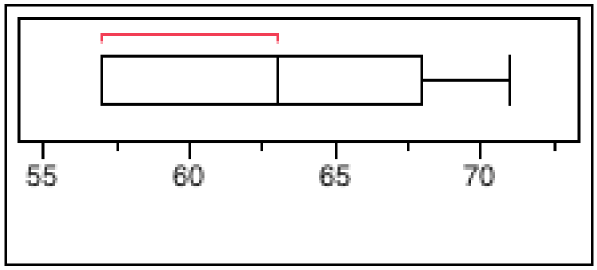


Looking at the numbers below the median, the median of those is 57. Q_1 = 57°F. Looking at the numbers above the median, the median of those is 68. Q_3 = 68°F. Now find the minimum and maximum. The minimum is 57°F and the maximum is 71°F. Thus, the five-number summary is:
Min = 57°F
Q_1 = 57°F
Med = Q_2 = 63°F
Q_3 = 68°F
Max = 71°F.
Also, the IQR = $Q_3 - Q_1$ = 68 – 57 = 11°F

Finally, draw a box plot for this data set as follows:

Figure 2.3.7: Box Plot

Temperatures in °F in Flagstaff, AZ, in early May 2013



Notice that the median is basically in the center of the box, which implies that the data is not skewed. However, the minimum value is the same as Q_1, so that implies there might be a little skewing, though not much.

Example 2.3.5: Find the Five-Number Summary and IQR and Draw a Box Plot (Even Number of Data Points)

The first 12 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F):

Table 2.3.8: Weather Data for Flagstaff, AZ, in May 2013

71	59	69	68	63	57
57	57	57	65	67	73

(Weather Underground, n.d.)

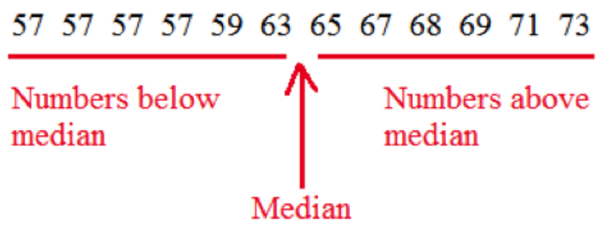
Find the five-number summary and IQR and draw a box plot.
To find the five-number summary, you must first put the data values in order from smallest to largest. 57, 57, 57, 57, 59, 63, 65, 67, 68, 69, 71, 73

Then find the median. The numbers 63 and 65 are in the middle of the data set, so the median is $\frac{63+65}{2} = 64^\circ F$

To find Q_1, look at the numbers below the median. Since the number 64 is the median, you include all the numbers below 64, including the 63 that you used to find the median.

To find Q_3, look at the numbers above the median. Since the number 64 is the median, you include all the numbers above 64, including the 65 that you used to find the median.

Figure 2.3.9: Finding the Median, Q_1, and Q_3.



Looking at the numbers below the median (57, 57, 57, 57, 59, 63), the median of those is $\frac{57+57}{2} = 57^{\circ}F$. $Q_1 = 57$°F. Looking at the numbers above the median (65, 67, 68, 69, 71, 73), the median of those is $\frac{68+69}{2} = 68.5^{\circ}F$. $Q_3 = 68.5$°F.

Now find the minimum and maximum. The minimum is 57°F and the maximum is 73°F.

Thus, the five-number summary is:

Min = 57°F

Q_1 = 57°F

Med = Q_2 = 64°F

Q_3 = 68.5°F

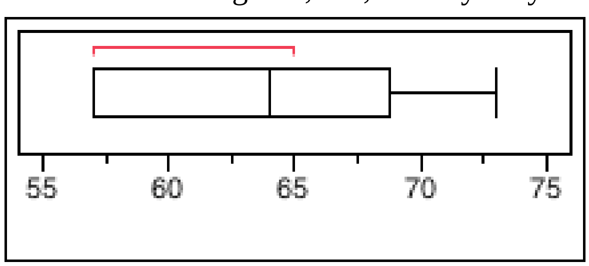
Max = 73°F.

Also, the IQR = $Q_3 - Q_1$ = 68.5 – 57 = 11.5°F

Finally, draw a box plot for this data set as follows:

Figure 2.3.10: Box Plot

Temperatures in °F in Flagstaff, AZ, in early May 2013



Notice that the median is basically in the center of the box, so that implies that the data is not skewed. However, the minimum value is the same as Q_1, so that implies there might be a little skewing, though not much.

It is important to understand how to find all descriptive statistics by hand and also by using a calculator.

Example 2.3.6: Finding the Descriptive Statistics Using the TI-83/84 Calculator

The first 11 days of May 2013 in Flagstaff, AZ, had the following high temperatures (in °F):

Table 2.3.11: Weather Data for Flagstaff, AZ, in May 2013

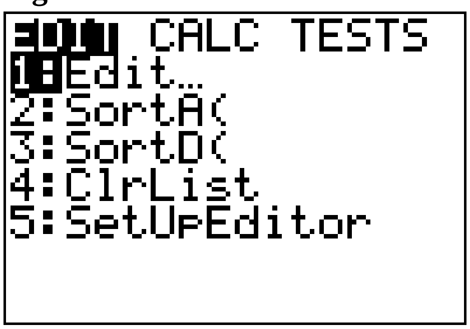
71	59	69	68	63	57
57	57	57	65	67	

(Weather Underground, n.d.)

Find the descriptive statistics for this data set using the TI-83/84 calculator.

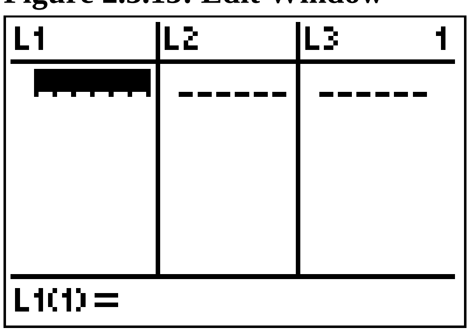
First you need to put the data into the calculator. To do this, press STAT. The STAT button is in the third row of buttons, next to the arrow keys.
Once you press STAT, you will see the following screen:

Figure 2.3.12: STAT Window



Choose 1:Edit… and you will see the following:

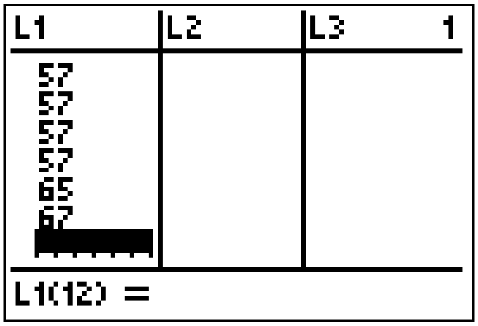
Figure 2.3.13: Edit Window



Note: If there is already data in list 1 (L1), then you should move the cursor up to L1 by using the arrow keys. Then, press clear and enter. This should clear all data from list 1 (L1).

Now type all of the data into list 1 (L1):

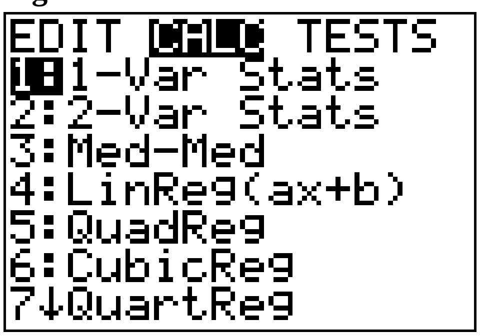
Figure 2.3.14: Data Typed Into L1



Note: Figure 2.3.14 only shows the last six data points entered, but all the data has been entered.

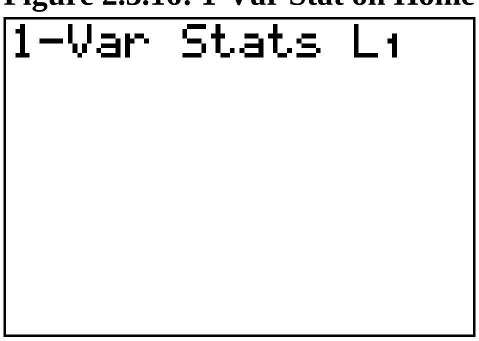
Next, press STAT again and move over to CALC using the right arrow button. You will see the following:

Figure 2.3.15: CALC Window



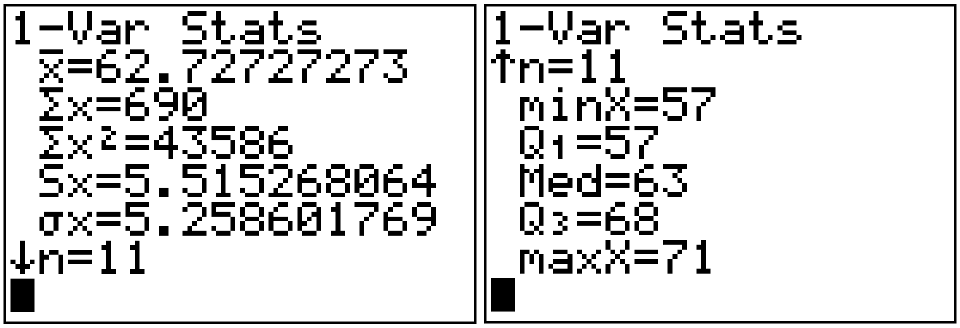
Choose 1:1-Var Stats. This will put 1-Var Stats on your home screen. Type in L1 (2nd 1), and the calculator will show the following:

Figure 2.3.16: 1-Var Stat on Home Screen



At this point press ENTER, and you will see the following: (Use the down arrow button to see the rest of the results.)

Figure 2.3.17: 1-Var Stat Results



Therefore, the mean is $\bar{x} = 62.7°F$, the standard deviation is $s = 5.515°F$, and the five-number summary is Min = 57°F, Q_1 = 57°F, Med = Q_2 = 63°F, Q_3 = 68°F, Max = 71°F. You can find the range by subtracting the max and min. You can find IQR by subtracting Q_3 and Q_1, and you can find the variance by squaring the standard deviation. You cannot find the mode from the calculator. Note that the calculator gives you the population standard deviation $\sigma = 5.259°F$. Notice it is different than the value for s, since they are calculated differently. The value the calculator gives you for the population standard deviation is not the actual true value. The calculator gives you both values because it does not know if you typed in a sample or a population. You can ignore the population standard deviation σ in almost all cases.

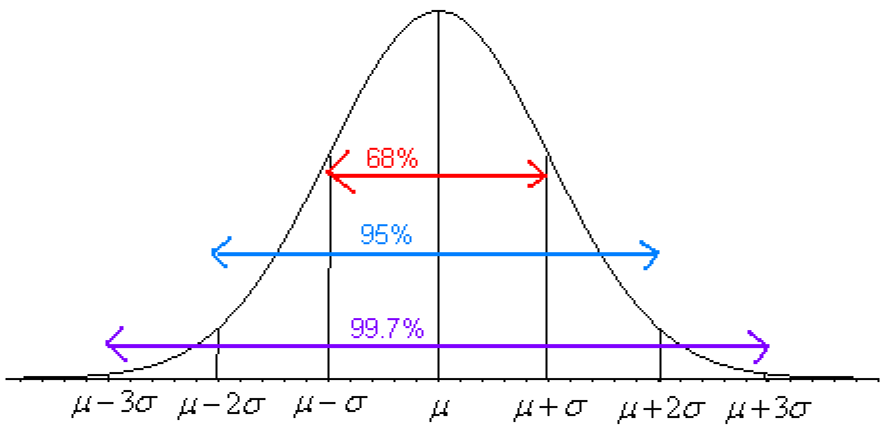
Section 2.4: The Normal Distribution

There are many different types of distributions (shapes) of quantitative data. In section 1.5 we looked at different histograms and described the shapes of them as symmetric, skewed left, and skewed right. There is a special symmetric shaped distribution called the normal distribution. It is high in the middle and then goes down quickly and equally on both ends. It looks like a bell, so sometimes it is called a bell curve. One property of the normal distribution is that it is symmetric about the mean. Another property has to do with what percentage of the data falls within certain standard deviations of the mean. This property is defined as the empirical Rule.

The Empirical Rule: Given a data set that is approximately normally distributed:
Approximately 68% of the data is within one standard deviation of the mean.
Approximately 95% of the data is within two standard deviations of the mean.
Approximately 99.7% of the data is within three standard deviations of the mean.

To visualize these percentages, see the following figure.

Figure 2.4.1: Empirical Rule



Note: The empirical rule is only true for approximately normal distributions.

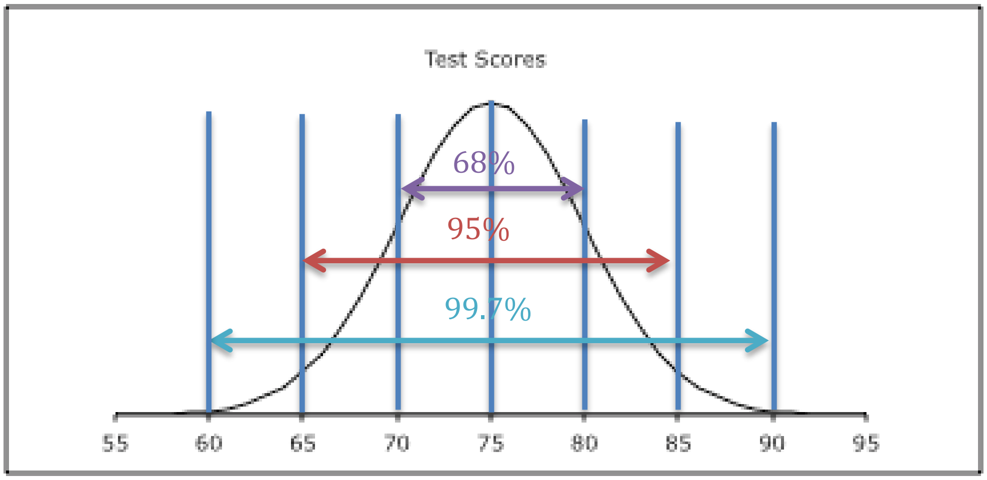
Example 2.4.1: Using the Empirical Rule

Suppose that your class took a test and the mean score was 75% and the standard deviation was 5%. If the test scores follow an approximately normal distribution, answer the following questions:

a. What percentage of the students had scores between 65 and 85?
b. What percentage of the students had scores between 65 and 75?
c. What percentage of the students had scores between 70 and 80?
d. What percentage of the students had scores above 85?

To solve each of these, it would be helpful to draw the normal curve that follows this situation. The mean is 75, so the center is 75. The standard deviation is 5, so for each line above the mean add 5 and for each line below the mean subtract 5. The graph looks like the following:

Figure 2.4.2: Empirical Rule for Example 2.4.1



a. From the graph we can see that 95% of the students had scores between 65 and 85.
b. The scores of 65 to 75 are half of the area of the graph from 65 to 85. Because of symmetry, that means that the percentage for 65 to 85 is ½ of the 95%, which is 47.5%.
c. From the graph we can see that 68% of the students had scores between 70 and 80.
d. For this problem we need a bit of math. If you looked at the entire curve, you would say that 100% of all of the test scores fall under it. So because of symmetry 50% of the test scores fall in the area above the mean and 50% of the test scores fall in the area below the mean. We know from part b that the percentage from 65 to 75 is 47.5%. Because of symmetry, the percentage from 75 to 85 is also 47.5%. So the percentage above 85 is 50% - 47.5% = 2.5%.

When we look at Example 2.4.1, we realize that the numbers on the scale are not as important as how many standard deviations a number is from the mean. As an example, the number 80 is one standard deviation from the mean. The number 65 is 2 standard deviations from the mean. However, 80 is above the mean and 65 is below the mean. Suppose we wanted to know how many standard deviations the number 82 is from the mean. How would we do that? The other numbers were easier because they were a whole number of standard deviations from the mean. We need a way to quantify this. We will use a z-score (also known as a z-value or standardized score) to measure how many standard deviations a data value is from the mean. This is defined as:

z-score: $z = \dfrac{x-\mu}{\sigma}$

where x = data value (raw score)

z = standardized value (z-score or z-value)

μ = population mean

σ = population standard deviation

Note: Remember that the z-score is always how many standard deviations a data value is from the mean of the distribution.

Suppose a data value has a z-score of 2.13. This tells us two things. First, it says that the data value is above the mean, since it is positive. Second, it tells us that you have to add more than two standard deviations to the mean to get to this value. Since most data (95%) is within two standard deviations, then anything outside this range would be considered a strange or unusual value. A z-score of 2.13 is outside this range so it is an unusual value. As another example, suppose a data value has a z-score of -1.34. This data value must be below the mean, since the z-score is negative, and you need to subtract more than one standard deviation from the mean to get to this value. Since this is within two standard deviations, it is an ordinary value.

An **unusual value** has a z-score < -2 or a z-score > 2

A **usual value** has a z-score between -2 and 2, that is $-2 < z-score < 2$.

You may encounter standardized scores on reports for standardized tests or behavior tests as mentioned previously.

Example 2.4.2: Calculating Z-Scores

Suppose that your class took a test the mean score was 75% and the standard deviation was 5%. If test scores follow an approximately normal distribution, answer the following questions:

a. If a student earned 87 on the test, what is that student's z-score and what does it mean?

$\mu = 75$, $\sigma = 5$, and $x = 87$

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma} \\
&= \frac{87-75}{5} \\
&= 2.40
\end{aligned}
$$

This means that the score of 87 is more than two standard deviations above the mean, and so it is considered to be an unusual score.

b. If a student earned 73 on the test, what is that student's z-score and what does it mean?

$\mu = 75$, $\sigma = 5$, and $x = 73$

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{73-75}{5} \\ &= -0.40 \end{aligned}$$

This means that the score of 73 is less than one-half of a standard deviation below the mean. It is considered to be a usual or ordinary score.

c. If a student earned 54 on the test, what is that student's z-score and what does it mean?

$\mu = 75$, $\sigma = 5$, and $x = 54$

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{54-75}{5} \\ &= -4.20 \end{aligned}$$

The means that the score of 54 is more than four standard deviations below the mean, and so it is considered to be an unusual score.

d. If a student has a z-score of 1.43, what actual score did she get on the test?

$\mu = 75$, $\sigma = 5$, and $z = 1.43$

This problem involves a little bit of algebra. Do not worry, it is not that hard. Since you are now looking for x instead of z, rearrange the equation solving for x as follows:

$$z = \frac{x-\mu}{\sigma}$$

$$z \bullet \sigma = \frac{x-\mu}{\cancel{\sigma}} \bullet \cancel{\sigma}$$

$$z\sigma = x - \mu$$

$$z\sigma + \mu = x - \cancel{\mu} + \cancel{\mu}$$

$$x = z\sigma + \mu$$

Now, you can use this formula to find x when you are given z.

$$x = z\sigma + \mu$$

$x = 1.43 \bullet 5 + 75$
$x = 7.15 + 75$
$x = 82.15$
Thus, the z-score of 1.43 corresponds to an actual test score of 82.15%.

e. If a student has a z-score of -2.34, what actual score did he get on the test?
$\mu = 75$, $\sigma = 5$, and $z = -2.34$
Use the formula for x from part d of this problem:
$x = z\sigma + \mu$
$x = -2.34 \bullet 5 + 75$
$x = -11.7 + 75$
$x = 63.3$
Thus, the z-score of -2.34 corresponds to an actual test score of 63.3%.

The Five-Number Summary for a Normal Distribution
Looking at the Empirical Rule, 99.7% of all of the data is within three standard deviations of the mean. This means that an approximation for the minimum value in a normal distribution is the mean minus three times the standard deviation, and for the maximum is the mean plus three times the standard deviation. In a normal distribution, the mean and median are the same. Lastly, the first quartile can be approximated by subtracting 0.67448 times the standard deviation from the mean, and the third quartile can be approximated by adding 0.67448 times the standard deviation to the mean. All of these together give the five-number summary.

In mathematical notation, the five-number summary for the normal distribution with mean μ and standard deviation σ is as follows:

Five-Number Summary for a Normal Distribution
$\text{min} = \mu - 3\sigma$
$Q_1 = \mu - 0.67448\sigma$
$\text{med} = \mu$
$Q_3 = \mu + 0.67448\sigma$
$\text{max} = \mu + 3\sigma$

Example 2.4.3: Calculating the Five-Number Summary for a Normal Distribution
Suppose that your class took a test and the mean score was 75% and the standard deviation was 5%. If the test scores follow an approximately normal distribution, find the five-number summary.

The mean is $\mu = 75\%$ and the standard deviation is $\sigma = 5\%$. Thus, the five-number summary for this problem is:

$$\min = 75 - 3(5) = 60\%$$
$$Q1 = 75 - 0.67448(5) \approx 71.6\%$$
$$\text{med} = 75\%$$
$$Q3 = 75 + 0.67448(5) \approx 78.4\%$$
$$\max = 75 + 3(5) = 90\%$$

Section 2.5: Correlation and Causation, Scatter Plots

The label on a can of Planters Cocktail Peanuts says, "Scientific evidence suggest but does not prove that eating 1.5 ounces per day of most nuts, such as peanuts, as part of a diet low in saturated fat and cholesterol & not resulting in increased caloric intake may reduce the risk of heart disease. See nutritional information for fat content (1.5 oz. is about 53 pieces)." Why is it written this way and what does this statement mean? There are many studies that exist that show that two variables are related to one another. The strength of a relationship between two variables is called **correlation**. Variables that are strongly related to each other have strong correlation. However, if two variables are correlated it does not mean that one variable caused the other variable to occur. The above example from the Planters Cocktail Peanuts label is an example of this. There is a strong correlation between eating a diet that is low in saturated fat and cholesterol and heart disease. But that correlation does not mean that eating a diet that is low in saturated fat and cholesterol will cause your risk of heart disease to go down. There could be many different variables that could cause both variables in question to go down or up. One example is that a person's genetic makeup could make them not want to eat fatty food and also not develop heart disease. No matter how strong a correlation is between two variables, you can never know for sure if one variable causes the other variable to occur without conducting experimentation. The only way to find out if eating a diet low in saturated fat and cholesterol actually lowers the risk of heart disease is to do an experiment. This is where you tell one group of people that they have to eat a diet low in saturated fat and cholesterol and another group of people that they have to eat a diet high in saturated fat and cholesterol, and then observe what happens to both groups over the years. You cannot morally do this experiment, so there is no way to prove the statement. That is why the word "may" is in the statement. We see many correlations like this one. Always be sure not to make a correlation statement into a causation statement.

Example 2.5.1: Correlation vs Causation

For each of the following scenarios answer the question and give an example of another variable that could explain the correlation.

a. There is a negative correlation between number of children a woman has and her life expectancy. Does that mean that having children causes a woman to die earlier?

   A correlation between two variables does not mean that one causes the other. A possible cause for both variables could be better health care. If there is better health care, then life expectancy goes up, and also with better health care birth control is more readily available.

b. There is a positive correlation between ice cream sales and the number of drownings at the beach. Does that mean that eating ice cream can cause a person to drown?

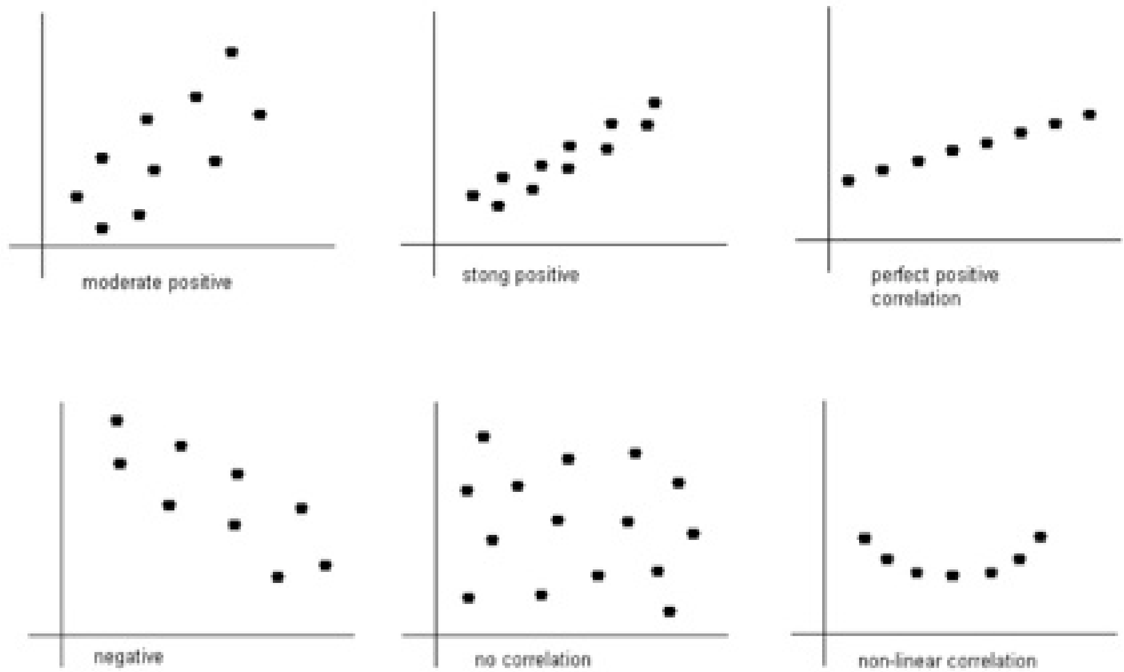
   A correlation between two variables does not mean that one causes the other. The cause for both could be that the temperature is going up. The higher the temperature, the more likely someone will buy ice cream and the more people at the beach.

c. There is a correlation between waist measures and wrist measures. Does this mean that your waist measurement causes your wrist measurement to change?

   A correlation between two variables does not mean that one causes the other. The cause of both could be a person's genetics, eating habits, exercise habits, etc.

How do we tell if there is a correlation between two variables? The easiest way is to graph the two variables together as ordered pairs on a graph called a **scatter plot**. To create a scatter plot, consider that one variable is the independent variable and the other is the dependent variable. This means that the dependent variable depends on the independent variable. We usually set up these two variables as ordered pairs where the independent variable is first and the dependent variable is second. Thus, when graphed, the independent variable is graphed along the horizontal axis and the dependent variable is graphed along the vertical axis. You do not connect the dots after plotting these ordered pairs. Instead look to see if there is a pattern, such as a line, that fits the data well. Here are some examples of scatter plots and how strong the linear correlation is between the two variables.

Figure 2.5.1: Scatter Plots Showing Types of Linear Correlation



Creating a scatter plot is not difficult. Just make sure that you set up your axes with scaling before you start to plot the ordered pairs.

Example 2.5.2: Creating a Scatter Plot

Data has been collected on the life expectancy and the fertility rate in different countries ("World health rankings," 2013). A random sample of 10 countries was taken, and the data is:

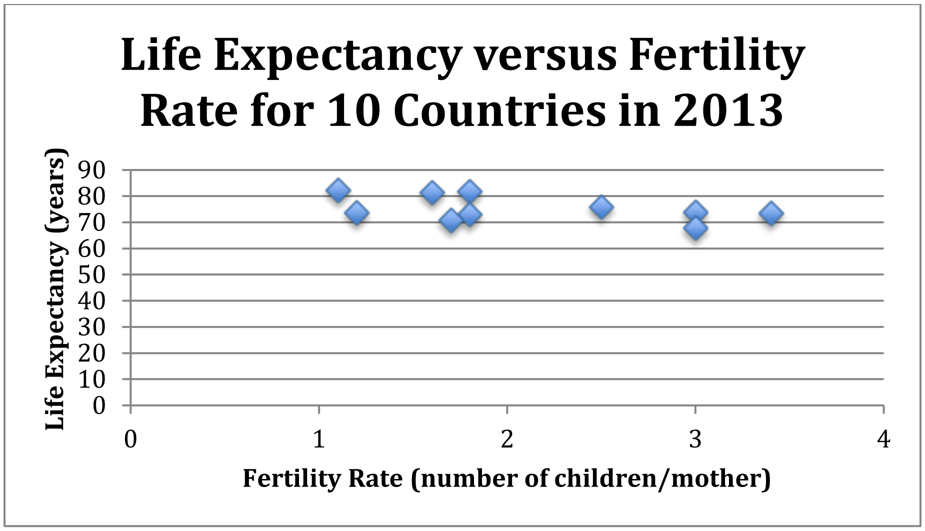
Table 2.5.2: Life Expectancy and Fertility Rate in 2013

Country	Life Expectancy (years)	Fertility Rate (number of children per mother)
SINGAPORE	82.3	1.1
MONACO	81.9	1.8
CANADA	81.5	1.6
ECUADOR	76	2.5
MALAYSIA	73.9	3
LITHUANIA	73.8	1.2
BELIZE	73.6	3.4
ALGERIA	73	1.8
TRINIDAD/TOB.	70.8	1.7
TAJIKISTAN	67.9	3

To make the scatter plot, you have to decide which variable is the independent variable and which one is the dependent variable. Sometimes it is obvious which variable is which, and in some case it does not seem to be obvious. In this case, it seems to make more sense to predict what the life expectancy is doing based on fertility rate, so choose life expectancy to be the dependent variable and fertility rate to be the independent variable. The horizontal axis needs to encompass 1.1 to 3.4, so have it range from zero to four, with tick marks every one unit. The vertical axis needs to encompass the numbers 70.8 to 81.9, so have it range from zero to 90, and have tick marks every 10 units.

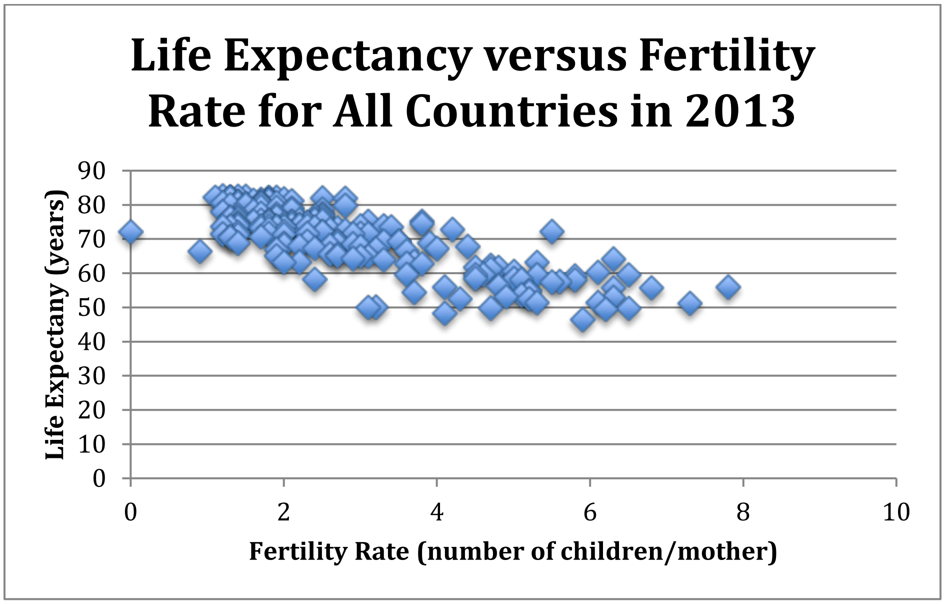
Note: Always start the vertical axis at zero to avoid exaggeration of the data.

Graph 2.5.3: Scatter Plot of Life Expectancy versus Fertility Rate



From the graph, you can see that there is somewhat of a downward trend, but it is not prominent. What this says is that as fertility rate increases, life expectancy decreases. The trend is not strong which could be due to not having enough data or this could represent the actual relationship between these two variables. Let's see what the scatter plot looks like with data from all countries in 2013 ("World health rankings," 2013).

Graph 2.5.4: Scatter Plot of Life Expectancy versus Fertility Rate for All Countries in 2013



Again, there is a downward trend. It looks a little stronger than the previous scatter plot and the trend looks more obvious. This correlation would probably be considered moderate negative correlation. It appears that there is a trend that the higher the fertility rate, the lower the life expectancy. Caution: just because there is a correlation between higher fertility rate and lower life expectancy, do not assume that having fewer children will mean that a person lives longer. The fertility rate does not necessarily cause the life expectancy to change. There are many other factors that could influence both, such as medical care and education. Remember a correlation does not imply causation.

Chapter 2 Homework

1. A study was conducted of Long Beach School District schools regarding how many require school uniforms. In 2006, of the 296 schools questioned, 184 said they required school uniforms. (Gentile & Imberman, 2009) Find the proportion of schools that require a school uniform.

2. A Center for Disease Control (CDC) study conducted in 2008, found that out of 32,601 children in Arizona, 507 had autism. (CDC, 2012) Find the proportion of children in Arizona who had autism in 2008.

3. The temperatures (in degrees Fahrenheit) for the first 10 days of July 2013 in Phoenix, AZ are given in the table below ("Weather underground," 2013).

112	108	111	108	106
111	112	113	107	104

   a. Find the mean, median, and mode for the data set.
   b. Find the range, variance, and standard deviation for the data set.
   c. Find the five-number summary and the interquartile range (IQR) for the data set.
   d. Draw a box-and-whiskers plot for the data set.

4. The number of traffic fatalities involving a driver with a blood alcohol content of 0.1 or more is given in the table for the southern states in 2013 ("Traffic fatalities by," 2013).

260	904	394	366	177	194
264	430	423	345	278	134

   a. Find the mean, median, and mode for the data set.
   b. Find the range, variance, and standard deviation for the data set.
   c. Find the five-number summary and the interquartile range (IQR) for the data set.
   d. Draw a box-and-whiskers plot for the data set.

5. A new sedan car in 2013 or 2014, which has a gas mileage in the city of 40 to 50 mpg, had the following prices ("Motor trend," 2013).

$38,700	$26,200	$39,780	$27,200	$38,700
$24,360	$39,250	$35,925	$35,555	$26,140
$24,995	$28,775	$24,200	$34,755	$25,990

   a. Find the mean, median, and mode for the data set.
   b. Find the range, variance, and standard deviation for the data set.
   c. Find the five-number summary and the interquartile range (IQR) for the data set.
   d. Draw a box-and-whiskers plot for the data set.

6. The prices of an airline flight from New York City to Los Angeles on September 7, 2013 around 8 am, and returning September 14, 2013 are given in the table below ("Expedia," 2013).

$317	$351	$378
$397	$327	$334
$337	$383	$327

   a. Find the mean, median, and mode for the data set.
   b. Find the range, variance, and standard deviation for the data set.
   c. Find the five-number summary and the interquartile range (IQR) for the data set.
   d. Draw a box-and-whiskers plot for the data set.

7. The gas prices (in $/gallon) at all gasoline stations in Flagstaff, AZ, on July 16, 2013 are given in the table below ("Arizona gas prices," 2013).

3.45	3.47	3.48	3.48	3.48	3.49
3.51	3.51	3.51	3.55	3.55	3.56
3.59	3.59	3.59	3.59	3.65	3.65
3.65	3.65	3.66	3.67	3.69	3.69
3.69	3.69	3.69	3.69	3.69	3.69
3.69					

   Using a calculator, find the mean, median, standard deviation, and five-number summary.

8. The city gas mileage (in mpg) of 2011 small pick-up trucks that are four-wheel drive are given in the table below ("Fuel efficiency guide," 2011).

17	18	17	14	16	16
14	14	15	17	18	17
14	16	16	14	14	15
14	18	18	16	14	

Using a calculator, find the mean, median, standard deviation, and five-number summary.

9. Sustainability Victoria, in Australia, surveys all Victorian local communities on waste and recycling services every year. A random sample of 10 local communities reported the number of households served in that community in 2008 and the data is given in the table below ("2001-02 to 2007-08," 2009).

7,551	4,907	45,439	46,000	46,000
49,732	38,264	39,195	40,374	40,500

a. Find the mean and median of the data set.
b. Find the mean and the median of the data set with the two lowest data values (7,551 and 4,907) removed.
c. Discuss what happened to the mean and the median when the two lowest data values (7,551 and 4,907) are removed.

10. Natural gas consumptions (in billions of cubic feet) for selected countries in South America are listed in the following table ("International energy statistics," 2013).

1629	87	885	199
312	8	202	961

a. Find the mean and median of the data set.
b. Find the mean and median with the highest data value of 1629 removed.
c. Discuss what happened to the mean and median when the highest data value (1629) is removed.

11. Suppose your child takes a test to evaluate whether or not your child is at risk for Attention Deficit Hyperactivity Disorder (ADHD). One assessment for ADHD is the Behavior Assessment System for Children, Second Edition (BASC-2) survey. After taking this survey a child is rated on several different qualities. One of the qualities is aggression, where a high score represents a tendency towards being aggressive. Suppose your child is in the 35th percentile on aggression.
a. What does this percentile mean?
b. What does this percentile mean about your child and this quality of ADHD?

12. You are planning to go to graduate school after you finish your bachelor's degree and you take the Graduate Record Examination (GRE). Your score on the mathematics section of the general GRE puts you in the 90^{th} percentile.
    a. What does this percentile mean?
    b. Did you pass (score of 70% or better) the mathematics section of the general GRE?

13. The IQ of a person follows a normal distribution and has a mean of 100 and a standard deviation of 15. Using this information, find the following:
    a. What percentage of the people have IQ scores between 85 and 115?
    b. What percentage of the people have IQ scores between 70 and 100?
    c. What percentage of the people have IQ scores between 130 and 145?
    d. What percentage of the people have IQ scores above 145?

14. The mean systolic blood pressure of people in the U.S. is 124 with a standard deviation of 16. Assume that systolic blood pressure follows a normal distribution.
    a. What percentage of the people in the U.S. have systolic blood pressure between 108 and 124?
    b. What percentage of the people in the U.S. have systolic blood pressure between 92 and 156?
    c. What percentage of the people in the U.S. have systolic blood pressure between 76 and 108?
    d. What percentage of the people in the U.S. have systolic blood pressure above 156?

15. The mean diastolic blood pressure of people in the U.S. is 77 with a standard deviation of 11. Assume that diastolic blood pressure follows a normal distribution.
    a. What percentage of the people in the U.S. have diastolic blood pressure between 77 and 88?
    b. What percentage of the people in the U.S. have diastolic blood pressure between 66 and 88?
    c. What percentage of the people in the U.S. have diastolic blood pressure between 55 and 99?
    d. What percentage of the people in the U.S. have diastolic blood pressure below 55

16. The mean height of men in the U.S. is 69.1 inches with a standard deviation of 2.9 inches. Assume that height follows a normal distribution.
    a. What percentage of the males in the U.S. have height between 74.9 and 77.8 inches?
    b. What percentage of the males in the U.S. have height between 60.4 and 77.8 inches?
    c. What percentage of the males in the U.S. have height between 63.3 and 72 inches?
    d. What percentage of the males in the U.S. have heights below 63.3 inches?

17. The IQ of a person follows a normal distribution and has a mean of 100 and a standard deviation of 15. Find the z-score for an IQ score of 134. Is this value unusual? Why or why not?

18. The mean systolic blood pressure of people in the U.S. is 124 with a standard deviation of 16. Assume that systolic blood pressure follows a normal distribution. Find the z-score for a systolic blood pressure of 135. Is this value unusual? Why or why not?

19. The mean diastolic blood pressure of people in the U.S. is 77 with a standard deviation of 11. Assume that diastolic blood pressure follows a normal distribution. Find the z-score for a diastolic blood pressure of 54. Is this value unusual? Why or why not?

20. The mean height of mean in the U.S. is 69.1 inches with a standard deviation of 2.9 inches. Assume that height follows a normal distribution. Find the z-score for a man who is 64 inches tall. Is this value unusual? Why or why not?

21. The IQ of a person follows a normal distribution and has a mean of 100 and a standard deviation of 15. Find the five-number summary.

22. The mean systolic blood pressure of people in the U.S. is 124 with a standard deviation of 16. Assume that systolic blood pressure follows a normal distribution. Find the five-number summary.

23. The mean diastolic blood pressure of people in the U.S. is 77 with a standard deviation of 11. Assume that diastolic blood pressure follows a normal distribution. Find the five-number summary.

24. The mean height of mean in the U.S. is 69.1 inches with a standard deviation of 2.9 inches. Assume that height follows a normal distribution. Find the five-number summary.

25. It can be shown that a man's height and a man's weight have a positive correlation. Does this mean that a man's height causes him to be a certain weight? Explain.

26. Engine size and city gas mileage have a negative correlation. Does this mean that the engine size causes the gas mileage of a car? Explain.

27. Suppose 10 men had their height and weight measured. The data is below. Draw a scatter plot of the data. Describe what relationship you can see from the graph.

Height (inches)	67	72	74	65	70	72	74	69	68	70
Weight (pounds)	185	202	226	165	221	217	218	189	201	185

28. Nine midsize 2011 hybrid cars' city gas mileage and engine size are recorded below ("Fuel efficiency guide," 2011). Draw a scatter plot of the data. Describe what relationship you can see from the graph.

Engine Size	4.4	2.5	2.4	5.0	2.5	2.5	2.5	2.4	1.8
City MPG	17	41	25	19	41	41	33	31	51

Chapter 3: Probability

We see probabilities almost every day in our real lives. Most times you pick up the newspaper or read the news on the internet, you encounter probability. There is a 65% chance of rain today, or a pre-election poll shows that 52% of voters approve of a ballot

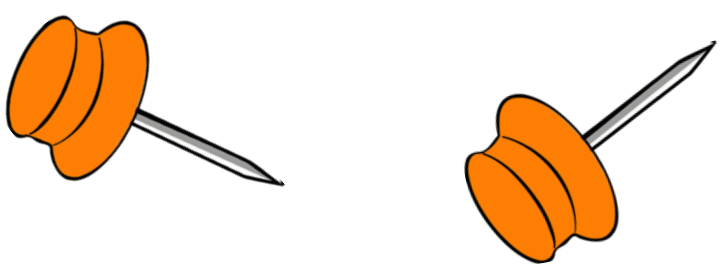
of the ideas students think they know about probability are incorrect. This is one area of item, are examples of probabilities. Did you ever wonder why a flush beats a full house in poker? It's because the probability of getting a flush is smaller than the probability of getting a full house. Probabilities can also be used to make business decisions, figure out insurance premiums, and determine the price of raffle tickets.

If an experiment has only three possible outcomes, does this mean that each outcome has a 1/3 chance of occurring? Many students who have not studied probability would answer yes. Unfortunately, they could be wrong. The answer depends on the experiment. Many math where their intuition is sometimes misleading. Students need to use experiments or mathematical formulas to calculate probabilities correctly.

Section 3.1: Basic Probabilities and Probability Distributions; Three Ways to Define Probabilities

Toss a thumb tack one time. Do you think the tack will land with the point up or the point down?

Figure 3.1.1: Which Way Will a Tack Fall?



(Thumbtack, n.d.)

We cannot predict which way the tack will land before we toss it. Sometimes it will land with the point up and other times it will land with the point down. Tossing a tack is a random experiment since we cannot predict what the outcome will be. We do know that there are only two possible outcomes for each trial of the experiment: lands points up or lands point down. If we repeat the experiment of tossing the tack many times we might be able to guess how likely it is that the tack will land point up.

A **random experiment** is an activity or an observation whose outcome cannot be predicted ahead of time

A **trial** is one repetition of a random experiment.

The **sample space** is the set of all possible outcomes for a random experiment.

An **event** is a subset of the sample space.

Do you think chances of the tack landing point up and the tack landing point down are the same? This is an example where your intuition may be wrong. Having only two possible outcomes does not mean each outcome has a 50/50 chance of happening. In fact we are going to see that the probability of the tack landing point up is about 66%.

To begin to answer this question, toss a tack 10 times. For each toss record whether the tack lands point up or point down.

Table 3.1.2: Toss a Tack Ten Times

Trial	1	2	3	4	5	6	7	8	9	10
Up/Down	Up	Down	Up	Up	Down	Up	Up	Down	Up	Down

The random experiment here is tossing the tack one time. The possible outcomes for the experiment are that the tack lands point up or the tack lands point down so the sample space is S = {point up, point down}. We are interested in the event E that the tack lands point up, E = {point up}.

Based on our data we would say that the tack landed point up six out of 10 times. The fraction, $\frac{6}{10}$, is called the relative frequency. Since $\frac{6}{10} = 0.60$ we would guess that probability of the tack landing point up is about 60%.

Let's repeat the experiment by tossing the tack ten more times.

Table 3.1.3: Toss a Tack Ten More Times

Trial	1	2	3	4	5	6	7	8	9	10
Up/Down	Up	Up	Up	Down	Down	Up	Up	Down	Up	Up

This time the tack landed point up seven out of 10 times or 70% of the time. If we tossed the tack another 10 times we might get a different result again. The probability of the tack landing point up refers to what happens when we toss the tack many, many times. Let's toss the tack 150 times and count the number of times it lands point up. Along the way we will look at the proportion of landing point up.

Table 3.1.4: Toss a Tack Many Times

Trials	Number Up	Total Number Of Up	Total Number Of Trials	Proportion
0-10	6	6	10	6/10=0.60
11-20	7	13	20	13/20=0.65
21-30	8	21	30	21/30=0.70
31-40	6	27	40	27/40=0.68
41-50	8	35	50	35/50=0.70
51-60	8	43	60	43/60=0.72
61-70	4	47	70	47/70=0.67
71-80	7	54	80	54/80=0.68
81-90	5	59	90	59/90=0.66
91-100	6	65	100	65/100=0.65
101-110	10	75	110	75/110=0.68
111-120	5	80	120	80/120=0.67
121-130	8	88	130	88/130=0.68
131-140	4	92	140	92/140=0.66
141-150	7	99	150	99/150=0.66

When we have a small number of trials the proportion varies quite a bit. As we start to have more trials the proportion still varies but not by as much. It appears that the proportion is around 0.66 or 66%. We would have to do about 100,000 trials to get a better approximation of the actual probability of the tack landing point up.

The tack landed point up 99 out of 150 trials. The probability P of event E is written as:

$$P(E) = \frac{\text{\# of trials with point up}}{\text{total number of trials}} = \frac{99}{150} \approx 0.66$$

We would say the probability that the tack lands point up is about 66%.

Equally Likely Outcomes:

In some experiments all the outcomes have the same chance of happening. If we roll a fair die the chances are the same for rolling a two or rolling a five. If we draw a single card from a well shuffled deck of cards, each card has the same chance of being selected. We call outcomes like these equally likely. Drawing names from a hat or drawing straws are other examples of equally likely outcomes. The tack tossing example did not have equally likely outcomes since the probability of the tack landing point up is different than the probability of the tack landing point down.

An experiment has **equally likely outcomes** if every outcome has the same probability of occurring.

For equally likely outcomes, the **probability of outcome A**, *P(A)*, is:

$$P(A) = \frac{\text{number of ways for A to occur}}{\text{total number of outcomes}}.$$

Round Off Rule: Give probabilities as a fraction or as a decimal number rounded to three decimal places.

Figure 3.1.5: Deck of Cards

(Pine, 2007)

Example 3.1.1: Simple Probabilities with Cards

Draw a single card from a well shuffled deck of 52 cards. Each card has the same chance of being drawn so we have equally likely outcomes. Find the following probabilities:

a. *P*(card is red)

$$P(\text{card is red}) = \frac{\text{number of red cards}}{\text{total number of cards}} = \frac{26}{52} = \frac{1}{2}$$

The probability that the card is red is $\frac{1}{2}$.

b. P(card is a heart)

$$P(\text{card is a heart}) = \frac{\text{number of hearts}}{\text{total number of cards}} = \frac{13}{52} = \frac{1}{4}$$

The probability that the card is a heart is $\frac{1}{4}$.

c. P(card is a red 5)

$$P(\text{card is a red 5}) = \frac{\text{number of red fives}}{\text{total number of cards}} = \frac{2}{52} = \frac{1}{26}$$

The probability that the card is a red five is $\frac{1}{26}$.

Example 3.1.2: Simple Probabilities with a Fair Die

Roll a fair die one time. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Find the following probabilities.

a. P(roll a four)

$$P(\text{roll a four}) = \frac{\text{number of ways to roll a four}}{\text{total number of ways to roll a die}} = \frac{1}{6}$$

The probability of rolling a four is $\frac{1}{6}$.

b. P(roll an odd number)

The event roll an odd number is E = {1, 3, 5}.

$$P(\text{roll an odd number}) = \frac{\text{number of ways to roll an odd number}}{\text{total number of ways to roll a die}} = \frac{3}{6} = \frac{1}{2}$$

The probability of rolling an odd number is $\frac{1}{2}$.

c. P(roll a number less than five)

The event roll a number less than five is F = {1, 2, 3, 4}.

$$P(\text{roll a number less than five}) = \frac{\text{number of ways to roll number less than five}}{\text{total number of ways to roll a die}} = \frac{4}{6} = \frac{2}{3}$$

The probability of rolling a number less than five is $\frac{2}{3}$.

Example 3.1.3: Simple Probability with Books

Figure 3.1.6: Books on a Shelf

(Bookshelf, 2011)

A small bookcase contains five math, three English and seven science books. A book is chosen at random. What is the probability that a math book is chosen.

Since the book is chosen at random each book has the same chance of being chosen and we have equally likely events.

$$P(\text{math book}) = \frac{\text{number of ways to choose a math book}}{\text{total number of books}} = \frac{5}{15} = \frac{1}{3}$$

The probability a math book was chosen is $\frac{1}{3}$.

Three Ways of Finding Probabilities:

There are three ways to find probabilities. In the tack tossing example we calculated the probability of the tack landing point up by doing an experiment and recording the outcomes. This was an example of an empirical probability. The probability of getting a red jack in a card game or rolling a five with a fair die can be calculated from mathematical formulas. These are examples of theoretical probabilities. The third type of probability is a subjective probability. Saying that there is an 80% chance that you will go to the beach this weekend is a subjective probability. It is based on experience or guessing.

A **theoretical probability** is based on a mathematical model where all outcomes are equally likely to occur.

An **empirical probability** is based on an experiment or observation and is the relative frequency of the event occurring.

A **subjective probability** is an estimate (a guess) based on experience or intuition.

Complements:

If there is a 75% chance of rain today, what are the chances it will not rain? We know that there are only two possibilities. It will either rain or it will not rain. Because the sum of the probabilities for all the outcomes in the sample space must be 100% or 1.00, we know that

P(will rain) + P(will not rain) = 100%.

Rearranging this we see that

P(will not rain) = 100% - P(will rain) = 100% - 75% = 25%.

The events E = {will rain} and F = {will not rain} are called complements.

The **complement** of event E, denoted by $\overline{E}$, is the set of outcomes in the sample space that are not in the event E. The probability of $\overline{E}$ is given by $P(\overline{E}) = 1 - P(E)$.

Example 3.1.4: Complements with Cards

Draw a single card from a well shuffled deck of 52 cards.

a. Look at the suit of the card. Here the sample space S = {spades, clubs, hearts, diamonds}. If event E = {spades} the complement $\overline{E}$ = {clubs, hearts, diamonds}.

b. Look at the value of the cards. Here the sample space is S = {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}. If the event E = {the number is less than 7} = {A, 2, 3, 4, 5, 6} the complement $\overline{E}$ = {7, 8, 9, 10, J, Q, K}.

Example 3.1.5: Complements with Trains

A train arrives on time at a particular station 85% of the time. Does this mean that the train is late 15% of the time? The answer is no. The complement of E = {on time} is not $\overline{E}$ = {late}. There is a third possibility. The train could be early. The sample space is S = {on time, early, late} so the complement of E = {on time} is $\overline{E}$ = {early or late}. Based on the given information we cannot find P(late) but we can find P(early or late) = 15%.

Impossible Events and Certain Events:

Recall that $P(A)=\dfrac{\text{number of ways for A to occur}}{\text{total number of outcomes}}$. What does it mean if we say the probability of the event is zero? $P(A)=\dfrac{\text{number of ways for A to occur}}{\text{total number of outcomes}}=0$. The only way for a fraction to equal zero is when the numerator is zero. This means there is no way for event A to occur. A probability of zero means that the event is impossible.

What does it mean if we say the probability of an event is one?
$P(A)=\dfrac{\text{number of ways for A to occur}}{\text{total number of outcomes}}=1$. The only way for a fraction to equal one is if the numerator and denominator are the same. The number of ways for A to occur is the same as the number of outcomes. There are no outcomes where A does not occur. A probability of 1 means that the event always happens.

$$\boxed{\begin{array}{l} P(A)=0 \text{ means that } A \text{ is impossible} \\ P(A)=1 \text{ means that } A \text{ is certain} \end{array}}$$

Probability Distributions:

A probability distribution (probability space) is a sample space paired with the probabilities for each outcome in the sample space. If we toss a fair coin and see which side lands up, there are two outcomes, heads and tails. Since the coin is fair these are equally likely outcomes and have the same probabilities. The probability distribution would be P(heads) = 1/2 and P(tails) = 1/2. This is often written in table form:

Table 3.1.7: Probability Distribution for a Fair Coin

Outcome	Heads	Tails
Probability	1/2	1/2

A **probability distribution** for an experiment is a list of all the possible outcomes and their corresponding probabilities.

Example 3.1.6: Probabilities for the Sum of Two Fair Dice

In probability problems when we roll two dice, it is helpful to think of the dice as being different colors. Let's assume that one die is red and the other die is green. We consider getting a three on the red die and a five on the green die different than getting a five on the red die and a three on the green die. In other words, when we list the outcomes the order matters. The possible outcomes of rolling two dice and looking at the sum are given in Table 3.1.8.

Table 3.1.8: All Possible Sums of Two Dice

1+1 = 2	1+2 = 3	1+3 = 4	1+4 = 5	1+5 = 6	1+6 = 7
2+1 = 3	2+2 = 4	2+3 = 5	2+4 = 6	2+5 = 7	2+6 = 8
3+1 = 4	3+2 = 5	3+3 = 6	3+4 = 7	3+5 = 8	3+6 = 9
4+1 = 5	4+2 = 6	4+3 = 7	4+4 = 8	4+5 = 9	4+6 = 10
5+1 = 6	5+2 = 7	5+3 = 8	5+4 = 9	5+5 = 10	5+6 = 11
6+1 = 7	6+2 = 8	6+3 = 9	6+4 = 10	6+5 = 11	6+6 = 12

Table 3.1.9: Probability Distribution for the Sum of Two Fair Dice

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
Reduced Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Example 3.1.7: Valid and Invalid Probability Distributions

Are the following valid probability distributions?

a. **Table 3.1.10:**

Outcome	A	B	C	D	E
Probability	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

This is a valid probability distribution. All the probabilities are between zero and one inclusive and the sum of the probabilities is 1.00.

b. **Table 3.1.11:**

Outcome	A	B	C	D	E	F
Probability	0.45	0.80	-0.20	-0.35	0.10	0.20

This is not a valid probability distribution. The sum of the probabilities is 1.00, but some of the probabilities are not between zero and one, inclusive.

c. **Table 3.1.12:**

Outcome	A	B	C	D
Probability	0.30	0.20	0.40	0.25

This is not a valid probability distribution. All of the probabilities are between zero and one, inclusive, but the sum of the probabilities is 1.15 not 1.00.

Odds:

Probabilities are always numbers between zero and one. Many people are not comfortable working with such small values. Another way of describing the likelihood of an event happening is to use the ratio of how often it happens to how often it does not happen. The ratio is called the odds of the event happening. There are two types of odds, odds for and odds against. Casinos, race tracks and other types of gambling usually state the odds against an event happening.

If the probability of an event *E* is *P(E)*, then the **odds for event *E***, *O(E)*, are given by:

$$O(E)=\frac{P(E)}{P(\overline{E})} \qquad \text{OR} \qquad O(E)=\frac{\text{number of ways for E to occur}}{\text{number of ways for E to not occur}}$$

Also, the **odds against event *E***, are given by:

$$O(\overline{E})=\frac{P(\overline{E})}{P(E)} \qquad \text{OR} \qquad O(E)=\frac{\text{number of ways for E to not occur}}{\text{number of ways for E to occur}}$$

Example 3.1.8: Odds in Drawing a Card

A single card is drawn from a well shuffled deck of 52 cards. Find the odds that the card is a red eight.

There are two red eights in the deck.

$$P(\text{red eight})=\frac{2}{52}=\frac{1}{26}$$

$$P(\text{not a red eight})=\frac{50}{52}=\frac{25}{26}.$$

$$O(\text{red eight}) = \frac{P(\text{red eight})}{P(\text{not a red eight})} = \frac{\frac{1}{26}}{\frac{25}{26}} = \frac{1}{\cancel{26}} \cdot \frac{\cancel{26}}{25} = \frac{1}{25}.$$

The odds of drawing a red eight are 1 to 25. This can also be written as 1:25.

Note: Do not write odds as a decimal or a percent.

Example 3.1.9: Odds in Roulette

Many roulette wheels have slots numbered 0, 00, and 1 through 36. The slots numbered 0 and 00 are green. The even numbered slots are red and the odd numbered slots are black. The game is played by spinning the wheel one direction and rolling a marble around the outer edge the other direction. Players bet on which slot the marble will fall into. What are the odds the marble will land in a red slot?

There are 38 slots in all. The slots 2, 4, 6, …, 36 are red so there are 18 red slots. The other 20 slots are not red.

$$P(\text{red}) = \frac{18}{38} = \frac{9}{19}$$

$$P(\text{not red}) = 1 - \frac{9}{19} = \frac{19}{19} - \frac{9}{19} = \frac{10}{19}$$

$$O(\text{red}) = \frac{P(\text{red})}{P(\text{not red})} = \frac{\frac{9}{19}}{\frac{10}{19}} = \frac{9}{\cancel{19}} \cdot \frac{\cancel{19}}{10} = \frac{9}{10}$$

The odds of the marble landing in a red slot are 9 to 10. This can also be written as 9:10.

Example 3.1.10: Odds Against an Event

Two fair dice are tossed and the sum is recorded. Find the odds against rolling a sum of nine.

The event *E*, roll a sum of nine is: *E* = {(3, 6), (4, 5), (5, 4), (6, 3)}

There are 36 ways to roll two dice and four ways to roll a sum of nine. That means there are 32 ways to roll a sum that is not nine.

$$P(\text{sum is nine}) = \frac{4}{36} = \frac{1}{9}$$

$$P(\text{sum is not nine}) = \frac{32}{36} = \frac{8}{9}$$

$$O(\text{against sum is nine}) = \frac{P(\text{sum is not nine})}{P(\text{sum is nine})} = \frac{\frac{8}{9}}{\frac{1}{9}} = \frac{8}{\cancel{9}} \cdot \frac{\cancel{9}}{1} = \frac{8}{1}$$

The odds against rolling a sum of nine are 8 to 1 or 8:1.

We can also find the probability of an event happening based on the odds for the event. Saying that the odds of an event are 3 to 5 means that the event happens three times for every five times it does not happen. If we add up the possibilities of both we get a sum of eight. So the event happens about three out of every eight times. We would say the probability is 3/8.

If the odds favoring event E are a to b, then: $P(E) = \frac{a}{a+b}$ and $P(\bar{E}) = \frac{b}{a+b}$.

Example 3.1.11: Finding the Probability from the Odds

A local little league baseball team is going to a tournament. The odds of the team winning the tournament are 3 to 7. Find the probability of the team winning the tournament.

$$P(\text{winning}) = \frac{3}{3+7} = \frac{3}{10} = 0.3$$

Section 3.2: Combining Probabilities with “And” and “Or”

Many probabilities in real life involve more than one outcome. If we draw a single card from a deck we might want to know the probability that it is either red or a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair and blue eyes. When we combine two outcomes to make a single event we connect the outcomes with the word “and” or the word “or.” It is very important in probability to pay attention to the words “and” and “or” if they appear in a problem. The word “and” restricts the field of possible outcomes to only those outcomes that simultaneously satisfy more than one event. The word "or" broadens the field of possible outcomes to those that satisfy one or more events.

Example 3.2.1: Counting Students

Figure 3.2.1: College Classroom

(Colwell, 2013)

Suppose a teacher wants to know the probability that a single student in her class of 30 students is taking either Art or English. She asks the class to raise their hands if they are taking Art and counts 13 hands. Then she asks the class to raise their hands if they are taking English and counts 21 hands. The teacher then calculates

$$P(\text{Art or English}) = \frac{13+21}{30} = \frac{33}{30}.$$

The teacher knows that this is wrong because probabilities must be between zero and one, inclusive. After thinking about it she remembers that nine students are taking both Art and English. These students raised their hands each time she counted, so the teacher counted them twice. When we calculate probabilities we have to be careful to count each outcome only once.

Mutually Exclusive Events:

An experiment consists of drawing one card from a well shuffled deck of 52 cards. Consider the events *E*: the card is red, *F*: the card is a five, and *G*: the card is a spade. It is possible for a card to be both red and a five at the same time but it is not possible for a card to be both red and a spade at the same time. It would be easy to accidentally count a red five twice by mistake. It is not possible to count a red spade twice.

Two events are **mutually exclusive** if they have no outcomes in common.

Example 3.2.2: Mutually Exclusive with Dice

Two fair dice are tossed and different events are recorded. Let the events *E*, *F* and *G* be as follows:

E = {the sum is five} = {(1, 4), (2, 3), (3, 2), (4, 1)}

F = {both numbers are even} = {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}

G = {both numbers are less than five} = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4,1), (4, 2), (4, 3), (4,4)}

a. Are events *E* and *F* mutually exclusive?

   Yes. *E* and *F* are mutually exclusive because they have no outcomes in common. It is not possible to add two even numbers to get a sum of five.

b. Are events *E* and *G* mutually exclusive?

   No. *E* and *G* are not mutually exclusive because they have some outcomes in common. The pairs (1, 4), (2, 3), (3, 2) and (4, 1) all have sums of 5 and both numbers are less than five.

c. Are events *F* and *G* mutually exclusive?

   No. *F* and *G* are not mutually exclusive because they have some outcomes in common. The pairs (2, 2), (2, 4), (4, 2) and (4, 4) all have two even numbers that are less than five.

Addition Rule for "Or" Probabilities:

The addition rule for probabilities is used when the events are connected by the word "or". Remember our teacher in Example 3.2.1 at the beginning of the section? She wanted to know the probability that her students were taking either art or English. Her problem was that she counted some students twice. She needed to add the number of students taking art to the number of students taking English and then subtract the number of students she counted twice. After dividing the result by the total number of students she will find the desired probability. The calculation is as follows:

$$\begin{aligned} P(\text{art or English}) &= \frac{\text{\# taking art + \# taking English - \# taking both}}{\text{total number of students}} \\ &= \frac{13+21-9}{30} \\ &= \frac{25}{30} \approx 0.833 \end{aligned}$$

The probability that a student is taking art or English is 0.833 or 83.3%.

When we calculate the probability for compound events connected by the word “or” we need to be careful not to count the same thing twice. If we want the probability of drawing a red card or a five we cannot count the red fives twice. If we want the probability a person is blonde-haired or blue-eyed we cannot count the blue-eyed blondes twice. The addition rule for probabilities adds the number of blonde-haired people to the number of blue-eyed people then subtracts the number of people we counted twice.

Addition Rule for “Or” Probabilities
If A and B are any events then, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
If A and B are mutually exclusive events then $P(A \text{ and } B) = 0$, so then
$P(A \text{ or } B) = P(A) + P(B)$.

Example 3.2.3: Additional Rule for Drawing Cards

A single card is drawn from a well shuffled deck of 52 cards. Find the probability that the card is a club or a face card.

There are 13 cards that are clubs, 12 face cards (J, Q, K in each suit) and 3 face cards that are clubs.

$$\begin{aligned} P(\text{club or face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club and face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \approx 0.423 \end{aligned}$$

The probability that the card is a club or a face card is approximately 0.423 or 42.3%.

Example 3.2.4: Addition Rule for Tossing a Coin and Rolling a Die

An experiment consists of tossing a coin then rolling a die. Find the probability that the coin lands heads up or the number is five.

Let H represent heads up and T represent tails up. The sample space for this experiment is S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

There are six ways the coin can land heads up, {H1, H2, H3, H4, H5, H6}.

There are two ways the die can land on five, {H5, T5}.

There is one way for the coin to land heads up and the die to land on five, {H5}.

$$\begin{aligned} P(\text{heads or five}) &= P(\text{heads}) + P(\text{five}) - P(\text{both heads and five}) \\ &= \frac{6}{12} + \frac{2}{12} - \frac{1}{12} \\ &= \frac{7}{12} \approx 0.583 \end{aligned}$$

The probability that the coin lands heads up or the number is five is approximately 0.583 or 58.3%.

Example 3.2.5: Addition Rule for Satisfaction of Car Buyers

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table.

Table 3.2.2: Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

Find the probability that a person bought a new car or was not satisfied.

$$\begin{aligned} P(\text{new car or not satisfied}) &= P(\text{new car}) + P(\text{not satisfied}) - P(\text{new car and not satisfied}) \\ &= \frac{120}{250} + \frac{75}{250} - \frac{28}{250} = \frac{167}{250} \approx 0.668 \end{aligned}$$

The probability that a person bought a new car or was not satisfied is approximately 0.668 or 66.8%.

Independent Events:

Sometimes we need to calculate probabilities for compound events that are connected by the word "and." We have two methods to choose from, independent events or conditional probabilities (Section 3.3). Tossing a coin multiple times or rolling dice are independent events. Each time you toss a fair coin the probability of getting heads is ½. It does not matter what happened the last time you tossed the coin. It's similar for dice. If you rolled double sixes last time that does not change the probability that you will roll double sixes this time. Drawing two cards without replacement is not an independent event. When you draw the first card and set it aside, the probability for the second card is now out of 51 cards not 52 cards.

> Two events are **independent events** if the occurrence of one event has no effect on the probability of the occurrence of the other event.

> **Multiplication Rule for "And" Probabilities: Independent Events**
> If events A and B are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Example 3.2.6: Independent Events for Tossing Coins

Suppose a fair coin is tossed four times. What is the probability that all four tosses land heads up?

The tosses of the coins are independent events. Knowing a head was tossed on the first trial does not change the probability of tossing a head on the second trial.

$$\begin{aligned} P(\text{four heads in a row}) &= P(\text{1st heads and 2nd heads and 3rd heads and 4th heads}) \\ &= P(\text{1st heads}) \cdot P(\text{2nd heads}) \cdot P(\text{3rd heads}) \cdot P(\text{4th heads}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{16} \end{aligned}$$

The probability that all four tosses land heads up is $\frac{1}{16}$.

Example 3.2.7: Independent Events for Drawing Marbles

A bag contains five red and four white marbles. A marble is drawn from the bag, its color recorded and the marble is returned to the bag. A second marble is then drawn. What is the probability that the first marble is red and the second marble is white?

Since the first marble is put back in the bag before the second marble is drawn these are independent events.

$$\begin{aligned} P(\text{1st red and 2nd white}) &= P(\text{1st red}) \cdot P(\text{2nd white}) \\ &= \frac{5}{9} \cdot \frac{4}{9} = \frac{20}{81} \end{aligned}$$

The probability that the first marble is red and the second marble is white is $\frac{20}{81}$.

Example 3.2.8: Independent Events for Faulty Alarm Clocks

Abby has an important meeting in the morning. She sets three battery-powered alarm clocks just to be safe. If each alarm clock has a 0.03 probability of malfunctioning, what is the probability that all three alarm clocks fail at the same time?

Since the clocks are battery powered we can assume that one failing will have no effect on the operation of the other two clocks. The functioning of the clocks is independent.

$$\begin{aligned} P(\text{all three fail}) &= P(\text{first fails}) \cdot P(\text{second fails}) \cdot P(\text{third fails}) \\ &= (0.03)(0.03)(0.03) \\ &= 2.7x10^{-5} \end{aligned}$$

The probability that all three clocks will fail is approximately 0.000027 or 0.0027%. It is very unlikely that all three alarm clocks will fail.

At Least Once Rule for Independent Events:

Many times we need to calculate the probability that an event will happen at least once in many trials. The calculation can get quite complicated if there are more than a couple of trials. Using the complement to calculate the probability can simplify the problem considerably. The following example will help you understand the formula.

Example 3.2.9: At Least Once Rule

The probability that a child forgets her homework on a given day is 0.15. What is the probability that she will forget her homework at least once in the next five days?

Assume that whether she forgets or not one day has no effect on whether she forgets or not the second day.

If P(forgets) = 0.15, then P(not forgets) = 0.85.

$$\begin{aligned}
&P(\text{forgets at least once in 5 tries})\\
&= P(\text{forgets 1, 2, 3, 4 or 5 times in 5 tries})\\
&= 1 - P(\text{forgets 0 times in 5 tries})\\
&= 1 - P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget})\\
&= 1 - (0.85)(0.85)(0.85)(0.85)(0.85)\\
&= 1 - (0.85)^5 = 0.556
\end{aligned}$$

The probability that the child will forget her homework at least one day in the next five days is 0.556 or 55.6%

The idea in Example 3.2.9 can be generalized to get the At Least Once Rule.

At Least Once Rule
If an experiment is repeated n times, the n trials are independent and the probability of event A occurring one time is P(A) then the probability that A occurs at least one time is: $P(A \text{ occurs at least once in n trials}) = 1 - P(\overline{A})^n$

Example 3.2.10: At Least Once Rule for Bird Watching

The probability of seeing a falcon near the lake during a day of bird watching is 0.21. What is the probability that a birdwatcher will see a falcon at least once in eight trips to the lake?

Let A be the event that he sees a falcon so P(A) = 0.21. Then,
$P(\overline{A}) = 1 - 0.21 = 0.79$.

$$\begin{aligned}
P(\text{at least once in eight tries}) &= 1 - P(\overline{A})^8\\
&= 1 - (0.79)^8\\
&= 1 - 0.152 = 0.848
\end{aligned}$$

The probability of seeing a falcon at least once in eight trips to the lake is approximately 0.848 or 84.8%.

Example 3.2.11: At Least Once Rule for Guessing on Multiple Choice Tests

A multiple choice test consists of six questions. Each question has four choices for answers, only one of which is correct. A student guesses on all six questions. What is the probability that he gets at least one answer correct?

Let A be the event that the answer to a question is correct. Since each question has four choices and only one correct choice, $P(\text{correct}) = \frac{1}{4}$.

That means $P(\text{not correct}) = 1 - \frac{1}{4} = \frac{3}{4}$.

$$\begin{aligned} P(\text{at least one correct in six trials}) &= 1 - P(\text{not correct})^6 \\ &= 1 - \left(\frac{3}{4}\right)^6 \\ &= 1 - 0.178 = 0.822 \end{aligned}$$

The probability that he gets at least one answer correct is 0.822 or 82.2%.

"And" Probabilities from Two-Way Tables:

"And" probabilities are usually done by one of two methods. If you know the events are independent you can use the rule $P(A \text{ and } B) = P(A) \cdot P(B)$. If the events are not independent you can use the conditional probabilities in Section 3.3. There is an exception when we have data given in a two-way table. We can calculate "and" probabilities without knowing if the events are independent or not.

Example 3.2.12: "And" Probability from a Two-Way Table

Continuation of Example 3.2.5:

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table.

Table 3.2.2: Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

A person is chosen at random. Find the probability that the person:

a. bought a new car and was satisfied.

$$P(\text{new car and satisfied}) = \frac{\text{number of new car and satisfied}}{\text{number of people}} = \frac{92}{250} = 0.368 = 36.8\%$$

b. bought a used car and was not satisfied.

$$P(\text{used car and not satisfied}) = \frac{\text{number of used and not satisfied}}{\text{number of people}} = \frac{47}{250} = 0.188 = 18.8\%$$

Section 3.3: Conditional Probabilities

What do you think the probability is that a man is over six feet tall? If you knew that both his parents were tall would you change your estimate of the probability? A conditional probability is a probability that is based on some prior knowledge.

A **conditional probability** is the probability that an event will occur if some other condition has already occurred. This is denoted by $P(A \mid B)$, which is read "the probability of *A* given *B*."

Example 3.3.1: Conditional Probability for Drawing Cards without Replacement

Two cards are drawn from a well shuffled deck of 52 cards without replacement. Find the following probabilities.

a. The probability that the second card is a heart given that the first card is a spade.

Without replacement means that the first card is set aside before the second card is drawn and we assume the first card is a spade. There are only 51 cards to choose from for the second card. Thirteen of those cards are hearts.

It's important to notice that the question only asks about the second card.

$$P(\text{2nd heart} \mid \text{1st spade}) = \frac{13}{51}$$

The probability that the second card is a heart given that the first card is a spade is $\frac{13}{51}$.

b. The probability that the first card is a face card and the second card an ace.

Notice that this time the question asks about both of the cards.

There are 12 face cards out of 52 cards when we draw the first card. We set the first card aside and assume that it is a face card. Then there are four aces out of the 51 remaining cards. We want to draw a face card <u>and</u> an ace so use multiplication.

$$P(\text{1st face card and 2nd ace}) = \frac{12}{52} \cdot \frac{4}{51} = \frac{48}{2652} \approx 0.018$$

The probability that the first card is a face card and the second card an ace is approximately 0.018 or 1.8%.

c. The probability that one card is a heart and the other a club.

There are two ways for this to happen. We could get a heart first and a club second or we could get the club first and the heart second.

$$\begin{aligned} P(\text{heart and club}) &= P(\text{heart 1st and club 2nd or club 1st and heart 2nd}) \\ &= P(\text{heart 1st and club 2nd}) + P(\text{club 1st and heart 2nd}) \\ &= \frac{13}{52} \cdot \frac{13}{51} + \frac{13}{52} \cdot \frac{13}{51} \\ &\approx 0.127 \end{aligned}$$

The probability that one card is a heart and the other a club is approximately 0.127 or 12.7%.

Example 3.3.2: Conditional Probability for Rolling Dice

Two fair dice are rolled and the sum of the numbers is observed. What is the probability that the sum is at least nine if it is known that one of the dice shows a five?

Since we are given that one of the dice shows a five this is a conditional probability. List the pairs of dice with one die showing a five. Be careful not to count (5,5) twice.

{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5), (5,1), (5,2), (5,3), (5,4), (5,6)}

List the pairs from above that have a sum of at least nine.

{(4,5), (5,5), (6,5), (5,4), (5,6)}

There are 11 ways for one die to show a five and five of these ways have a sum of at least nine.

$$P(\text{sum at least 9} \mid \text{one die is a 5}) = \frac{5}{11}$$

The probability that the sum is at least nine if it is known that one of the dice shows a five is $\frac{5}{11}$.

Multiplication Rule for "And" Probabilities: Any Events
For events A and B, $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

Conditional Probability
For events A and B, $P(B \mid A) = \dfrac{P(A \text{ and } B)}{P(A)}$

Example 3.3.3: Conditional Probability for Satisfaction of Car Buyers

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table from Section 3.2.

Table 3.2.2: Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

What is the probability that a person is satisfied if it is known that the person bought a used car?

This is a conditional probability because we already know that the person bought a used car.

$$P(\text{satisfied} \mid \text{used car}) = \frac{P(\text{satisfied and used})}{P(\text{used})}$$

$$= \frac{\frac{83}{250}}{\frac{130}{250}} = \frac{83}{\cancel{250}} \bullet \frac{\cancel{250}}{130} = \frac{83}{130} \approx 0.638$$

The probability that a person is satisfied if it is known that the person bought a used car is approximately 0.638 or 63.8%.

Example 3.3.4: Conditional Probability for Residence and Class Standing

A survey of 350 students at a university revealed the following data about class standing and place of residence.

Table 3.3.1: Housing by Class

Residence\Class	Freshman	Sophomore	Junior	Senior	Row Totals
Dormitory	89	34	46	15	184
Apartment	32	17	22	48	119
With Parents	13	31	3	0	47
Column Totals	134	82	71	63	350

What is the probability that a student is a sophomore if that student lives in an apartment?

This is a conditional probability because we are given that the student lives in an apartment.

$$P(\text{sophomore} \mid \text{apartment}) = \frac{P(\text{sophomore and apartment})}{P(\text{apartment})}$$

$$= \frac{\frac{17}{350}}{\frac{119}{350}} = \frac{17}{\cancel{350}} \bullet \frac{\cancel{350}}{119} = \frac{17}{119} \approx 0.143$$

The probability that a student is a sophomore given that the student lives in an apartment is approximately 0.143 or 14.3%.

Section 3.4: Expected Value and Law of Large Numbers

Would you buy a lottery ticket with the numbers 1, 2, 3, 4, 5? Do you think that a winning ticket with five consecutive numbers is less likely than a winning ticket with the numbers 2, 14, 18, 23 and 32? If you are playing a slot machine in Las Vegas and you have lost the last 10 times, do you keep playing the same machine because you are "due for a win?" Have you ever wondered how a casino can afford to offer meals and rooms at such cheap rates? Should you play a game of chance at a carnival? How much should an organization charge for raffle tickets for their next fund raiser? All of these questions can be answered using probabilities.

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. If the probability that each of these values occurs is $p_1, p_2, p_3, \ldots, p_n$, respectively, then the **expected value** of the random variable is $E(x) = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots + x_n p_n$.

Example 3.4.1: Expected Value for Raffle Tickets

Valley View Elementary is trying to raise money to buy tablets for their classrooms. The PTA sells 2000 raffle tickets at $3 each. First prize is a flat-screen TV worth $500. Second prize is an android tablet worth $375. Third prize is an e-reader worth $200. Five $25 gift certificates will also be awarded. What are the expected winnings for a person who buys one ticket?

We need to write out the probability distribution before we find the expected value.

A total of eight tickets are winners and the other 1992 tickets are losers.

Table 3.4.1: Probability Distribution for the Valley View Raffle

Outcome	Win $500	Win $375	Win $200	Win $25	Win $0
Probability	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{5}{2000}=\frac{1}{400}$	$\frac{1992}{2000}=\frac{249}{250}$

Now use the formula for the expected value.

$$
\begin{aligned}
E &= 500\left(\frac{1}{2000}\right)+375\left(\frac{1}{2000}\right)+200\left(\frac{1}{2000}\right)+25\left(\frac{1}{400}\right)+0\left(\frac{249}{250}\right)\\
&= 0.60
\end{aligned}
$$

It costs \$3 to buy a ticket but we only win an average of \$0.60 per ticket. That means the expected winnings per ticket are \$0.60 - \$3 = -\$2.40.

We would expect to lose an average of \$2.40 for each ticket bought. This means that the school will earn an average of \$2.40 for each ticket bought for a profit of \$2.40 • 2000 = \$4800.

Example 3.4.2: Expected Value for Profit from a Purchase

A real estate investor buys a parcel of land for \$150,000. He estimates the probability that he can sell it for \$200,000 to be 0.40, the probability that he can sell it for \$160,000 to be 0.45 and the probability that he can sell it for \$125,000 to be 0.15. What is the expected profit for this purchase?

Find the profit for each situation first. \$200,000 – \$150,000 = \$50,000 profit, \$160,000 - \$150,000 = \$10,000 profit, and \$125,000 - \$150,000 = -\$25,000 profit (loss).

The probability distribution is

Table 3.4.2: Probability Distribution for a Real Estate Purchase

Outcome	\$50,000	\$10,000	-\$25,000
Probability	0.40	0.45	0.15

$$\begin{aligned} E &= 50,000(0.40) + 10,000(0.45) + (-25,000)(0.15) \\ &= \$20,750 \end{aligned}$$

The expected profit from the purchase is \$20,750.

Example 3.4.3: Expected Value for Life Insurance

The cost of a \$50,000 life insurance policy is \$150 per year for a person who is 21-years old. Assume the probability that a person will die at age 21 is 0.001. What is the company's expected profit if the company sells 10,000 policies to 21-year olds?

There are two outcomes. If the person lives the insurance company makes a profit of \$150. The probability that the person lives is 1-0.001=0.999. If the person dies the company takes in \$150 and pays out \$50,000 for a loss of \$49,850.

Table 3.4.3: Probability Distribution for Life Insurance

Outcome	\$150	-\$49,850
Probability	0.999	0.001

The expected value for one policy is:

$$E(x) = \$150(0.999) + (-\$49{,}850)(0.001) = \$149.80$$

If the company sells 10,000 policies at a profit of \$149.80 each, the total expected profit is $\$149.80(10{,}000) = \$1{,}498{,}000$.

A game that has an expected value of zero is called a **fair game.**

Example 3.4.4: Expected Value for a Carnival Game

A carnival game consists of drawing two balls without replacement from a bag containing five red and eight white balls. If both balls are red you win \$6.00. If both balls are white you lose \$1.50. Otherwise you lose \$1.00. Is this a fair game? What would you expect would happen if you played the game many times?

First we need to find the probability distribution. These are conditional probabilities since the balls are drawn without replacement.

$$P(\text{both red}) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156} = \frac{5}{39}, \quad P(\text{both white}) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156} = \frac{14}{39}$$

$$\begin{aligned} P(\text{1 red and 1 white}) &= P(\{\text{red then white}\} \text{ or } \{\text{white then red}\}) \\ &= P(\text{red then white}) + P(\text{white then red}) \\ &= \frac{5}{13} \cdot \frac{8}{12} + \frac{8}{13} \cdot \frac{5}{12} \\ &= \frac{80}{156} = \frac{20}{39} \end{aligned}$$

Check that the sum of the probabilities is 1.00. $\frac{5}{39} + \frac{14}{39} + \frac{20}{39} = \frac{39}{39} = 1.00$

Thus, the probability distribution is valid and is shown below:

Table 3.4.4: Probability Distribution for a Carnival Game

Outcome	Win $6.00	Lose $1.50	Lose $1.00
Probability	$\frac{5}{39}$	$\frac{14}{39}$	$\frac{20}{39}$

Now find the expected value: $E = \frac{5}{39}(6.00) + \frac{14}{39}(-1.50) + \frac{20}{39}(-1.00) \approx -0.28$

Since the expected value is not zero this is not a fair game. Drawing two balls out of the bag is a random experiment so we cannot predict what will happen if we play the game once. We can predict what will happen if we play the game many times. We would expect to lose an average of $0.28 for every game we play. That means that the carnival will make an average of $0.28 for every game played. In this example we would refer to the carnival as the "house."

Have you ever wondered how casinos make money when they advertise a 99% payback on their slot machines? The games in a casino are not fair games since the expected value is not zero. The expected value of the game for a gambler is a small negative number like -$0.01. For a particular game the gambler may win or the gambler may lose. It's a random experiment and we cannot predict the outcome. What we can predict is what will happen if the gambler continues to play the game many times. If the expected value is -$0.01, the gambler will expect to lose an average of $0.01 for every game played. If he/she plays 100 games, he/she will expect to lose 100(0.01) = $1.00. Every penny the gambler loses the casino keeps. If hundreds of gamblers play hundreds of games each, every day of the year all those pennies add up to millions of dollars. The casino is referred to as the "house" and the $0.01 that the house expects to win for each game played is called the "house edge."

The **house edge** is the amount that the house can expect to earn for each dollar bet.

Figure 3.4.5: Roulette Wheel


(Ogle, 2009)

Example 3.4.5: House Edge in Roulette

A roulette wheel consists of 38 slots numbered 0, 00, and 1 through 36, evenly spaced around a wheel. The wheel is spun one direction and a ball is rolled around the wheel in the opposite direction. Eventually the ball will drop into one of the numbered slots. A player bets $1 on a single number. If the ball lands in the slot for that number, the player wins $35, otherwise the player loses the $1. Find the house edge for this type of bet.

The house edge is the expected value so we need to find the probability distribution and then the expected value.

There are 38 slots. One slot wins and the other 37 slots lose so

$P(\text{win}) = \frac{1}{38}$ and $P(\text{lose}) = \frac{37}{38}$.

The probability distribution is

Table 3.4.6: Probability Distribution for a Roulette Wheel

Outcome	Win $35	Lose $1
Probability	$\frac{1}{38}$	$\frac{37}{38}$

The expected value is:

$$E = \$35\left(\frac{1}{38}\right) + (-1)\left(\frac{37}{38}\right) \approx -\$0.0526$$

The expected value of the game is -$0.0526. This means that the player would expect to lose an average of 5.26 cents for each game played. The house would win an average of 5.26 cents for each game played.

The house edge is 5.26 cents.

Gamblers' Fallacy and Streaks:

Often times a gambler on a losing streak will keep betting in the belief that his/her luck must soon change. Consider flipping a fair coin. Each toss of the coin is independent of all the other tosses. Assume the coin has landed heads up the last eight times. Some people erroneously believe that the coin is more likely to land tails up on the next toss. In reality, the coin still has a 50% chance of landing tails up. It does not matter what happened the last eight tosses.

The **gambler's fallacy** is the mistaken belief that a streak of bad luck makes a person due for a streak of good luck.

Example 3.4.6: Streaks

Toss a fair coin seven times and record which side lands up. For example HHTHTTT would represent getting a head on the first, second, and fourth tosses and a tail on the other tosses. Is a streak of all heads less likely than the other possible outcomes?

As we will see in Section 3.5, there are 128 possible ways to toss a coin seven times. Some of the possibilities are HHTTHHT, HTHTHTH, HHHHTTT, and HTTHTTH. Because tossing coins are independent events and the coin is fair each of these 128 possibilities has the same probability.

$P(\text{HHTTHHT}) = \frac{1}{128}$, $P(\text{HTHTHTH}) = \frac{1}{128}$, etc… This also means that the probability of getting all heads is $P(\text{HHHHHHH}) = \frac{1}{128}$. Getting a streak of all heads has exactly the same probability as any other possible outcome.

Law of Large Numbers:

When studying probabilities, many times the law of large numbers will apply. If you want to observe what the probability is of getting tails up when flipping a coin, you could do an experiment. Suppose you flip a coin 20 times and the coin comes up tails nine times. Then, using an empirical probability, the probability of getting tails is 9/20 = 45%. However, we know that the theoretical probability for getting tails should be 1/2 = 50%. Why is this different? It is because there is error inherent to sampling methods. However, if you flip the coin 100 times or 1000 times, and use the information to calculate an empirical probability for getting tails up, then the probabilities you will observe will become closer to the theoretical probability of 50%. This is the law of large numbers.

The **law of large numbers** means that with larger numbers of trials of an experiment the observed empirical probability of an event will approach the calculated theoretical probability of the same event.

Section 3.5: Counting Methods

Recall that $P(A) = \dfrac{\text{number of ways for } A \text{ to occur}}{\text{total number of outcomes}}$ for theoretical probabilities. So far the problems we have looked at had rather small total number of outcomes. We could easily count the number of elements in the sample space. If there are a large number of elements in the sample space we can use counting techniques such as permutations or combinations to count them.

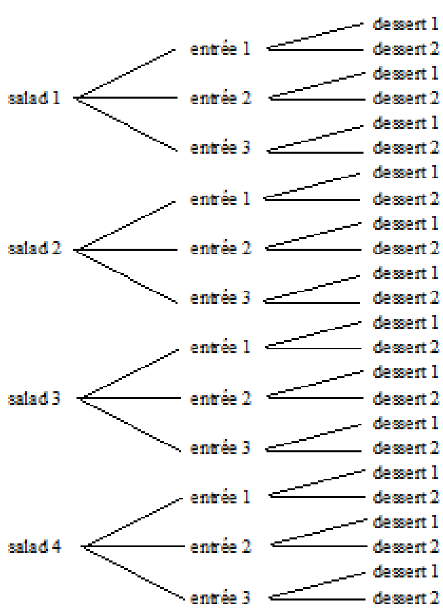
Multiplication Principle and Tree Diagrams:

The simplest of the counting techniques is the multiplication principle. A tree diagram is a useful tool for visualizing the multiplication principle.

Example 3.5.1: Multiplication Principle for a Three Course Dinner

Let's say that a person walks into a restaurant for a three course dinner. There are four different salads, three different entrees, and two different desserts to choose from. Assuming the person wants to eat a salad, an entrée and a desert, how many different meals are possible?

Figure 3.5.1: Tree Diagram for Three-Course Dinner



Looking at the tree diagram we can see that the total number of meals is 4 x 3 x 2 = 24 meals.

Multiplication Principle: If there are n_1 ways to of choosing the first item, n_2 ways of choosing the second item after the first item is chosen, n_3 ways of choosing the third item after the first two have been chosen, and so on until there are n_k ways of choosing the last item after the earlier choices, then the total number of choices overall is given by $n_1 \bullet n_2 \bullet n_3 \bullet \ldots \bullet n_k$.

Example 3.5.2: Multiplication Principle for Lining up People

Let's look at the number of ways that four people can line up. We can choose any of the four people to be first. Then there are three people who can be second and two people who can be third. At this point there is only one person left to be last. Using the multiplication principle there are 4 x 3 x 2 x 1 = 24 ways for four people to line up.

This type of calculation occurs frequently in counting problems so we have some notation to simplify the problem.

The **factorial** of n, read "n factorial" is $n! = n(n-1)(n-2)...(2)(1)$.
By definition, 0! = 1.

Example 3.5.3: Factorials

5! = 5 x 4 x 3 x 2 x 1 = 120

8! = 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 40,320

Factorials get very large very fast. 20! = 2.43 x 10^{18} and 40! = 8.16 x 10^{47}. 70! is larger than most calculators can handle.

The multiplication principle may seem like a very simple idea but it is very powerful. Many complex counting problems can be solved using the multiplication principle.

Example 3.5.4: Multiplication Principle for License Plates

Some license plates in Arizona consist of three digits followed by three letters. How many license plates of this type are possible if:

a. both digits and letters can be repeated?

There are 10 digits (0, 1, 2, 3, …, 9) and 26 letters.

$$\left(\underline{10}\cdot\underline{10}\cdot\underline{10}\right)\cdot\left(\underline{26}\cdot\underline{26}\cdot\underline{26}\right)=17,576,000 \text{ license plates}$$

digits letters

b. letters can be repeated but digits cannot?

$$\left(\underline{10}\cdot\underline{9}\cdot\underline{8}\right)\cdot\left(\underline{26}\cdot\underline{26}\cdot\underline{26}\right)=12,654,720 \text{ license plates}$$

digits letters

c. the first digit cannot be zero and both digits and letters can be repeated?

$$\left(\underline{9}\cdot\underline{10}\cdot\underline{10}\right)\cdot\left(\underline{26}\cdot\underline{26}\cdot\underline{26}\right)=15,818,400 \text{ license plates}$$

digits letters

d. neither digits nor numbers can be repeated.

$$\left(\underline{10}\cdot\underline{9}\cdot\underline{8}\right)\cdot\left(\underline{26}\cdot\underline{25}\cdot\underline{24}\right)=11,232,000 \text{ license plates}$$

digits letters

Permutations:

Consider the following counting problems. 1) In how many ways can three runners finish a race? 2) In how many ways can a group of three people be chosen to work on a project? What is the difference between these two problems? In the first problem the order that the runners finish the race matters. In the second problem the order in which the three people are chosen is not important, only which three people are chosen matters.

A **permutation** is an arrangement of a set of items.
The number of permutations of n items taking r at a time is given by:

$$P(n,r)=\frac{n!}{(n-r)!}$$

Note: Many calculators can calculate permutations directly. Look for a function that looks like ${}_nP_r$ *or* $P(n,r)$.

Example 3.5.5: Permutation for Race Cars

Let's look at a simple example to understand the formula for the number of permutations of a set of objects. Assume that 10 cars are in a race. In how many ways can three cars finish in first, second and third place? The order in which the cars finish is important. Use the multiplication principle. There are 10 possible cars to finish first. Once a car has finished first, there are nine cars to finish second. After the second car is finished, any of the eight remaining cars can finish third. 10 x 9 x 8 = 720. This is a permutation of 10 items taking three at a time.

Using the permutation formula:

$$P(10,3)=\frac{10!}{(10-3)!}=\frac{10!}{7!}=\frac{10\bullet 9\bullet 8\bullet \cancel{7}\bullet \cancel{6}\bullet \cancel{5}\bullet \cancel{4}\bullet \cancel{3}\bullet \cancel{2}\bullet \cancel{1}}{\cancel{7}\bullet \cancel{6}\bullet \cancel{5}\bullet \cancel{4}\bullet \cancel{3}\bullet \cancel{2}\bullet \cancel{1}}=10\bullet 9\bullet 8=720$$

Using the multiplication principle:

$\underline{10} \bullet \underline{9} \bullet \underline{8} = 720$

There are 720 different ways for cars to finish in the top three places.

Example 3.5.6: Permutation for Orchestra Programs

The school orchestra is planning to play six pieces of music at their next concert. How many different programs are possible?

This is a permutation because they are arranging the songs in order to make the program.

Using the permutation formula:

$$P(6,6)=\frac{6!}{(6-6)!}=\frac{6!}{0!}=\frac{720}{1}=720$$

Using the multiplication principle:

$\underline{6}\cdot\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}=720$

There are 720 different ways of arranging the songs to make the program.

Example 3.5.7: Permutation for Club Officers

The Volunteer Club has 18 members. An election is held to choose a president, vice-president and secretary. In how many ways can the three officers be chosen?

The order in which the officers are chosen matters so this is a permutation.

Using the permutation formula:

$$P(18,3)=\frac{18!}{(18-3)!}=\frac{18!}{15!}=18\bullet 17\bullet 16=4896$$

Note: All digits in 18! in the numerator from 15 down to one will cancel with the 15! in the denominator.

Using the multiplication principle:

$$\underset{\text{Pres.}}{\underline{18}} \cdot \underset{\text{V.P.}}{\underline{17}} \cdot \underset{\text{Sec.}}{\underline{16}} = 4896$$

There are 4896 different ways the three officers can be chosen.

Another notation for permutations is nPr. So, $P(18,3)$ can also be written as ${}_{18}P_3$. Most scientific calculators have an nPr button or function.

Combinations:

Example 3.5.8: Formula for Combinations

Choose a committee of two people from persons A, B, C, D and E. By the multiplication principle there are $5 \cdot 4 = 20$ ways to arrange the two people.

AB	AC	AD	AE	BA	BC	BD	BE	CA	CB
CD	CE	DA	DB	DC	DE	EA	EB	EC	ED

Committees AB and BA are the same committee. Similarly for committees CD and DC. Every committee is counted twice. $\frac{20}{2} = 10$ so there are 10 possible different committees.

Now choose a committee of three people from persons A, B, C, D and E. There are $5 \cdot 4 \cdot 3 = 60$ ways to pick three people in order. Think about the committees with persons A, B and C. There are $3! = 6$ of them.

ABC ACB BAC BCA CAB CBA

Each of these is counted as one of the 60 possibilities but they are the same committee. Each committee is counted six times so there are $\frac{60}{6} = 10$ different committees.

In both cases we divided the number of permutations by the number of ways to rearrange the people chosen.

The number of permutations of n people taking r at a time is *P(n,r)* and the number of ways to rearrange the people chosen is *r!*. Putting these together we get

$$\frac{\text{\# permutations of n items taking r at a time}}{\text{\# ways to arrange r items}} = \frac{P(n,r)}{r!} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{1}}$$

$$= \frac{n!}{(n-r)!} \cdot \frac{1}{r!}$$

$$= \frac{n!}{(n-r)!r!}$$

A **combination** is a selection of objects in which the order of selection does not matter.

The number of combinations of n items taking r at a time is: $C(n,r) = \frac{n!}{r!(n-r)!}$

Note: Many calculators can calculate combinations directly. Look for a function that looks like $_nC_r$ *or* $C(n,r)$.

Example 3.5.9: Combination for Picking Books

A student has a summer reading list of eight books. The student must read five of the books before the end of the summer. In how many ways can the student read five of the eight books?

The order of the books is not important, only which books are read. This is a combination of eight items taking five at a time.

$$C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8\bullet 7\bullet 6\bullet \cancel{5\bullet 4\bullet 3\bullet 2\bullet 1}}{\cancel{5\bullet 4\bullet 3\bullet 2\bullet 1}\bullet 3\bullet 2\bullet 1} = \frac{8\bullet 7\bullet \cancel{6}}{\cancel{3\bullet 2}\bullet 1} = 8\bullet 7 = 56$$

There are 56 ways to choose five of the books to read.

Example 3.5.10: Combination for Halloween Candy

A child wants to pick three pieces of Halloween candy to take in her school lunch box. If there are 13 pieces of candy to choose from, how many ways can she pick the three pieces?

This is a combination because it does not matter in what order the candy is chosen.

$$C(13,3) = \frac{13!}{3!(13-3)!} = \frac{13!}{3!10!} = \frac{13 \bullet 12 \bullet 11 \bullet \cancel{10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}{3 \bullet 2 \bullet 1 \bullet \cancel{10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}$$

$$= \frac{13 \bullet 12 \bullet 11}{3 \bullet 2 \bullet 1} = \frac{1716}{6} = 286$$

There are 286 ways to choose the three pieces of candy to pack in her lunch.

Note: The difference between a combination and a permutation is whether order matters or not. If the order of the items is important, use a permutation. If the order of the items is not important, use a combination.

Example 3.5.11: Permutation or Combination for Bicycle Serial Numbers

A serial number for a particular model of bicycle consists of a letter followed by four digits and ends with two letters. Neither letters nor numbers can be repeated. How many different serial numbers are possible?

This is a permutation because the order matters.

Use the multiplication principle to solve this. There are 26 letters and 10 digits possible.

$$\underline{26} \cdot \underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{25} \cdot \underline{24} = 78,624,000$$

There are 78,624,000 different serial numbers of this form.

Example 3.5.12: Permutation or Combination for Choosing Men and Women

A class consists of 15 men and 12 women. In how many ways can two men and two women be chosen to participate in an in-class activity?

This is a combination since the order in which the people is chosen is not important.

Choose two men: $C(15,2) = \frac{15!}{2!(15-2)!} = \frac{15!}{2!13!} = 105$

Choose two women: $C(12,2) = \frac{12!}{2!(12-2)!} = \frac{12!}{2!10!} = 66$

We want 2 men <u>and</u> 2 women so multiply these results.

$$105(66) = 6930$$

There are 6930 ways to choose two men and two women to participate.

Probabilities Involving Permutations and Combinations:

Now that we can calculate the number of permutations or combinations, we can use that information to calculate probabilities.

Example 3.5.13: Probability with a Combination for Choosing Students

There are 20 students in a class. Twelve of the students are women. The names of the students are put into a hat and five names are drawn. What is the probability that all of the chosen students are women?

This is a combination because the order of choosing the students is not important.

$$P(\text{all females}) = \frac{\text{\# ways to pick 5 women}}{\text{\# ways to pick 5 students}}$$

The number of way to choose 5 women is

$$C(12,5) = 792$$

The number of ways to choose 5 students is

$$C(20,5) = 15{,}504$$

$$P(\text{all females}) = \frac{\text{\# ways to pick 5 women}}{\text{\# ways to pick 5 students}} = \frac{792}{15{,}504} = 0.051$$

The probability that all the chosen students are women is 0.051 or 5.1%.

Example 3.5.14: Probability with a Permutation for a Duck Race

The local Boys and Girls Club holds a duck race to raise money. Community members buy a rubber duck marked with a numeral between 1 and 30, inclusive. The box of 30 ducks is emptied into a creek and allowed to float downstream to the finish line. What is the probability that ducks numbered 5, 18 and 21 finish in first, second and third, respectively?

This is a permutation since the order of finish is important.

$$P(\text{5, 18 \& 21 finish 1st, 2nd \& 3rd}) = \frac{\text{\# ways 5, 18 \& 21 finish 1st, 2nd \& 3rd}}{\text{\# ways any ducks can finish 1st, 2nd \& 3rd}}$$

There is only one way that the ducks can finish with #5 in first, #18 in second and #21 in third.

The number of ways any ducks can finish in first, second and third is

$$P(30,3) = 24,360$$

$$P(5\text{, 18 \& 21 finish 1st, 2nd \& 3rd}) = \frac{\text{\# ways 5, 18 \& 21 finish 1st, 2nd \& 3rd}}{\text{\# ways any ducks can finish 1st, 2nd \& 3rd}}$$

$$= \frac{1}{24,360} \approx 4.10x10^{-5}$$

The probability that ducks numbered 5, 18 and 21 finish in first, second and third, respectively, is approximately 0.000041 or 0.0041%.

Example 3.5.15: Probability with a Permutation for Two-Card Poker Hands

A poker hand consists of two cards. What is the probability that the poker hand consists of two jacks or two fives?

It's not possible to get two jacks and two fives at the same time so these are mutually exclusive events.

The number of ways to get two jacks is

$C(4,2) = 6$.

The number of ways to get two fives is

$C(4,2) = 6$

The number of ways to get two jacks or two fives is

6 + 6 = 12.

The total number of ways to get a 2-card poker hand is

$C(52,2) = 1326$

$$P(\text{2 jacks or 2 fives}) = \frac{\text{number of ways to get 2 jacks or 2 fives}}{\text{number of ways to choose 2 cards}} = \frac{12}{1326} \approx 0.009$$

The probability of getting two jacks or two fives is approximately 0.009 or 0.9%.

Example 3.5.16: Probability with a Combination for Rotten Apples

A basket contains 10 good apples and two bad apples. If a distracted shopper reaches into the basket and picks three apples without looking, what is the probability he gets one bad apple?

This is a combination since the order in which the apples were picked is not important. He picks three apples total. If one apple is bad the other two must be good. Find the probability of one bad apple and two good apples.

$$P(\text{one bad and two good apples}) = \frac{\text{\# ways to get one bad and two good apples}}{\text{\# ways to get three apples}}$$

The number of ways to get one bad and two good apples is

$$C(2,1)\cdot C(10,2) = 2\cdot 45 = 90$$

The number of ways to get three apples is

$$C(10,3) = 120$$

$$P(\text{one bad and two good apples}) = \frac{\text{\# ways to get one bad and two good apples}}{\text{\# ways to get three apples}}$$

$$= \frac{90}{120} = 0.75$$

The probability of getting one bad apple out of three apples is 0.75 or 75%.

Chapter 3 Homework

1. A random experiment consists of drawing a single card from a well-shuffled deck and recording the suit. Write the sample space for this experiment.

2. A random experiment consists of drawing a single card from a well-shuffled deck and recording the number. Write the sample space for this experiment.

3. A random experiment consists of tossing a fair coin five times and recording the number of tails. Write the sample space for this experiment.

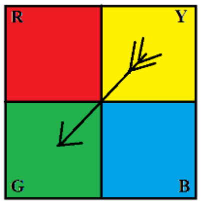
4. A random experiment consists of tossing a fair coin four times and recording the number of heads. Write the sample space for this experiment.

5. A random experiment consists of tossing three fair coins and recording whether each coin lands heads up or tails up. Write the sample space for this experiment.

6. A random experiment consists of tossing four fair coins and recording whether each coin lands heads up or tails up. Write the sample space for this experiment.

7. A random experiment consists of tossing a fair die and then tossing a fair coin. The number showing on top of the die and whether the coin lands heads up or tails up is recorded. Write the sample space for this experiment.

8. A spinner in the following figure is spun and a coin is tossed. The color of the spinner and whether the coin is heads up or tails up is recorded. Write the sample space for this experiment.



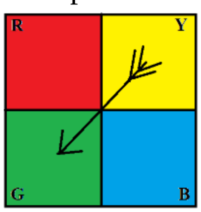
9. A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that:
   a. the card is a seven.
   b. the card is a face card.
   c. the card is a number between 2 and 5, inclusive.

10. A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that:
    a. the card is a jack.
    b. the card is a club.
    c. the card is a number less than 4, including aces.

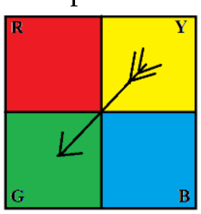
11. Two fair dice are tossed. What is the probability that the sum of the numbers is 5?

12. Two fair dice are tossed. What is the probability that the sum of the numbers is 10?

13. The spinner is spun once. What is the probability that it lands on green?



14. The spinner is spun once. What is the probability that it lands on blue?



15. An urn contains 10 red balls, 15 white balls, and 20 black balls. A single ball is selected at random. What is the probability that the ball is:
    a. red?
    b. white?
    c. not black?
    d. black or white?

16. A candy dish contains 12 chocolate candies, 18 butterscotch candies, eight caramels, and 15 peppermints. A single candy is selected at random. What is the probability that the candy is:
    a. a butterscotch candy?
    b. not a caramel candy?
    c. a chocolate or a peppermint candy?

17. Three fair coins are tossed at the same time. What is the probability of getting exactly two heads? *Hint: use problem #5.*

18. Four fair coins are tossed at the same time. What is the probability of getting exactly one head? *Hint: use problem #6.*

19. An instructor collected the following data from the students in her classes.

Class / Year	Freshman	Sophomore	Total
MAT 121	43	15	58
MAT 142	35	28	63
MAT 187	27	32	59
Total	105	75	180

If a student is selected at random, what is the probability that:
    a. The student is a freshman?
    b. The student is taking MAT 142?

20. A real estate agent has kept records of the number of bedrooms in the houses he has sold for the last five years. The data is listed in the following table.

Year/ Number of Bedrooms	one or two	three	four	five or more	Total
2009	5	12	25	3	45
2010	7	15	22	3	47
2011	6	18	28	6	58
2012	6	16	30	2	54
2013	5	17	28	4	54
Total	29	78	133	18	258

a. Find the probability that a randomly selected house sold by the agent had four bedrooms.
b. Find the probability that a randomly selected house sold by the agent was sold in 2010.
c. Find the probability that a randomly selected house sold by the agent had less than four bedrooms.
d. Find the probability that a randomly selected house sold by the agent was sold after 2011.

21. A student believes he has an 80% chance of passing his English class. What is the probability he will not pass the class?

22. There is a 45% chance that it will snow today. What is the probability that it will not snow today?

23. A single die is rolled. What is the probability of not rolling a five?

24. A student is randomly chosen from a class of 30 students. If seven of the students are majoring in business, what is the probability that the randomly chosen student is not majoring in business?

25. There is a 35% chance that a bus will arrive early at a bus stop. Explain why you cannot assume that there is a 65% probability that the bus will be late at the same bus stop.

26. Are each of the following valid probability distributions or not? For each one, explain why or why not.

a.

Outcome	A	B	C	D	E
Probability	0.2	0.4	-0.2	0.4	0.2

b.

Outcome	A	B	C	D	E
Probability	0.2	0.4	0.1	0.4	0.2

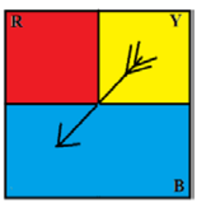
c.

Outcome	A	B	C	D	E
Probability	0.20	0.30	0.10	0.15	0.25

27. Three fair coins are tossed and the number of heads recorded. Write the probability distribution for this random experiment. *Hint: use problem #5.*

28. Four fair coins are tossed and the number of heads recorded. Write the probability distribution for this experiment. *Hint: use problem #6.*

29. The spinner in the following figure is spun once and the color recorded. Write the probability distribution for this random experiment.



30. An urn contains three red balls, four blue balls, and five green balls. A ball is selected at random and its color recorded. Write out the probability distribution for this experiment.

31. A single card is drawn from a well-shuffled deck of 52 cards. The card is either an even number, an odd number or a face card. If we consider aces to be ones, write the probability distribution for this random experiment.

32. Two fair dice are rolled. Find the odds for rolling a sum of five.

33. Two fair dice are rolled. Find the odds for rolling a sum of 10.

34. Two fair dice are rolled. Find the odds against rolling a sum of six.

35. Two fair dice are rolled. Find the odds against rolling a sum of eight.

36. A single card is drawn from a well-shuffled deck of 52 cards. Find the odds that the card is a nine.

37. A single card is drawn from a well-shuffled deck of 52 cards. Find the odds against the card being a face card.

38. A candy dish contains 12 chocolate candies, 18 butterscotch candies, 8 caramels, and 15 peppermints. A single candy is selected at random. What are the odds that the candy is a chocolate candy?

39. An urn contains 10 red balls, 15 white balls, and 20 black balls. A single ball is selected at random. Find the odds against drawing a white ball.

40. A real estate agent has kept records of the number of bedrooms in the houses he has sold for the last year. The data is listed in the following table.

Number of Bedrooms	1 or 2	3	4	5 or more
Number of Houses	5	12	25	3

If a house sold by the agent is selected at random, find:

a. the odds for the house having four bedrooms.
b. the odds for the house having less than three bedrooms.
c. the odds against the house having five or more bedrooms.
d. the odds against the house having more than three bedrooms.

41. Suppose the odds that Josh will win a tennis match against Jesse are 2 to 1. What is the probability that Josh will win?

42. The odds of winning a particular carnival game are 27 to 5. Find the probability of winning the game.

43. The odds of winning a particular carnival game are 15 to 28. Find the probability of losing the game.

44. The odds that a randomly selected student is a male are 21 to 25. Find the probability that a randomly selected student is a male.

45. A single card is drawn from a well-shuffled deck of 52 cards. Are the events E = the card is an even number and F = the card is a heart mutually exclusive? Explain why or why not.

46. A single card is drawn from a well-shuffled deck of 52 cards. Are the events E = the card is a face card and F = the card is a seven mutually exclusive? Explain why or why not.

47. An instructor randomly selects a student from his class. Are the events E = the student is taking History and F = the student is a freshman mutually exclusive? Explain why or why not?

48. A campus security officer randomly selects a car in the parking lot. Are the event E = the car is a Toyota and F = the car is red mutually exclusive? Explain why or why not.

49. Suppose you roll a single fair die. What is the probability of getting a four or a five?

50. Suppose you roll a single fair die. What is the probability of getting a two or an even?

51. Suppose you draw one card from a standard deck of cards. What is the probability that the card is an ace or a diamond?

52. Suppose you draw one card from a standard deck of cards. What is the probability that the card is an ace or the king of diamonds?

53. A teacher asks a class of 40 students about the classes they are taking. 17 of the students are taking math, 31 are taking English and 15 are taking both math and English. What is the probability that a randomly selected student is taking either math or English?

54. A teachers looks over her class and notices a few trends. Out of the 60 students in the class, 23 have brown hair, seven have green eyes, and three have both brown hair and green eyes. What is the probability that a randomly selected student has either brown hair or green eyes?

55. A student observes 46 vehicles in the CCC parking lot. He notices that 12 of the vehicles are red and that 19 of the vehicles are 4-wheel drive. If the probability that a randomly chosen vehicle is red or has 4-wheel drive is 0.609, what is the probability that the car is red and has 4-wheel drive?

56. Are drawing a card from a deck and tossing a coin independent events? Explain why or why not.

57. A single card is drawn from a deck. Are drawing a red card and drawing a heart independent events? Explain why or why not.

58. A jar contains three red, four blue, and five white marbles. A marble is drawn and its color is recorded. The marble is put back in the jar and a second marble is drawn. Is drawing two marbles in this manner independent events or not? Explain why or why not.

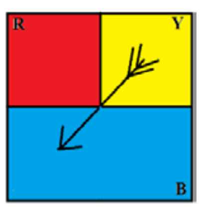
59. A jar contains three red, four blue, and five white marbles. A marble is drawn and its color is recorded. The marble is not put back in the jar before a second marble is drawn. Is drawing two marbles in this manner independent events or not? Explain why or why not.

60. A fair coin is tossed ten times. What is the probability of getting ten heads?

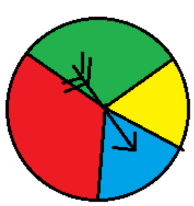
61. A fair coin is tossed then a fair die is rolled. What is the probability of getting a head and a number less than three?

62. A card is drawn from a well-shuffled deck, its number recorded, and the card returned to the deck. A second card is then drawn.
    a. What is the probability of getting a jack first and a spade second?
    b. What is the probability of getting a jack and a spade in any order?

63. The spinner shown in the following figure is spun three times. What is the probability that the spinner lands on red first, blue second, and yellow third?



64. The spinner shown in the following figure is spun four times. What is the probability that the spinner lands on red the first three times and blue the fourth time?



65. A mother figures that her son will forget his homework about 20% of the time. What is the probability that he will forget his homework at least once in the next 10 school days?

66. A bird watcher expects to see a falcon about 35% of the times he visits a nearby park. If he visits the park 12 times in the next month, what is the probability that he will see a falcon at least once?

67. A police officer watching a particular stretch of highway in Montana finds that about one out of every five drivers is speeding. What is the probability that at least one of the next eight cars he sees is speeding?

68. An instructor expects 10% of the class to earn an A on the final exam. If there are 23 students in the class, what is the probability that at least one student earns an A on the final exam?

69. A researcher surveys 150 young athletes asking about the last sport the athlete played and whether or not the athlete was injured playing that sport. The data is summarized in the following table.

Sport	Injured	Not Injured	Total
Gymnastics	16	34	50
Soccer	5	30	35
Football	27	18	45
Skiing	7	13	20
Total	55	95	150

If an athlete is selected at random, what is the probability that the athlete:

a. played soccer and was not injured?
b. did gymnastics and was injured?
c. played football or was injured?

70. An instructor collected the following data from the students in her classes.

Class / Year	Freshman	Sophomore	Total
MAT 121	43	15	58
MAT 142	35	28	63
MAT 187	27	32	59
Total	105	75	180

If a student is selected at random, what is the probability that:

a. the student is a freshman and taking MAT 121?
b. the student is a sophomore and taking MAT 187?
c. the student is a freshman or taking MAT 142?

71. Suppose you draw two cards from a standard deck of cards without replacement. What is the probability that:
    a. both cards are Kings?
    b. both cards are face cards?
    c. the first card is a five and the second card is a Jack?
    d. the first card is a Queen and the second card is a number less than five (count Aces as ones)?

72. Suppose you draw two cards from a standard deck of cards without replacement. What is the probability that:
    a. you draw an Ace on the first card and a seven on the second card?
    b. you draw a heart on the first card and a spade on the second card?
    c. you draw an eight on the first card and a face card on the second card?
    d. you draw two hearts in a row?

73. Suppose you pick two candies randomly from a box of candies and eat them. There are four chocolates, four caramels, and four mints. What is the probability that both candies will be chocolates?

74. A kindergarten class has 15 boys and 13 girls. The teacher calls on three students, one at a time, to line up at the board. What is the probability that the teacher calls up a boy followed by two girls?

75. A small parking lot has five black cars, seven white cars, three red cars, and four blue cars. If two cars leave in random order, what is the probability that a red car will leave first, followed by a black car?

76. A cooler contains six colas, eight root beers, and four ginger ales. Three kids grab a drink at random, one at a time.
    a. What is the probability that the first kid grabs a cola, the second kid grabs a ginger ale, and the third kid grabs a cola?
    b. What is the probability that the third kid grabs a root beer given that the first two grabbed colas?

77. An instructor collected the following data from the students in her classes.

Class / Year	Freshman	Sophomore	Total
MAT 121	43	15	58
MAT 142	35	28	63
MAT 187	27	32	59
Total	105	75	180

If a student is selected at random, find:
    a. the probability that the student is a freshman given the student is taking MAT 121.
    b. the probability that the student is taking MAT 142 given that the student is a sophomore.
    c. the probability that the student is a freshman and taking MAT 187.

78. A real estate agent has kept records of the number of bedrooms in the houses he has sold for the last five years. The data is listed in the table.

Year/ Number of Bedrooms	one or two	three	four	five or more	Total
2009	5	12	25	3	45
2010	7	15	22	3	47
2011	6	18	28	6	58
2012	6	16	30	2	54
2013	5	17	28	4	54
Total	29	78	133	18	258

If a house sold by the agent is randomly selected, find:

a. the probability that the house has three bedrooms given it was sold in 2012.
b. the probability that the house has five or more bedrooms and was sold in 2009.
c. the probability that the house has four bedrooms and was sold in 2013.
d. the probability that the house was sold in 2011 given that is has four bedrooms.

79. A single card is drawn from a well-shuffled deck of 52 cards. If the card is an ace, you win $12; otherwise, you lose $1.50.
    a. What is the expected value of this game?
    b. Explain what the expected value means in terms of the game.
    c. Is this a fair game or not?

80. Four thousand tickets are sold at $1 each for a charity raffle. Tickets are drawn at random without replacement. There is one $800 first prize, two $300 second prizes and eight $50 third prizes.
    a. What is the expected value of this raffle if you buy one ticket?
    b. Explain what the expected value means in terms of the raffle.
    c. Is this a fair game or not?

81. You and a friend are playing some of the games at the county fair. A particular game consists of drawing a single card from a standard deck. You win $5 if you draw an ace, $2 if you draw a face card, and $0.25 if any other card is drawn. It costs $1 to draw a card. Your friend thinks it is a great idea to play this game.
    a. Calculate the expected value for this game.
    b. Should your friend play the game or not?
    c. Using simple English explain to your friend why he should or should not play the game.

82. A casino game has an expected value of -$0.15. Explain what this means in a complete sentence. Do not use the words “expected value” in your explanation.

83. Two coins are flipped. You win $3 if either two heads or two tails turns up; you lose $4 if a head and a tail turn up. What is the expected value of the game?

84. Two dice are tossed at the same time. If the sum of the numbers is less than eight you win \$5.50; if the sum is exactly eight you win \$8; and if the sum is greater than eight you win \$3. It costs \$5.00 to play the game. Use the expected value to determine if you should play this game or not. Explain why or why not.

85. A lottery game consists of picking five numbers between 1 and 36, inclusive. You want to buy a ticket with the numbers 1, 2, 3, 4, and 5. Your friend laughs at you and says that those five numbers are really unlikely. What should you say to your friend to explain why he is wrong?

86. You and your friend are playing some slot machines in a casino. Your friend has lost the last 15 times he played. He asks to borrow some money because he feels his luck is about to change, that he is due to win after all those loses. What should you say to your friend to explain why he is wrong?

87. Suppose you want to create a computer password that has to be 10 digits long with each digit being chosen from the numbers 0 through 9. How many different passwords are available?

88. Jack wants to buy a new set of yard furniture. If there are three choices of tables, five choices of gliders, six choices of cushions, and two choices of umbrellas, how many different sets of yard furniture are possible?

89. The chess club is having a Sundae Night to raise money. There are three flavors of ice cream and six different toppings. A sundae consists of two scoops of ice cream and one topping. The ice cream scoops do not have to be the same flavor. How many different sundaes are possible?

90. The new iPhone comes in five different colors. There are three different sizes of memory available. The store has a selection of ten different cases to fit it. Assuming a person wants to buy an iPhone and a case, how many different phone/case combinations are possible?

91. How many ways can you line up 5 people?

92. Ten runners participate in the 100 meter hurdles. In how many ways can the medals for the top three finishers be awarded?

93. The Science club has 25 members. In how many ways can a President, Vice President, Secretary, and Treasurer be chosen?

94. Five friends are taking a road trip in a car that seats five. The car has a standard transmission so only two of the friends can drive it. How many seating arrangements are possible?

95. A night club owner is trying to arrange nine acts for a show. There are five musical numbers and four comedians. In how many ways can the show be arranged if:
    a. the acts can be in any order?
    b. the acts must alternate between musical numbers and comedians, starting with a musical number?
    c. the show must open and close with musical numbers but the remaining acts may be in any order?
96. Stanley wants to rearrange six science books and four history books on his shelf. How many arrangements are possible if:
    a. the books can be in any order?
    b. the science books are on the left and the history books are on the right?
    c. the four history books are in the middle with three science books on each end?

97. In how many ways can a five-card poker hand be drawn from a standard deck of cards?

98. In how many way can a two-card poker hand be drawn from a standard deck of cards?

99. The science club consists of 18 men and 12 women. Five members are chosen to staff the club's booth at the Science in the Park Festival. In how many ways can the five members be chosen if:
    a. any of the members can be chosen?
    b. all five members are women?
    c. exactly three men are chosen?

100. There are 28 graduate students in the department of Math and Statistics. Eighteen of the students are majoring in math and the other 10 are majoring in statistics. The department wants to send four of the graduate students to a conference. How many ways can the four students be chosen if:
    a. any of the students can be chosen?
    b. two students from each major must be chosen?
    c. at least two must be majoring in math.

101. A barrel contains 20 good peaches and four rotten peaches. A person selects three peaches at random. In how many ways can the person select:
    a. three rotten peaches?
    b. three good peaches?
    c. two good and one rotten peach?

102. A toddler is playing with some magnetic letters on the refrigerator. He has the letters l, t, a, b, and e. If he arranges the letters in a line to make a word, what is the probability that he makes the word "table"?

103. The theatre club has 14 female and nine male members. Two members are selected at random to do an acting exercise. What is the probability that both members are males?

104. Three friends, Al, Ted, and Bert run a foot race with five other boys. What is the probability that:
    a. Ted finishes first, Bert finishes second and Al finishes third?
    b. the three friends all finish in the top three places?

105. A night club owner is trying to arrange nine acts for a show. There are five musical numbers and four comedians. If the owner randomly arranges the acts, what is the probability that:

   a. the acts alternate between musical numbers and comedians, starting with a musical number?
   b. the show opens and closes with musical numbers?

106. The science club consists of 18 men and 12 women. Five members are chosen to staff the club's booth at the Science in the Park Festival. What is the probability that:

   a. all five members are women?
   b. exactly three men are chosen?

107. There are 28 graduate students in the department of Math and Statistics. Eighteen of the students are majoring in math and the other ten are majoring in statistics. The department wants to send four of the graduate students to a conference. What is the probability that:

   a. two students from each major are chosen to attend the conference?
   b. at least two who are majoring in math are chosen to attend the conference?

108. A barrel contains 20 good peaches and four bad peaches. A person selects three peaches at random. what is the probability of getting:

   a. three bad peaches?
   b. three good peaches?
   c. two good and one bad peach?

Chapter 4: Growth

Population growth is a current topic in the media today. The world population is growing by over 70 million people every year. Predicting populations in the future can have an impact on how countries plan to manage resources for more people. The tools needed to help make predictions about future populations are growth models like the exponential function. This chapter will discuss real world phenomena, like population growth and radioactive decay, using three different growth models.

The growth functions to be examined are linear, exponential, and logistic growth models. Each type of model will be used when data behaves in a specific way and for different types of scenarios. Data that grows by the same amount in each iteration uses a different model than data that increases by a percentage.

Section 4.1: Linear Growth

Starting at the age of 25, imagine if you could save $20 per week, every week, until you retire, how much money would you have stuffed under your mattress at age 65? To solve this problem, we could use a linear growth model. Linear growth has the characteristic of growing by the same amount in each unit of time. In this example, there is an increase of $20 per week; a constant amount is placed under the mattress in the same unit of time.

If we start with $0 under the mattress, then at the end of the first year we would have $\$20 \times 52 = \$1040$. So, this means you could add $1040 under your mattress every year. At the end of 40 years, you would have $\$1040 \times 40 = \$41,600$ for retirement. This is not the best way to save money, but we can see that it is calculated in a systematic way.

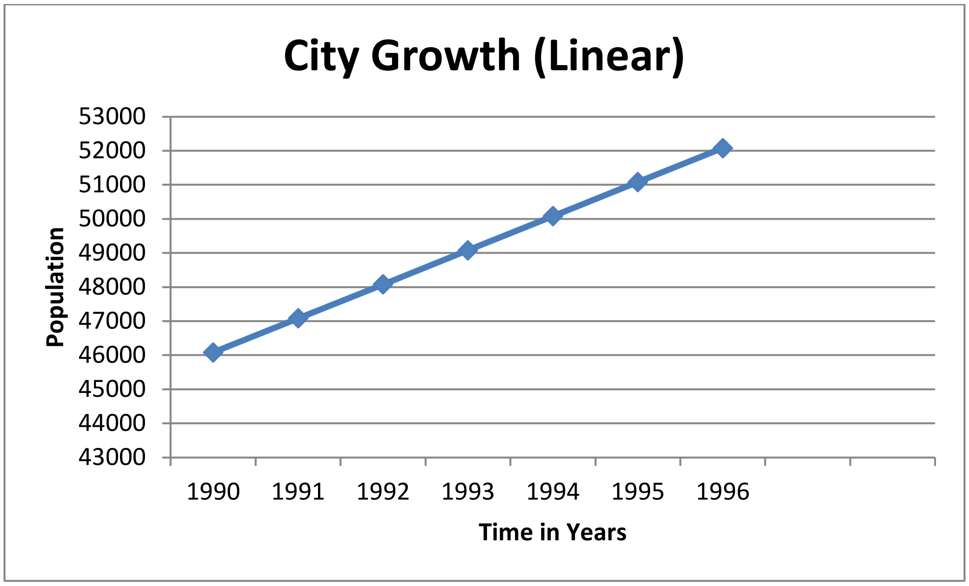
Linear Growth: A quantity grows linearly if it grows by a constant amount for each unit of time.

Example 4.1.1: City Growth

Suppose in Flagstaff Arizona, the number of residents increased by 1000 people per year. If the initial population was 46,080 in 1990, can you predict the population in 2013? This is an example of linear growth because the population grows by a constant amount. We list the population in future years below by adding 1000 people for each passing year.

	1990	1991	1992	1993	1994	1995	1996
Year	0	1	2	3	4	5	6
Population	46,080	47,080	48,080	49,080	50,080	51,080	52,080

Figure 4.1.1: Graph of Linear Population Growth
This is the graph of the population growth over a six year period in Flagstaff, Arizona. It is a straight line and can be modeled with a linear growth model.



The population growth can be modeled with a linear equation. The initial population P_0 is 48,080. The future population depends on the number of years, *t*, after the initial year. The model is $P(t) = 46{,}080 + 1000t$

To predict the population in 2013, we identify how many years it has been from 1990, which is year zero. So $n = 23$ for the year 2013.

$$P(23) = 46{,}080 + 1000(23) = 69{,}080$$

The population of Flagstaff in 2013 would be 69,080 people.

Linear Growth Model: Linear growth begins with an initial population called P_0. In each time period or generation *t*, the population changes by a constant amount called the common difference *d*. The basic model is:

$$P(t) = P_0 + td$$

Example 4.1.2: Antique Frog Collection

Dora has inherited a collection of 30 antique frogs. Each year she vows to buy two frogs a month to grow the collection. This is an additional 24 frogs per year. How many frogs will she have is six years? How long will it take her to reach 510 frogs?

The initial population is $P_0 = 30$ and the common difference is $d = 24$. The linear growth model for this problem is:

$P(t) = 30 + 24t$

The first question asks how many frogs Dora will have in six years so, $t = 6$.

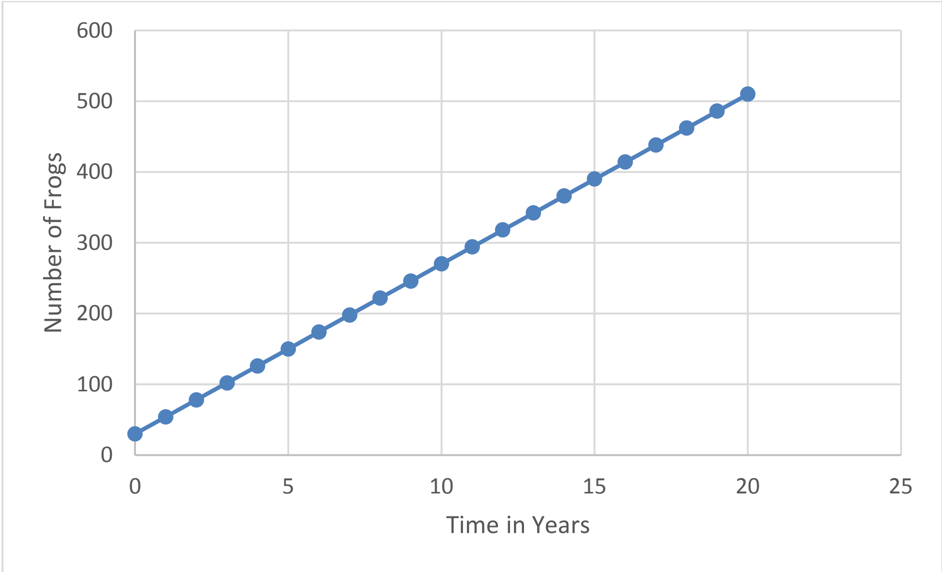
$P(6) = 30 + 24(6) = 30 + 144 = 174$ frogs.

The second question asks for the time it will take for Dora to collect 510 frogs. So, $P(t) = 510$ and we will solve for t.

$$510 = 30 + 24t$$
$$480 = 24t$$
$$20 = t$$

It will take 20 years to collect 510 antique frogs.

Figure 4.1.2: Graph of Antique Frog Collection



Note: The graph of the number of antique frogs Dora accumulates over time follows a straight line.

Example 4.1.3: Car Depreciation

Assume a car depreciates by the same amount each year. Joe purchased a car in 2010 for $16,800. In 2014 it is worth $12,000. Find the linear growth model. Predict how much the car will be worth in 2020.

$P_0 = 16,800$ and $P(4) = 12,000$

To find the linear growth model for this problem, we need to find the common difference d.

$$P(t) = P_0 + td$$

$$12,000 = 16,800 + 4d$$

$$-4800 = 4d$$

$$-1200 = d$$

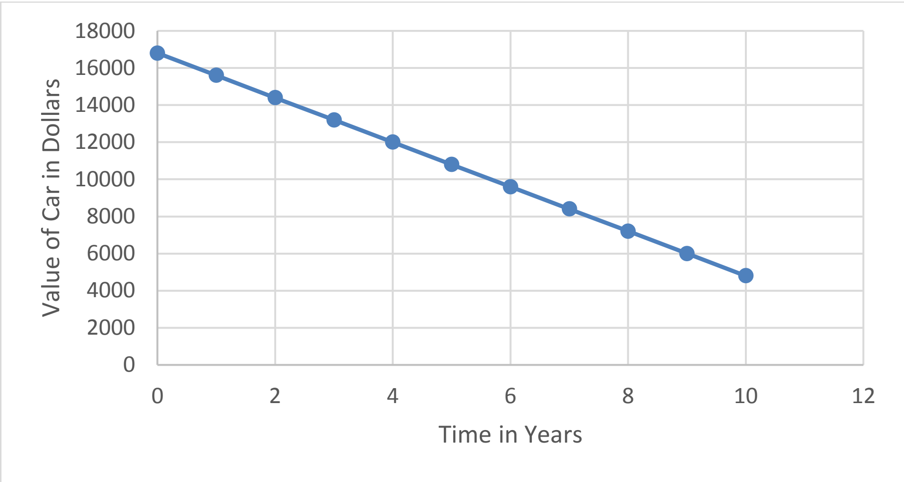
The common difference of depreciation each year is $d = \$-1200$. Thus, the linear growth model for this problem is: $P(t) = 16,800 - 1200t$

Now, to find out how much the car will be worth in 2020, we need to know how many years that is from the purchase year. Since it is ten years later, $t = 10$.

$$P(10) = 16,800 - 1200(10) = 16,800 - 12,000 = 4,800$$

The car is worth $4800 in 2020.

Figure 4.1.3: Graph of Car Value Depreciation



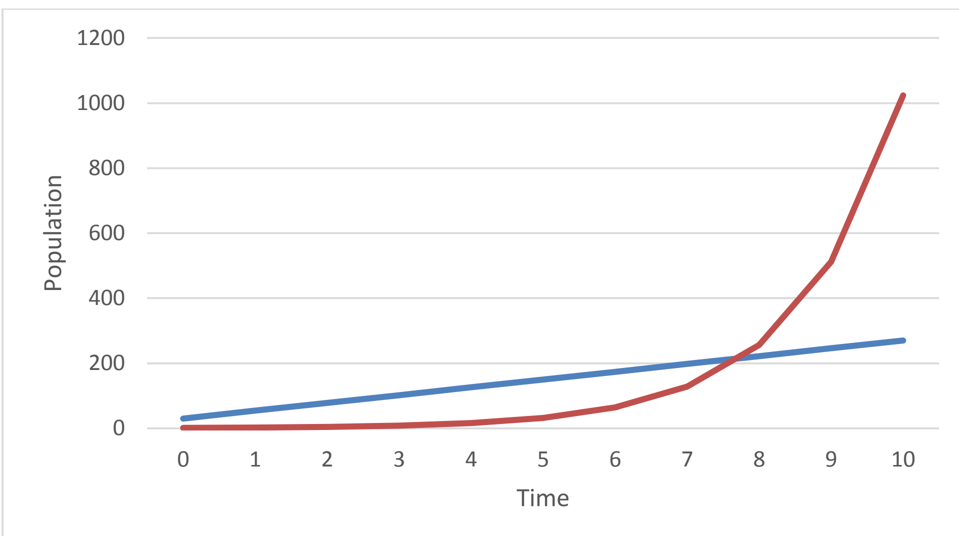
Note: The value of the car over time follows a decreasing straight line.

Section 4.2: Exponential Growth

The next growth we will examine is exponential growth. Linear growth occur by adding the same amount in each unit of time. Exponential growth happens when an initial population increases by the same percentage or factor over equal time increments or generations. This is known as relative growth and is usually expressed as percentage. For example, let's say a population is growing by 1.6% each year. For every 1000 people in the population, there will be $1000 \times 0.016 = 16$ more people added per year.

Exponential Growth: A quantity grows exponentially if it grows by a constant factor or rate for each unit of time.

Figure 4.2.1: Graphical Comparison of Linear and Exponential Growth
In this graph, the blue straight line represents linear growth and the red curved line represents exponential growth.



Example 4.2.1: City Growth

A city is growing at a rate of 1.6% per year. The initial population in 2010 is $P_0 = 125,000$. Calculate the city's population over the next few years.

The relative growth rate is 1.6%. This means an additional 1.6% is added on to 100% of the population that already exists each year. This is a factor of 101.6%.

Population in 2011 = $125,000(1.016)^1 = 127,000$

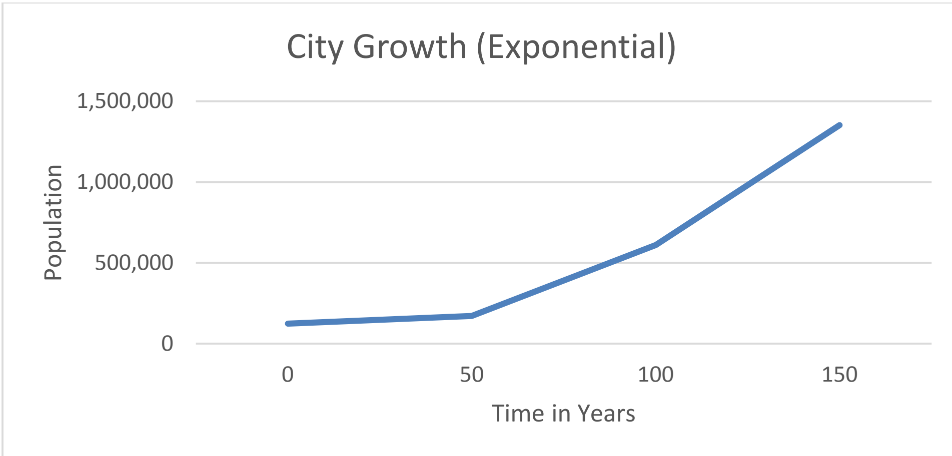
Population in 2012 = $127,000(1.016) = 125,000(1.016)^2 = 129,032$

Population in 2013 = $129,032(1.016) = 125,000(1.016)^3 = 131,097$

We can create an equation for the city's growth. Each year the population is 101.6% more than the previous year.

$P(t) = 125,000(1+0.016)^t$

Figure 4.2.2: Graph of City Growth



Note: The graph of the city growth follows an exponential growth model.

Example 4.2.2: A Shrinking Population

St. Louis, Missouri has declined in population at a rate of 1.6 % per year over the last 60 years. The population in 1950 was 857,000. Find the population in 2014. (Wikipedia, n.d.)

$P_0 = 857,000$

The relative growth rate is 1.6%. This means 1.6% of the population is subtracted from 100% of the population that already exists each year. This is a factor of 98.4%.

Population in 1951 = 857,000(0.984)^1 = 843,288

Population in 1952 = 843,228(0.984) = 857,000(0.984)^2 = 829,795

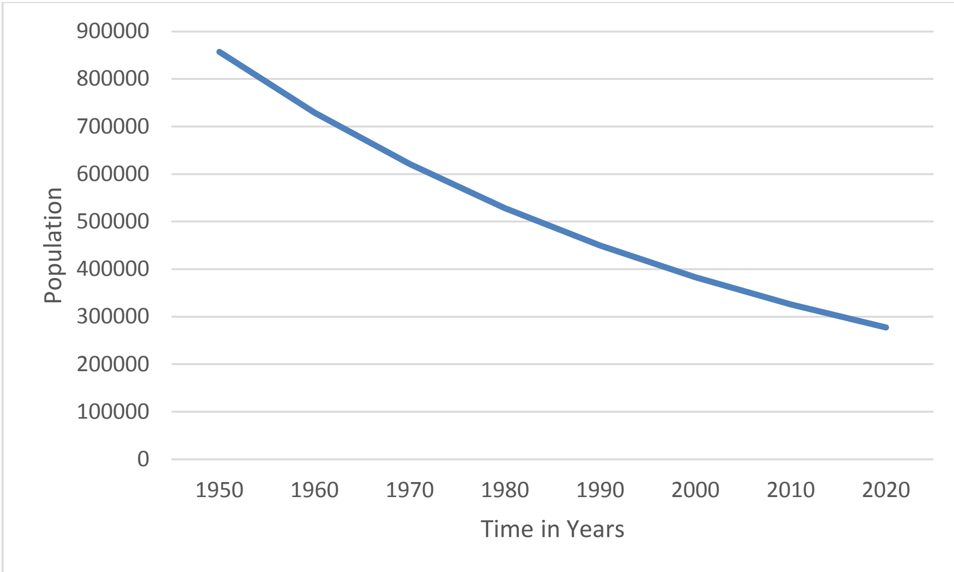
Population in 1953 = 828,795(0.984) = 857,000(0.984)^3 = 816,519

We can create an equation for the city's growth. Each year the population is 1.6% less than the previous year. $P(t) = 857,000(1-0.016)^t$

So the population of St. Louis Missouri in 2014, when $t = 64$, is:

$$\begin{aligned} P(64) &= 857,000(1-0.016)^{64} \\ &= 857,000(0.984)^{64} \\ &= 305,258 \end{aligned}$$

Figure 4.2.3: Graph of St. Louis, Missouri Population Decline



Note: The graph of the population of St. Louis, Missouri over time follows a declining exponential growth model.

Exponential Growth Model $P(t) = P_0(1+r)^t$

P_0 is the initial population,

r is the relative growth rate.

t is the time unit.

r is positive if the population is increasing and negative if the population is decreasing

Example 4.2.3: Inflation

The average inflation rate of the U.S. dollar over the last five years is 1.7% per year. If a new car cost $18,000 five years ago, how much would it cost today? (U.S. Inflation Calculator, n.d.)
To solve this problem, we use the exponential growth model with *r* = *1*.7%.

$P_0 = 18,000$ and $t = 5$

$P(t) = 18,000(1+0.017)^t$

$P(5)=18,000(1+0.017)^{5}=19,582.91$

This car would cost $\$19,582.91$ today.

Example 4.2.4: Ebola Epidemic in Sierra Leone

In May of 2014 there were 15 cases of Ebola in Sierra Leone. By August, there were 850 cases. If the virus is spreading at the same rate (exponential growth), how many cases will there be in February of 2015? (McKenna, 2014)

To solve this problem, we have to find three things; the growth rate per month, the exponential growth model, and the number of cases of Ebola in February 2015. First calculate the growth rate per month. To do this, use the initial population $P_0 = 15$, in May 2014. Also, in August, three months later, the number of cases was 850 so, $P(3) = 850$.

Use these values and the exponential growth model to solve for *r*.

$P(t) = P_0(1+r)^t$

$850 = 15(1+r)^3$

$56.67 = (1+r)^3$

$\sqrt[3]{56.67} = \sqrt[3]{(1+r)^3}$

$3.84 = 1+r$

$2.84 = r$

The growth rate is 284% per month. Thus, the exponential growth model is:

$P(t) = 15(1+2.84)^t = 15(3.84)^t$

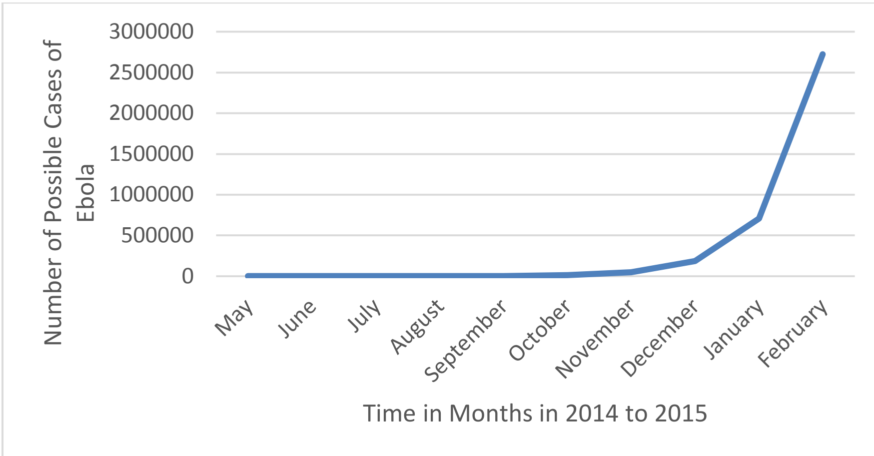
Now, we use this to calculate the number of cases of Ebola in Sierra Leone in February 2015, which is 9 months after the initial outbreak so, $t = 9$.

$P(9) = 15(3.84)^9 = 2,725,250$

If this same exponential growth rate continues, the number of Ebola cases in Sierra Leone in February 2015 would be 2,725,250.

This is a bleak prediction for the community of Sierra Leone. Fortunately, the growth rate of this deadly virus should be reduced by the world community and World Health Organization by providing the needed means to fight the initial spread.

Figure 4.2.4: Graph of Ebola Virus is Sierra Leone



Note: The graph of the number of possible Ebola cases in Sierra Leone over time follows an increasing exponential growth model.

Example 4.2.5: Population Decline in Puerto Rico

According to a new forecast, the population of Puerto Rico is in decline. If the population in 2010 is 3,978,000 and the prediction for the population in 2050 is 3,697,000, find the annual percent decrease rate. (Bloomberg Businessweek, n.d.) To solve this problem we use the exponential growth model**.** We need to solve for *r*.

$$P(t) = P_0(1+r)^t \text{ where t = 40 years}$$
$$P(40) = 3,697,000 \text{ and } P_0 = 3,978,000$$
$$3,697,000 = 3,978,000(1+r)^{40}$$
$$0.92936 = (1+r)^{40}$$
$$\sqrt[40]{0.92936} = \sqrt[40]{(1+r)^{40}}$$
$$0.99817 = 1+r$$
$$-0.0018 = r$$

The annual percent decrease is 0.18%.

Section 4.3: Special Cases: Doubling Time and Half-Life

Example 4.3.1: April Fool's Joke

Let's say that on April 1st I say I will give you a penny, on April 2nd two pennies, four pennies on April 3rd, and that I will double the amount each day until the end of the month. How much money would I have agreed to give you on April 30? With $P_0 = \$0.01$, we get the following table:

Table 4.3.1: April Fool's Joke

Day	Dollar Amount
April 1 = P_0	0.01
April 2 = P_1	0.02
April 3 = P_2	0.04
April 4 = P_3	0.08
April 5 = P_4	0.16
April 6 = P_5	0.32
….	….
April 30 = P_{29}	?

In this example, the money received each day is 100% more than the previous day. If we use the exponential growth model $P(t) = P_0(1+r)^t$ with r = 1, we get the doubling time model.

$P(t) = P_0(1+1)^t = P_0(2)^t$

We use it to find the dollar amount when $t = 29$ which represents April 30

$P(29) = 0.01(2)^{29} = \$5,368,709.12$

Surprised?

That is a lot of pennies.

Doubling Time Model:

Example 4.3.2: E. coli Bacteria

A water tank up on the San Francisco Peaks is contaminated with a colony of 80,000 E. coli bacteria. The population doubles every five days. We want to find a model for the population of bacteria present after *t* days. The amount of time it takes the population to double is five days, so this is our time unit. After *t* days have passed, then $\frac{t}{5}$ is the number of time units that have passed. Starting with the initial amount of 80,000 bacteria, our doubling model becomes:

$$P(t) = 80,000(2)^{\frac{t}{5}}$$

Using this model, how large is the colony in two weeks' time? We have to be careful that the units on the times are the same; 2 weeks = 14 days.

$$P(14) = 80,000(2)^{\frac{14}{5}} = 557,152$$

The colony is now 557,152 bacteria.

Doubling Time Model: If D is the doubling time of a quantity (the amount of time it takes the quantity to double) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0(2)^{\frac{t}{D}}$

Example 4.3.3: Flies

The doubling time of a population of flies is eight days. If there are initially 100 flies, how many flies will there be in 17 days? To solve this problem, use the doubling time model with $D = 8$ and $P_0 = 100$ so the doubling time model for this problem is:

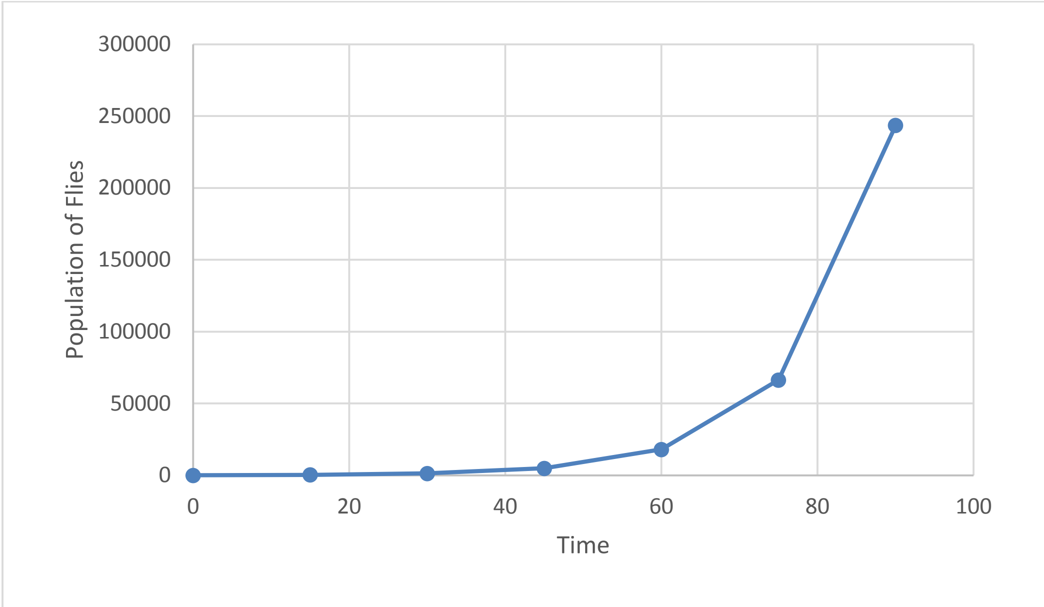
$$P(t) = 100(2)^{\frac{t}{8}}$$

When t = 17 days,

$$P(17) = 100(2)^{\frac{17}{8}} = 436$$

There are 436 flies after 17 days.

Figure 4.3.2: Graph of Fly Population in Three Months



Note: The population of flies follows an exponential growth model.

Sometimes we want to solve for the length of time it takes for a certain population to grow given their doubling time. To solve for the exponent, we use the *log* button on the calculator.

Example 4.3.4: Bacteria Growth

Suppose that a bacteria population doubles every six hours. If the initial population is 4000 individuals, how many hours would it take the population to increase to 25,000?

$P_0 = 4000$ and $D = 6$, so the doubling time model for this problem is:

$P(t) = 4000(2)^{\frac{t}{6}}$ Now, find t when $P(t) = 25,000$

$25,000 = 4000(2)^{\frac{t}{6}}$

$$\frac{25,000}{4000} = \frac{\cancel{4000}(2)^{\frac{t}{6}}}{\cancel{4000}}$$

$6.25 = (2)^{\frac{t}{6}}$ Now, take the log of both sides of the equation.

$\log 6.25 = \log(2)^{\frac{t}{6}}$ The exponent comes down using rules of logarithms.

$\log 6.25 = \left(\frac{t}{6}\right)\log 2$ Now, calculate $\log 6.25$ and $\log 2$ with your calculator.

$0.7959 = \left(\frac{t}{6}\right) \times 0.3010$

$\frac{0.7959}{0.3010} = \frac{t}{6}$

$2.644 = \frac{t}{6}$

$t = 15.9$

The population would increase to 25,000 bacteria in approximately 15.9 hours.

Rule of 70:

There is a simple formula for approximating the doubling time of a population. It is called the rule of 70 and it is an approximation for growth rates less than 15%. Do not use this formula if the growth rate is 15% or greater.

> **Rule of 70**: For a quantity growing at a constant percentage rate (not written as a decimal), *R*, per time period, the doubling time is approximately given by
> $\text{Doubling time D} \approx \frac{70}{R}$

Example 4.3.5: Bird Population

A bird population on a certain island has an annual growth rate of 2.5% per year. Approximate the number of years it will take the population to double. If the initial population is 20 birds, use it to find the bird population of the island in 17 years.

To solve this problem, first approximate the population doubling time.

$\text{Doubling time } D \approx \frac{70}{2.5} = 28$ years.

With the bird population doubling in 28 years, we use the doubling time model to find the population is 17 years.

$P(t) = 20(2)^{\frac{t}{28}}$

When $t = 17$ years

$P(16) = 20(2)^{\frac{17}{28}} = 30.46$

There will be 30 birds on the island in 17 years.

Example 4.3.6: Cancer Growth Rate

A certain cancerous tumor doubles in size every six months. If the initial size of the tumor is four cells, how many cells will there in three years? In seven years?

To calculate the number of cells in the tumor, we use the doubling time model. Change the time units to be the same. The doubling time is six months = 0.5 years.

$P(t) = 4(2)^{\frac{t}{0.5}}$

When $t = 3$ years

$P(3) = 4(2)^{\frac{3}{0.5}} = 256$ cells

When $t = 7$ years

$P(7) = 4(2)^{\frac{7}{0.5}} = 65,536$ cells

Example 4.3.7: Approximating Annual Growth Rate

Suppose that a certain city's population doubles every 12 years. What is the approximate annual growth rate of the city?

By solving the doubling time model for the growth rate, we can solve this problem.

$D \approx \frac{70}{R}$

$R \bullet D \approx \frac{70}{\cancel{R}} \bullet \cancel{R}$

$RD \approx 70$

$\frac{R\cancel{D}}{\cancel{D}} \approx \frac{70}{D}$

Annual growth rate $R \approx \frac{70}{D}$

$R = \frac{70}{12} = 5.83\%$

The annual growth rate of the city is approximately 5.83%

Exponential Decay and Half-Life Model:

The half-life of a material is the time it takes for a quantity of material to be cut in half. This term is commonly used when describing radioactive metals like uranium or plutonium. For example, the half-life of carbon-14 is 5730 years.

If a substance has a half-life, this means that half of the substance will be gone in a unit of time. In other words, the amount decreases by 50% per unit of time. Using the exponential growth model with a decrease of 50%, we have

$$P(t) = P_0(1-0.5)^t = P_0(0.5)^t = P_0\left(\frac{1}{2}\right)^t$$

Example 4.3.8: Half-Life

Let's say a substance has a half-life of eight days. If there are 40 grams present now, how much is left after three days? We want to find a model for the quantity of the substance that remains after *t* days. The amount of time it takes the quantity to be reduced by half is eight days, so this is our time unit. After *t* days have passed, then $\frac{t}{8}$ is the number of time units that have passed. Starting with the initial amount of 40, our half-life model becomes:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{8}}$$

With $t = 3$

$$P(3) = 40\left(\frac{1}{2}\right)^{\frac{3}{8}} = 30.8$$

There are 30.8 grams of the substance remaining after three days.

Half-Life Model: If H is the half-life of a quantity (the amount of time it takes the quantity be cut in half) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$

Example 4.3.9: Lead-209

Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 40 milligrams of this isotope is created in an experiment, how much is left after 14 hours? Use the half-life model to solve this problem.

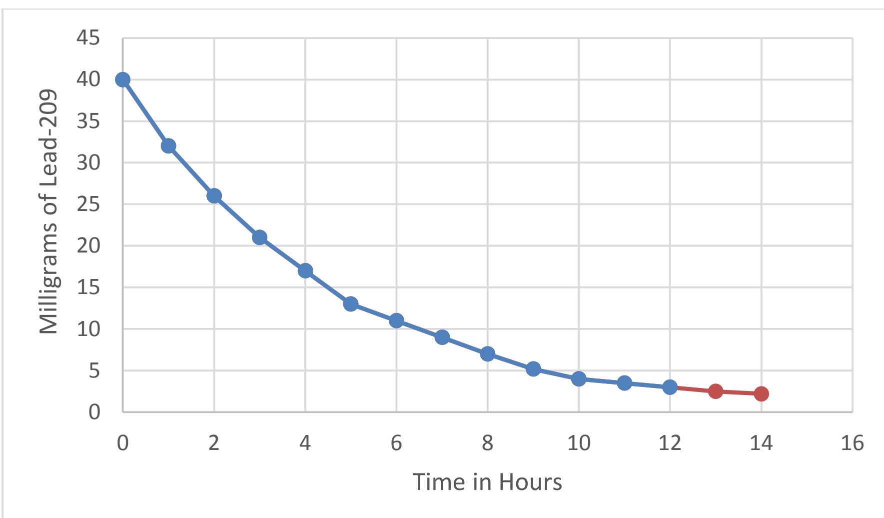
$P_0 = 40$ and $H = 3.3$, so the half-life model for this problem is:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{3.3}} \text{ when } t = 14 \text{ hours,}$$

$$P(14) = 40\left(\frac{1}{2}\right)^{\frac{14}{3.3}} = 2.1$$

There are 2.1 milligrams of Lead-209 remaining after 14 hours.

Figure 4.3.3: Lead-209 Decay Graph



Note: The milligrams of Lead-209 remaining follows a decreasing exponential growth model.

Example 4.3.10: Nobelium-259

Nobelium-259 has a half-life of 58 minutes. If you have 1000 grams, how much will be left in two hours? We solve this problem using the half-life model. Before we begin, it is important to note the time units. The half-life is given in minutes and we want to know how much is left in two hours. Convert hours to minutes when using the model: two hours = 120 minutes.

$P_0 = 1000$ and $H = 58$ minutes, so the half-life model for this problem is:

$$P(t) = 1000\left(\frac{1}{2}\right)^{\frac{t}{58}}$$

When $t = 120$ minutes,

$$P(120) = 1000\left(\frac{1}{2}\right)^{\frac{120}{58}} = 238.33$$

There are 238 grams of Nobelium-259 is remaining after two hours.

Example 4.3.11: Carbon-14

Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls. In this problem, we want to estimate the age of the scrolls. In 1947, 76% of the carbon-14 remained. This means that the amount remaining at time t divided by the original amount of carbon-14, P_0, is equal to 76%. So, $\frac{P(t)}{P_0} = 0.76$ we use this fact to solve for *t*.

$$P(t) = P_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\frac{P(t)}{P_0} = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$0.76 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$ Now, take the log of both sides of the equation.

$\log 0.76 = \log\left(\frac{1}{2}\right)^{\frac{t}{5730}}$ The exponent comes down using rules of logarithms.

$\log 0.76 = \left(\frac{t}{5730}\right)\log\frac{1}{2}$ Now, calculate $\log 0.76$ and $\log\frac{1}{2}$ with your calculator.

$$-0.1192 = \left(\frac{t}{5730}\right)\times(-0.3010)$$

$$\frac{-0.1192}{-0.3010} = \frac{t}{5730}$$

$$0.3960 = \frac{t}{5730}$$

$t = 2269.08$. The Dead Sea Scrolls are well over 2000 years old.

Example 4.3.12: Plutonium

Plutonium has a half-life of 24,000 years. Suppose that 50 pounds of it was dumped at a nuclear waste site. How long would it take for it to decay into 10 lbs?

$P_0 = 50$ and $H = 24,000$, so the half-life model for this problem is:

$$P(t) = 50\left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

Now, find t when $P(t) = 10$.

$$10 = 50\left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

$$\frac{10}{50} = \left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

$0.2 = \left(\frac{1}{2}\right)^{\frac{t}{24,000}}$ Now, take the log of both sides of the equation.

$\log 0.2 = \log\left(\frac{1}{2}\right)^{\frac{t}{24,000}}$ The exponent comes down using rules of logarithms.

$\log 0.2 = \left(\frac{t}{24,000}\right)\log\frac{1}{2}$ Now, calculate $\log 0.2$ and $\log\frac{1}{2}$ with your calculator.

$$-0.6990 = \left(\frac{t}{24,000}\right)\times(-0.3010)$$

$$\frac{-0.6990}{-0.3010} = \frac{t}{24,000}$$

$$2.322 = \frac{t}{24,000}$$

$$t = 55,728$$

The quantity of plutonium would decrease to 10 pounds in approximately 55,728 years.

Rule of 70 for Half-Life:

There is simple formula for approximating the half-life of a population. It is called the rule of 70 and is an approximation for decay rates less than 15%. Do not use this formula if the decay rate is 15% or greater.

Rule of 70: For a quantity decreasing at a constant percentage (not written as a decimal), R, per time period, the half-life is approximately given by:

$$\text{Half-life } H \approx \frac{70}{R}$$

Example 4.3.13: Elephant Population

The population of wild elephants is decreasing by 7% per year. Approximate the half -life for this population. If there are currently 8000 elephants left in the wild, how many will remain in 25 years?
To solve this problem, use the half-life approximation formula.

$$\text{Half-life } \mathrm{H} \approx \frac{70}{7} = 10 \text{ years}$$

$P_0 = 7000,\ H = 10$ years, so the half-life model for this problem is:

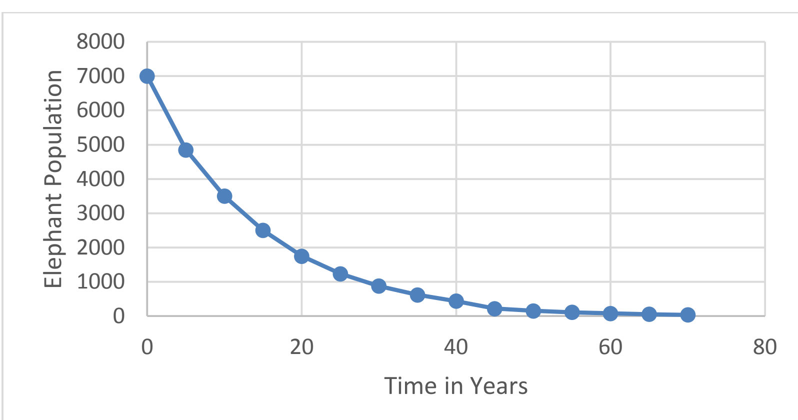
$$P(t) = 7000\left(\frac{1}{2}\right)^{\frac{t}{10}}$$

When $t = 25$,

$$P(25) = 7000\left(\frac{1}{2}\right)^{\frac{25}{10}} = 1237.4$$

There will be approximately 1237 wild elephants left in 25 years.

Figure 4.3.4: Elephant Population over a 70 Year Span.



Note: The population of elephants follows a decreasing exponential growth model.

Review of Exponent Rules and Logarithm Rules	
Rules of Exponents	**Rules of Logarithm for the Common Logarithm (Base 10)**
Definition of an Exponent $a^n = a \cdot a \cdot a \cdot a \cdot ..... \cdot a$ (n a's multiplied together)	**Definition of a Logarithm** $10^y = x$ if and only if $\log x = y$
Zero Rule $a^0 = 1$	
Product Rule $a^m \cdot a^n = a^{m+n}$	**Product Rule** $\log(xy) = \log x + \log y$
Quotient Rule $\dfrac{a^m}{a^n} = a^{m-n}$	**Quotient Rule** $\log\left(\dfrac{x}{y}\right) = \log x - \log y$
Power Rule $\left(a^n\right)^m = a^{n \cdot m}$	**Power Rule** $\log x^r = r \log x \quad (x > 0)$
Distributive Rules $(ab)^n = a^n b^n, \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\log 10^x = x \log 10 = x$
Negative Exponent Rules $a^{-n} = \dfrac{1}{a^n}, \left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$10^{\log x} = x \quad (x > 0)$

Section 4.4: Natural Growth and Logistic Growth

In this chapter, we have been looking at linear and exponential growth. Another very useful tool for modeling population growth is the natural growth model. This model uses base *e*, an irrational number, as the base of the exponent instead of $(1+r)$. You may remember learning about *e* in a previous class, as an exponential function and the base of the natural logarithm.

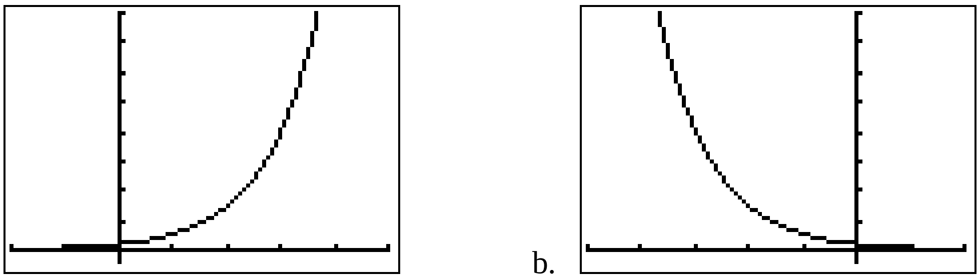
The Natural Growth Model: $P(t) = P_0 e^{kt}$ where P_0 is the initial population, k is the growth rate per unit of time, and t is the number of time periods.

Given $P_0 > 0$, if *k > 0*, this is an exponential growth model, if *k < 0*, this is an exponential decay model.

Figure 4.4.1: Natural Growth and Decay Graphs

a. Natural growth function $P(t) = e^t$

b. Natural decay function $P(t) = e^{-t}$



a. b.

Example 4.4.1: Drugs in the Bloodstream

When a certain drug is administered to a patient, the number of milligrams remaining in the bloodstream after *t* hours is given by the model $P(t) = 40e^{-.25t}$

How many milligrams are in the blood after two hours?

To solve this problem, we use the given equation with $t = 2$

$$P(2) = 40e^{-0.25(2)}$$
$$P(2) = 24.26$$

There are approximately 24.6 milligrams of the drug in the patient's bloodstream after two hours.

In the next example, we can see that the exponential growth model does not reflect an accurate picture of population growth for natural populations.

Example 4.4.2: Ants in the Yard

Bob has an ant problem. On the first day of May, Bob discovers he has a small red ant hill in his back yard, with a population of about 100 ants. If conditions are just right red ant colonies have a growth rate of 240% per year during the first four years. If Bob does nothing, how many ants will he have next May? How many in five years?

We solve this problem using the natural growth model.

$P(t)=100e^{2.4t}$

In one year, $t=1$, we have

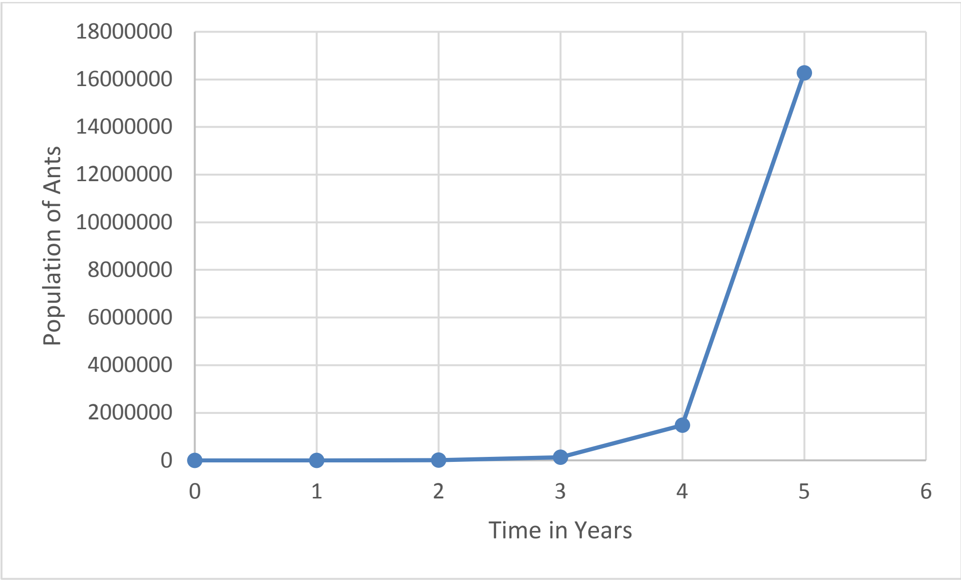
$P(1)=100e^{2.4(1)}=1102$ ants

In five years, $t=5$, we have

$P(5)=100e^{2.4(5)}=16,275,479$ ants

That is a lot of ants! Bob will not let this happen in his back yard!

Figure 4.4.2: Graph of Ant Population Growth in Bob's Yard.



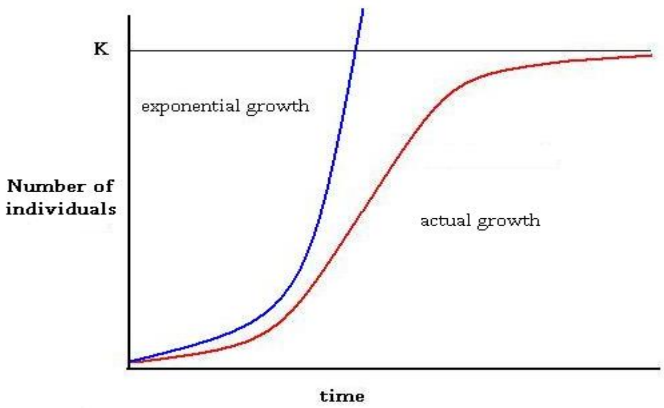
Note: The population of ants in Bob's back yard follows an exponential (or natural) growth model.

The problem with exponential growth is that the population grows without bound and, at some point, the model will no longer predict what is actually happening since the amount of resources available is limited. Populations cannot continue to grow on a purely physical level, eventually death occurs and a limiting population is reached.

Another growth model for living organisms in the logistic growth model. The logistic growth model has a maximum population called the carrying capacity. As the population grows, the number of individuals in the population grows to the carrying capacity and stays there. This is the maximum population the environment can sustain.

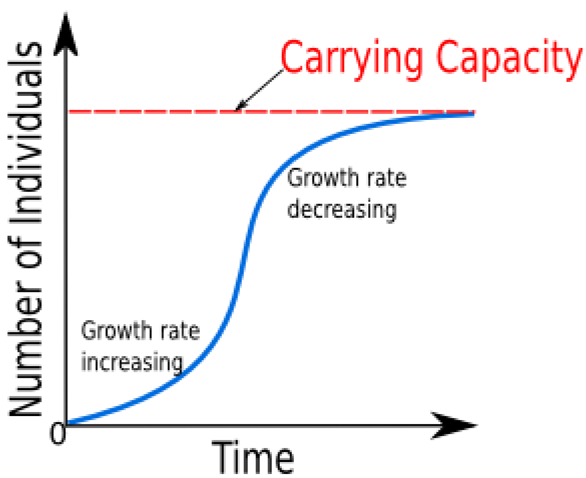
Logistic Growth Model: $P(t) = \dfrac{M}{1+ke^{-ct}}$ where *M*, *c*, and *k* are positive constants and *t* is the number of time periods.

Figure 4.4.3: Comparison of Exponential Growth and Logistic Growth
The horizontal line K on this graph illustrates the carrying capacity. *However, this book uses M to represent the carrying capacity rather than K.*



(Logistic Growth Image 1, n.d.)

Figure 4.4.4: Logistic Growth Model



(Logistic Growth Image 2, n.d.)

The graph for logistic growth starts with a small population. When the population is small, the growth is fast because there is more elbow room in the environment. As the population approaches the carrying capacity, the growth slows.

Example 4.4.3: Bird Population

The population of an endangered bird species on an island grows according to the logistic growth model.

$$P(t) = \frac{3640}{1+25e^{-0.04t}}$$

Identify the initial population. What will be the bird population in five years? What will be the population in 150 years? What will be the population in 500 years?

We know the initial population, P_0, occurs when $t = 0$.

$$P_0 = P(0) = \frac{3640}{1+25e^{-0.04(0)}} = \frac{3640}{1+25e^{-0.04(0)}} = 140$$

Calculate the population in five years, when $t = 5$.

$$P(5) = \frac{3640}{1+25e^{-0.04(5)}} = \frac{3640}{1+25e^{-0.04(5)}} = 169.6$$

The island will be home to approximately 170 birds in five years.

Calculate the population in 150 years, when $t = 150$.

$$P(150) = \frac{3640}{1+25e^{-0.04(150)}} = \frac{3640}{1+25e^{-0.04(150)}} = 3427.6$$

The island will be home to approximately 3428 birds in 150 years.

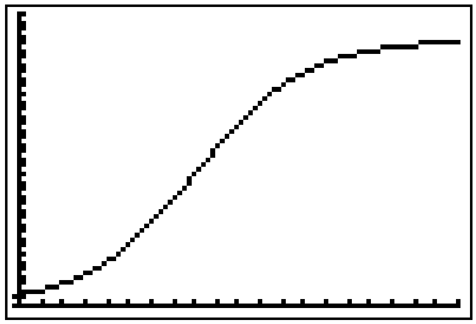
Calculate the population in 500 years, when $t = 500$.

$$P(500) = \frac{3640}{1+25e^{-0.04(500)}} = \frac{3640}{1+25e^{-0.04(500)}} = 3640.0$$

The island will be home to approximately 3640 birds in 500 years.

This example shows that the population grows quickly between five years and 150 years, with an overall increase of over 3000 birds; but, slows dramatically between 150 years and 500 years (a longer span of time) with an increase of just over 200 birds.

Figure 4.4.5: Bird Population over a 200-Year Span



Example 4.4.4: Student Population at Northern Arizona University

The student population at NAU can be modeled by the logistic growth model below, with initial population taken from the early 1960's. We will use 1960 as the initial population date.

$$P(t) = \frac{30,000}{1+5e^{-0.06t}}$$

Determine the initial population and find the population of NAU in 2014. What will be NAU's population in 2050? From this model, what do you think is the carrying capacity of NAU?

We solve this problem by substituting in different values of time.

When $t = 0$, we get the initial population P_0.

$$P_0 = P(0) = \frac{30,000}{1+5e^{-0.06(0)}} = \frac{30,000}{6} = 5000$$

The initial population of NAU in 1960 was 5000 students.

In the year 2014, 54 years have elapsed so, $t = 54$.

$$P(54) = \frac{30,000}{1+5e^{-0.06(54)}} = \frac{30,000}{1+5e^{-3.24}} = \frac{30,000}{1.19582} = 25,087$$

There are 25,087 NAU students in 2014.

In 2050, 90 years has elapsed so, $t = 90$.

$$P(90) = \frac{30,000}{1+5e^{-0.06(90)}} = \frac{30,000}{1+5e^{-5.4}} = 29,337$$

There are 29,337 NAU students in 2050.

Finally, to predict the carrying capacity, look at the population 200 years from 1960, when $t = 200$.

$$P(200) = \frac{30,000}{1+5e^{-0.06(200)}} = \frac{30,000}{1+5e^{-12}} = \frac{30,000}{1.00003} = 29,999$$

Thus, the carrying capacity of NAU is 30,000 students.

It appears that the numerator of the logistic growth model, M, is the carrying capacity.

Carrying Capacity: Given the logistic growth model $P(t) = \dfrac{M}{1+ke^{-ct}}$, the carrying capacity of the population is M. M, the carrying capacity, is the maximum population possible within a certain habitat.

Example 4.4.5: Fish Population

Suppose that in a certain fish hatchery, the fish population is modeled by the logistic growth model where t is measured in years.

$$P(t) = \frac{12,000}{1+11e^{-0.2t}}$$

What is the carrying capacity of the fish hatchery? How long will it take for the population to reach 6000 fish?

The carrying capacity of the fish hatchery is $M = 12,000$ fish.

Now, we need to find the number of years it takes for the hatchery to reach a population of 6000 fish. We must solve for t when $P(t) = 6000$.

$$6000 = \frac{12,000}{1+11e^{-0.2t}}$$

$$\left(1+11e^{-0.2t}\right) \bullet 6000 = \frac{12,000}{1+11e^{-0.2t}} \bullet \left(1+11e^{-0.2t}\right)$$

$$\left(1+11e^{-0.2t}\right)6000 = 12,000$$

$$\frac{\left(1+11e^{-0.2t}\right)\cancel{6000}}{\cancel{6000}}=\frac{12,000}{6000}$$

$$1+11e^{-0.2t}=2$$

$$11e^{-0.2t}=1$$

$e^{-0.2t}=\dfrac{1}{11}=0.090909$ Take the natural logarithm (ln on the calculator) of both sides of the equation.

$\ln e^{-0.2t}=\ln 0.090909$ $\quad$ $\ln e^{-0.2t}=-0.2t$ by the rules of logarithms.

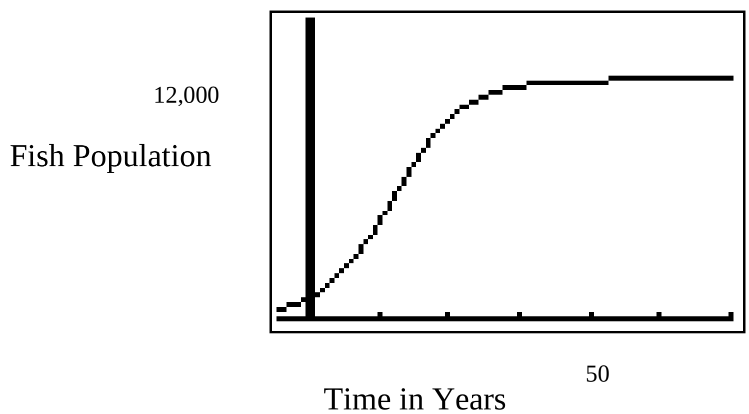
$$-0.2t=\ln 0.090909$$

$$t=\frac{\ln 0.090909}{-0.2}$$

$$t=11.999$$

It will take approximately 12 years for the hatchery to reach 6000 fish.

Figure 4.4.6: Fish Population over a 30-Year Period.



Chapter 4 Homework

1. Goodyear AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 72,800. The city's population grew by 3800 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Goodyear. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)

2. Gilbert AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 229,800. The city's population grew by 9200 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Gilbert. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)

3. Apache County AZ, is shrinking according to the census bureau. In 2012, the population was about 71,700. The county's population decreased by 1147 people from 2012 to 2013. If the decline is linear, create a population model for Apache County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

4. Cochise County AZ, is shrinking according to the census bureau. In 2012, the population was about 129,472. The county's population decreased by 2600 people from 2012 to 2013. If the decline is linear, create a population model for Cochise County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

5. Mohave County AZ, is shrinking according to the census bureau. In 2012, the population was about 203,030. The county's population decreased by 1200 people from 2012 to 2013. If the decline is linear, create a population model for Mohave County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

6. The 2012 Kia Sedona LX has one of the largest depreciation values of any car. Suppose a 2012 Kia Sedona sold for $24,900, and its value depreciates by $3400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

7. The 2013 Chevy Impala has one of the largest depreciation values of any car. Suppose a 2013 Chevy Impala sold for $27,800 and its value depreciates by $3600 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

8. The 2013 Jaguar XJ AWD has one of the largest depreciation values of any car. Suppose a 2013 Jaguar XJ AWD sold for $74,500 and its value depreciates by $10,400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

9. The 2012 Jeep Liberty Limited Sport 2WD has one of the largest depreciation values of any car. Suppose a 2012 Jeep Liberty Limited Sport 2WD sold for $23,400 and its value depreciates by $3,040 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

10. Suppose that in January, the maximum water depth at Lake Powell, Arizona was 528 feet. The water evaporates at an average rate of 1.2 feet per month. Find a model for the rate at which the water evaporates. If it does not rain at all, what will be the depth of Lake Powell in May and in September?

11. Suppose that the maximum water depth at Lake Tahoe, California in 2014 was 1644 feet. Because of the drought, the water level has been decreasing at an average rate of 6.2 feet per year. Find a model for the rate at which the water level decreases. If it there is no precipitation at all, what will be the depth of Lake Tahoe be in two years, and in five years?

12. Suppose the homes in Arizona have appreciated an average of 8% per year in the last five years. If the average home in a suburb sold for $225,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

13. Suppose the homes in Massachusetts have appreciated an average of 13% per year in the last five years. If the average home in a suburb sold for $205,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

14. Suppose the homes in Michigan have depreciated an average of 17% per year in the last five years. If the average home in a suburb sold for $215,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

15. Suppose the homes in Nevada have depreciated an average of 15% per year in the last five years. If the average home in a suburb sold for $318,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

16. The cost of a home in Flagstaff AZ was $89,000 in 1992. In 2007, the same home appraised for $349,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?

17. The cost of a home in Bullhead City AZ was $109,000 in 1992. In 2007, the same home appraised for $352,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?

18. The population of West Virginia is in decline. The population in 2014 was 1,850,326 and the population had decreased by 0.14% from 2010. How many people were living in West Virginia in 2010? Create a model for this population. If the decline continues at this rate, how many people will reside in West Virginia in 2020? (Wikipedia, n.d.)

19. Assume that the population of Arizona grew by 2.4% per year between the years 2000 to 2010. The number of Native American living in Arizona was 257,426 in 2010. How many Native Americans were living in Arizona in 2000? Create a model for this population. If the growth continues at this rate, how many Native Americans will reside in Arizona in 2020?

20. Assume the population of the U.S. grew by 0.96% per year between the years 2000 and 2010. The number of Hispanic Americans was 55,740,000 in 2010. How many Hispanic Americans were living in the U.S. in 2000? Create a model for this population. If the growth continues at this rate, how many Hispanic Americans will reside in The U.S. in 2020?

21. Assume the population of Michigan decreased by 0.6% per year between the years 2000 to 2010. The population of Michigan was 9,970,000 in 2010. How many people were living in Michigan in 2000? Create a model for this population. If the growth continues at this rate, how many people will reside in Michigan in 2020?

22. The doubling time of a population of aphids is 12 days. If there are initially 200 aphids, how many aphids will there be in 17 days?

23. The doubling time of a population of rabbits is six months. If there are initially 26 rabbits, how many rabbits will there be is 17 months?

24. The doubling time of a population of shrews is three months. If there are initially 32 shrews, how many shrews will there be is 21 months?

25. The doubling time of a population of hamsters is 1.2 years. If there are initially 43 hamsters, how many hamsters will there be is 7 years?

26. A certain cancerous tumor doubles in size in three months. If the initial size of the tumor is two cells, how many months will it take for the tumor to grow to 60,000 cells? How many cells will there be in 1.5 years? In three years?

27. A certain cancerous tumor doubles in size in 1.5 months. If the initial size of the tumor is eight cells, how many months will it take for the tumor to grow to 40,000 cells? How many cells will there be in six months? In 2.5 years?

28. A bird population on a certain island has an annual growth rate of 1.5% per year. Approximate the number of years it will take the population to double. If the initial population is 130 birds, use it to find the bird population of the island in 14 years.

29. The beaver population on Kodiak Island has an annual growth rate of 1.2% per year. Approximate the number of years it will take the population to double. If the initial population is 32 beavers, use it to find the population of beavers on the island in 20 years.

30. The black-footed ferret population in Arizona has an annual growth rate of 0.5% per year. Approximate the number of years it will take the population to double. If the initial population is 230 ferrets, use it to find the ferret population in AZ in 12 years.

31. The Mexican gray wolf population in southern Arizona increased from 72 individuals to 92 individuals from 2012 to 2013. What is the annual growth rate? Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Mexican gray wolves in 10 years.

32. There is a small population of Sonoran pronghorn antelope in a captive breeding program on the Cabeza Prieta National Wildlife Refuge in southwest Arizona. Recently, the population increased from 122 individuals to 135 individuals from 2012 to 2013. Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Sonoran pronghorn antelope in 10 years.

33. Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 56 milligrams of this isotope is created in an experiment, how much is left after 12 hours?

34. Titanium-44 has a half-life of 63 years. If there is 560 grams of this isotope, how much is left after 1200 years?

35. Uranium-232 has a half-life of 68.9 years. If there is 160 grams of this isotope, how much is left after 1000 years?

36. Nickel-63 has a half-life of 100 years. If there is 16 grams of this isotope, how much is left after 145 years?

37. Radium-226 has a half-life of 1600 years. If there is 56 grams of this isotope, how much is left after 145,000 years?

38. The population of wild desert tortoises is decreasing by 72% per year. Approximate the half-life for this population. If there are currently 100,000 tortoises left in the wild, how many will remain in 20 years?

39. The population of pygmy owls is decreasing by 4.4% per year. Approximate the half-life for this population. If there are currently 41 owls left in the wild, how many will remain in five years?

40. Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 2004, carbonized plant remains were found where human artifacts were unearthed along the Savannah River in Allendale County. Analysis indicated that the plant remains contained 0.2% of their original carbon-14. Estimate the age of the plant remains. (Wikipedia, n.d.)

41. The population of a bird species on an island grows according to the logistic model below.
$$P(t) = \frac{43,240}{1+12e^{-0.05t}}$$
Identify the initial population. What will be the bird population in five years? In 150 years? In 500 years? What is the carrying capacity of the bird population on the island?

42. Suppose the population of students at Arizona State University grows according to the logistic model below with the initial population taken from 2009.
$$P(t) = \frac{85,240}{1+0.6e^{-0.15t}}$$
Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at ASU?

43. Suppose the population of students at Ohio State University grows according to the logistic model below with the initial population taken from 1970.
$$P(t) = \frac{90,000}{1+4e^{-0.06t}}$$
Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at OSU?

44. Suppose that in a certain shrimp farm, the shrimp population is modeled by the logistic model below where t is measured in years.

    $$P(t) = \frac{120,000}{1+25e^{-0.12t}}$$

    Find the initial population. How long will it take for the population to reach 6000 shrimp? What is the carrying capacity of shrimp on the farm?

45. Suppose that in a certain oyster farm, the oyster population is modeled by the logistic model below where *t* is measured in years.

    $$P(t) = \frac{18,000}{1+19e^{-0.36t}}$$

    Find the initial population. How long will it take for the population to reach 2000 oysters? What is the carrying capacity of oysters on the farm?

Chapter 5: Finance

Most adults have to deal with the financial topics in this chapter regardless of their job or income. Understanding these topics helps us to make wise decisions in our private lives as well as in any business situations.

Section 5.1: Basic Budgeting

Budgeting is an important step in managing your money and spending habits. To create a budget you need to identify how much money you are spending. Some expenses to keep in mind when creating a budget are rent, car payment, fuel, auto insurance, utilities, groceries, cell phone, personal, gym membership, entertainment, gifts, dining out, medical expenses, etc.

There are several apps out there that can help you budget your money. Just a few examples are Mint, Manilla, and Check. These are all free apps that help you keep track of bills and your accounts. Your bank also keeps track of your spending and what categories each item falls under. Log into your bank account online and look for "Track Spending" or a similar item. Many banks give you a pie chart showing you how much you spent in each category in the last month. You can edit your categories, change the number of months, and sometimes even set a budget goal.

Table 5.1.1: Example of Budget in Excel

When you are creating a monthly budget, many experts say if you want to have control of your money, you should know where every dollar is going. In order to keep track of this, a written budget is essential. Below is one example of a budget in Excel. This was a free template from the "Life After College" blog. There are hundreds of free templates out there so you should find the template that suits you the best – or create your own Excel budget!

Four-Step Budget Template		**Last Updated:**
Brought to you by Jenny Blake		{Insert date here}
Life After College Blog:	http://lifeaftercollege.org	
Life After College Book:	http://amzn.to/jennyblake	
Related post:	http://bit.ly/VuSJB	
Note: Enter amounts in Column B and the totals will automatically calculate.		
Step 1: Income		
This includes: paychecks, side jobs, anything that brings money into your bank account.		***Notes:***
Income Source: {Fill In Name}		
Income Source: {Fill In Name}		
Income Source: {Fill in Name}		
TOTAL	$0	
Step 2: Must-Have Expenses		
This includes: Rent, utilities, cell phone bills, anything that will incur late fees. Includes other essentials like groceries and automatic savings account deductions. Saving is a must!		**Notes:**
Rent or Mortgage		
Utilities		
Cell Phone Bill		
Savings 1		
Savings 2		
Other (add rows as needed)		
TOTAL	$0	
Step 3: Nice-to-Have Expenses		
This includes: things that you KNOW you spend money on every month like going out to eat. This does not include: one-off purchases (like a TV), major shopping trips, major travel (unless you take frequent weekend trips).		***Notes:***
Going out to eat (estimate)		
Fill in...		
TOTAL	$0	
Step 4: Allowance		
Subtract your total expenses from your income to get your allowance. This is the money left-over each month for you to spend as you'd like - shopping, weekend trips, etc. *For bigger purchases, you may want to start a separate savings account and add that deduction to your "must have" column.* *To get a weekend allowance, divide this number by four. If you're really serious, take the "weekend budget" amount out in cash to monitor your spending even more closely.*		
TOTAL	**$0**	

("Four-Step Budget Template," n.d.)

Example 5.1.1: Budgeting

You make \$32,000 a year and want to save 10% of your income every year. How much should you put into savings every month?

$$\$32,000 \cdot 0.10 = \$3200$$

You want to save \$3200 a year.

$$\frac{\$3200}{12} = \$266.67$$

You should be saving \$266.67 a month or \$133.33 a paycheck if you are paid biweekly.

Section 5.2: Simple Interest

Money is not free to borrow! We will refer to money in terms of **present value P,** which is an amount of money at the present time, and **future value F,** which is an amount of money in the future. Usually, if someone loans money to another person in present value, and are promised to be paid back in future value, then the person who loaned the money would like the future value to be more than the present value. That is because the value of money declines over time due to inflation. Therefore, when a person loans money, they will charge interest. They hope that the interest will be enough to beat inflation and make the future value more than the present value.

Simple interest is interest that is only calculated on the initial amount of the loan. This means you are paying the same amount of interest every year. An example of simple interest is when someone purchases a U.S. Treasury Bond.

Simple Interest: Interest that is only paid on the principal.

Simple Interest Formula: $F = P(1 + rt)$
where,
F = Future value
P = Present value
r = Annual percentage rate (APR) changed to a decimal
t = Number of years

Example 5.2.1: Simple Interest—Using a Table

Sue borrows $2000 at 5% annual simple interest from her bank. How much does she owe after five years?

Table 5.2.1: Simple Interest Using a Table

Year	Interest Earned	Total Balance Owed
1	$2000*.05 = $100	$2000 + $100 = $2100
2	$2000*.05 = $100	$2100 + $100 = $2200
3	$2000*.05 = $100	$2200 + $100 = $2300
4	$2000*.05 = $100	$2300 + $100 = $2400
5	$2000*.05 = $100	$2400 + $100 = $2500

After 5 years, Sue owes $2500.

Example 5.2.2: Simple Interest—Using the Formula

Chad got a student loan for $10,000 at 8% annual simple interest. How much does he owe after one year? How much interest will he pay for that one year?

$P = \$10{,}000$, $r = 0.08$, $t = 1$

$$\boldsymbol{F = P(1+rt)}$$

$$F = 10000(1+0.08(1)) = \$10,800$$

Chad owes $10,800 after one year. He will pay $10800 - $10000 = $800 in interest.

Example 5.2.3: Simple Interest—Finding Time

Ben wants to buy a used car. He has $3000 but wants $3500 to spend. He invests his $3000 into an account earning 6% annual simple interest. How long will he need to leave his money in the account to accumulate the $3500 he wants?

$F = \$3500$, $P = \$3000$, $r = 0.06$

$$\boldsymbol{F = P(1+rt)}$$

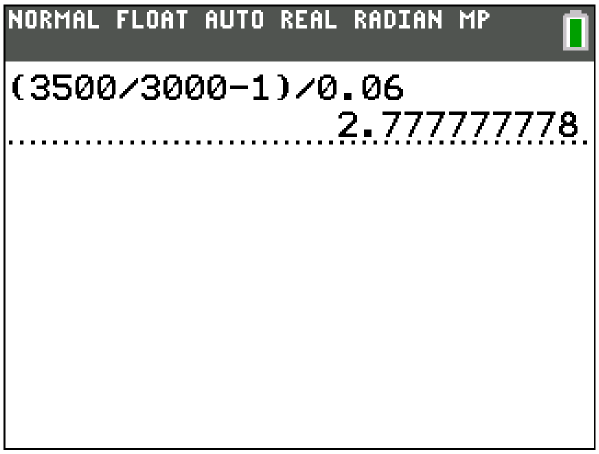
$$3500 = 3000\left(1+0.06t\right)$$

$$\frac{3500}{3000} = 1+0.06t$$

$$\frac{3500}{3000} - 1 = 0.06t$$

$$\frac{\frac{3500}{3000} - 1}{0.06} = t$$

Figure 5.2.2: Calculation to Find t on a TI 83/84 Calculator



$t \approx 2.8$ years

Ben would need to invest his $3000 for about 2.8 years until he would have $3500 to spend on a used car.

Note: As shown above, wait to round your answer until the very last step so you get the most accurate answer.

Section 5.3: Compound Interest

Most banks, loans, credit cards, etc. charge you compound interest, not simple interest. Compound interest is interest paid both on the original principal and on all interest that has been added to the original principal. Interest on a mortgage or auto loan is compounded monthly. Interest on a savings account can be compounded quarterly (four times a year). Interest on a credit card can be compounded weekly or daily!

Table 5.3.1: Compounding Periods

Compounding type	**Number of compounding periods per year**
Annually	1
Semiannually	2
Quarterly	4
Monthly	12
Daily	365

Compound Interest: Interest paid on the principal AND the interest accrued.

Example 5.3.1: Compound Interest—Using a Table

Suppose you invest $3000 into an account that pays you 7% interest per year for four years. Using compound interest, after the interest is calculated at the end of each year, then that amount is added to the total amount of the investment. Then the following year, the interest is calculated using the new total of the loan.

Table 5.3.2: Compound Interest Using a Table

Year	**Interest Earned**	**Total of Loan**
1	$3000*0.07 = $210	$3000 + $210 = $3210
2	$3210 *0.07 = $224.70	$3210 + $224.70 = $3434.70
3	$3434.70*0.07 = $240.43	$3434.70 + $240.43 = $3675.13
4	$3675.13 *0.07 = $257.26	$3675.13 + $257.26 = $3932.39
Total	$932.39	

So, after four years, you have earned $932.39 in interest for a total of $3932.39.

Compound Interest Formula: $F = P\left(1+\frac{r}{n}\right)^{n\cdot t}$ where,

F = Future value
P = Present value
r = Annual percentage rate (APR) changed into a decimal
t = Number of years
n = Number of compounding periods per year

Example 5.3.2: Comparing Simple Interest versus Compound Interest

Let's compare a savings plan that pays 6% simple interest versus another plan that pays 6% annual interest compounded quarterly. If we deposit $8,000 into each savings account, how much money will we have in each account after three years?

6% Simple Interest: P = \$8,000, r = 0.06, t = 3

$$F = P(1 + rt)$$

$$F = 8000(1 + 0.06 \cdot 3)$$

$$F = 9440$$

Thus, we have $9440.00 in the simple interest account after three years.

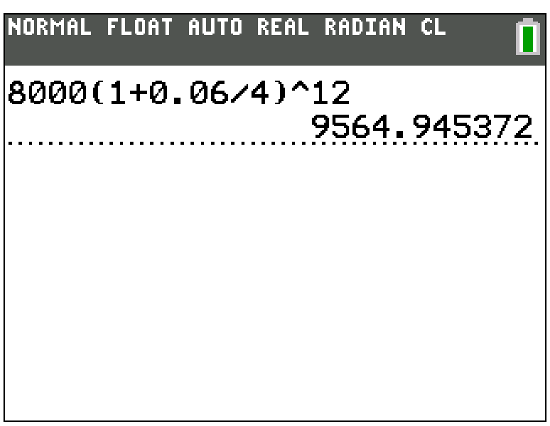
6% Interest Compounded Quarterly: P = $8,000, r = 0.06, t = 3, n=4

$$F = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$$

$$F = 8000\left(1 + \frac{0.06}{4}\right)^{4 \cdot 3}$$

$$F = 8000\left(1 + \frac{0.06}{4}\right)^{12}$$

Figure 5.3.3: Calculation for F for Example 5.3.2



$$F = 9564.95$$

So, we have $9564.95 in the compounded quarterly account after three years.

With simple interest we earn \$1440.00 on our investment, while with compound interest we earn \$1564.95.

Example 5.3.3: Compound Interest—Compounded Monthly

In comparison with Example 5.3.2 consider another account with 6% interest compounded monthly. If we invest \$8000 in this account, how much will there be in the account after three years?

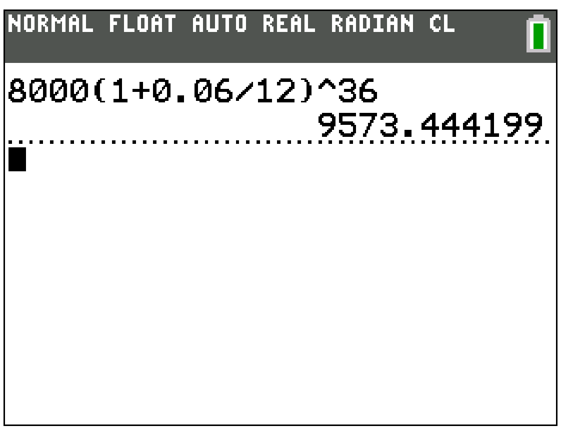
$P = \$8,000$, $r = 0.06$, $t = 3$, $n = 12$

$$F = P\left(1+\frac{r}{n}\right)^{n\cdot t}$$

$$F = 8000\left(1+\frac{0.06}{12}\right)^{12\cdot 3}$$

$$F = 8000\left(1+\frac{0.06}{12}\right)^{36}$$

Figure 5.3.4: Calculation for F for Example 5.3.3



$$F = 9573.44$$

Thus, we will have \$9573.44 in the compounded monthly account after three years.

Interest compounded monthly earns you \$9573.44 - \$9564.95 = \$8.49 more than interest compounded quarterly.

Example 5.3.4: Compound Interest—Savings Bond

Sophia's grandparents bought her a savings bond for \$200 when she was born. The interest rate was 3.28% compounded semiannually, and the bond would mature in 30 years. How much will Sophia's bond be worth when she turns 30?

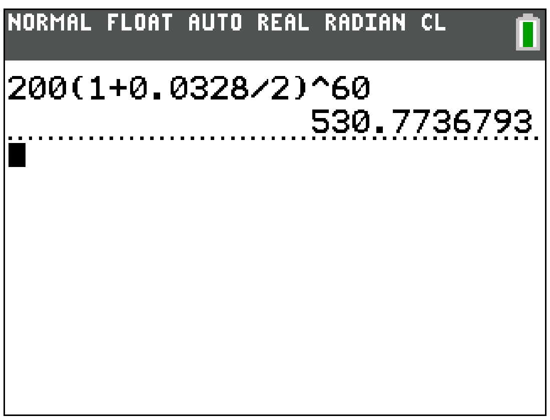
$P = \$200$, $r = 0.0328$, $t = 30$, $n = 2$

$$F = P\left(1+\frac{r}{n}\right)^{n\cdot t}$$

$$F = 200\left(1+\frac{0.0328}{2}\right)^{2\cdot 30}$$

$$F = 200\left(1+\frac{0.0328}{2}\right)^{60}$$

Figure 5.3.5: Calculation for F for Example 5.3.4



Sophia's savings bond will be worth \$530.77 after 30 years.

Continuous Compounding: Interest is compounded infinitely many times per year.

Continuous Compounding Interest Formula: $F = Pe^{rt}$
where,
F = Future value
P = Present value
r = Annual percentage rate (APR) changed into a decimal
t = Number of years

Example 5.3.5: Continuous Compounding Interest

Isabel invested her inheritance of $100,000 into an account earning 5.7% interest compounded continuously for 20 years. What will her balance be after 20 years?

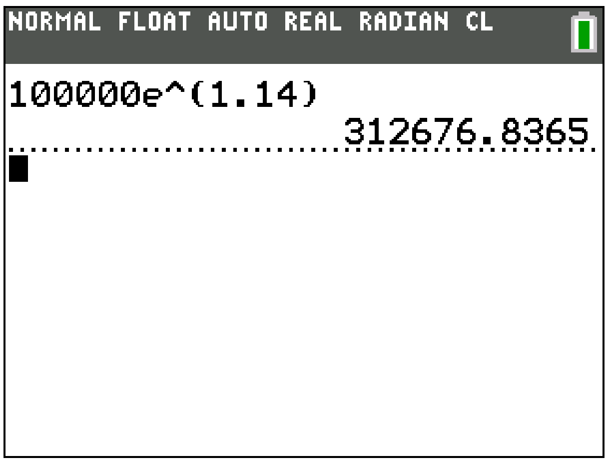
P = \$100,000, r = 0.057, t = 20

$$F = Pe^{rt}$$

$$F = 100000e^{0.057 \cdot 20}$$

$$F = 100000e^{1.14}$$

Figure 5.3.6: Calculation for F for Example 5.3.5



$$F = 312,676.84$$

Isabel’s balance will be \$312,676.84 after 20 years.

Annual Percentage Yield (APY): the actual percentage by which a balance increases in one year.

Example 5.3.6: Annual Percentage Yield (APY)

Find the Annual Percentage Yield for an investment account with
a. 7.7% interest compounded monthly
b. 7.7% interest compounded daily
c. 7.7% interest compounded continuously.

To find APY, it is easiest to examine an investment of \$1 for one year.

a. P = \$1, r = 0.077, t = 1, n = 12

$$F = 1\left(1+\frac{0.077}{12}\right)^{12\cdot 1} = 1.079776$$

The percentage the \$1 was increased was 7.9776%. The APY is 7.9776%.

b. P = \$1, r = 0.077, t = 1, n = 365

$$F = 1\left(1+\frac{0.077}{365}\right)^{365\cdot 1} = 1.080033$$

The percentage the \$1 was increased was 8.0033%. The APY is 8.0033%.

c. P = \$1, r = 0.077, t = 1

$$F = 1e^{0.0077\cdot 1} = 1.080042$$

The percentage the \$1 was increased was 8.0042%. The APY is 8.0042%.

Section 5.4: Savings Plans

Sometimes it makes better financial sense to put small amounts of money away over time to purchase a large item instead of taking out a loan with a high interest rate. When looking at depositing money into a savings account on a periodic basis we need to use the savings plan formula.

Savings Plan Formula: $F = PMT\left[\dfrac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$

where,
F = Future value
PMT = Periodic payment
r = Annual percentage rate (APR) changed to a decimal
t = Number of years
n = Number of payments made per year

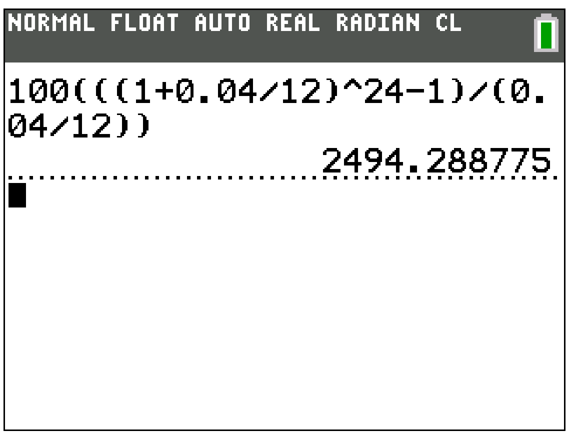
Example 5.4.1: Savings Plan—Vacation

Henry decides to save up for a big vacation by depositing \$100 every month into an account earning 4% per year. How much money will he have at the end of two years? *PMT* = \$100, *r* = .04, *t* = 2, *n* = 12

$$F = PMT\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

$$F = 100\left[\frac{\left(1+\frac{0.04}{12}\right)^{12\cdot 2}-1}{\frac{0.04}{12}}\right]$$

Figure 5.4.1: Calculation for F for Example 5.4.1



$F = 2494.29$

Henry will have \$2,494.29 for his vacation.

For some problems, you will have to find the payment instead of the future value. In that case, it is helpful to just solve the savings plan formula for *PMT*. Since *PMT* is multiplied by a fraction, to solve for *PMT*, you can just multiply both sides of the formula by that fraction. You should just think of the savings plan formula in two different forms, one solving for future value, *F*, and one solving for payment, *PMT*.

$$F = PMT\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

$$F \bullet \left[\frac{\frac{r}{n}}{\left(1+\frac{r}{n}\right)^{nt}-1}\right] = PMT\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right] \bullet \left[\frac{\frac{r}{n}}{\left(1+\frac{r}{n}\right)^{nt}-1}\right]$$

$$F \bullet \left[\frac{\frac{r}{n}}{\left(1+\frac{r}{n}\right)^{nt}-1}\right] = PMT$$

$$PMT = F \bullet \left[\frac{\frac{r}{n}}{\left(1+\frac{r}{n}\right)^{nt}-1}\right]$$

Example 5.4.2: Savings Plan—Finding Payment

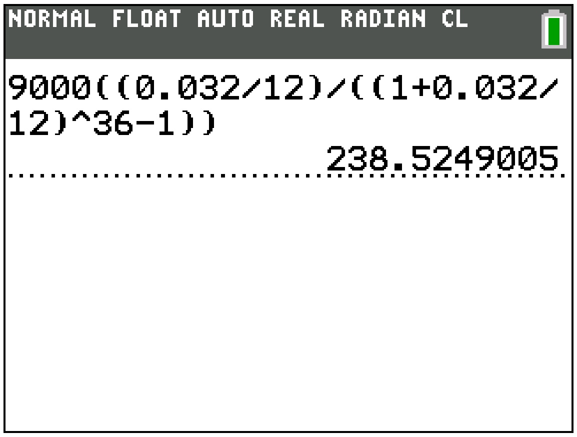
Joe wants to buy a pop-up trailer that costs $9,000. He wants to pay in cash so he wants to make monthly deposits into an account earning 3.2% APR. How much should his monthly payments be to save up the $9,000 in 3 years?

F = $9,000, r = .032, t = 3, n = 12

$$PMT = F \bullet \left[\frac{\frac{r}{n}}{\left(1+\frac{r}{n}\right)^{nt}-1}\right]$$

$$PMT = 9,000 \bullet \left[\frac{\frac{0.032}{12}}{\left(1+\frac{0.032}{12}\right)^{12\cdot 3}-1}\right]$$

Figure 5.4.2: Calculation for PMT for Example 5.4.2



$PMT = 238.52$

Joe has to make monthly payments of $238.52 for 3 years to save up the $9,000.

Example 5.4.3: Savings Plan—Finding Time

Sara has $300 a month she can deposit into an account earning 6.8% APR. How long will it take her to save up the $10,000 she needs?

F = $10,000, PMT = $300, r = 0.068, n = 12

Note: We will use the original savings plan formula which solves for the future value, F to solve this problem.

$$F = PMT\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

$$10,000 = 300\left[\frac{\left(1+\frac{0.068}{12}\right)^{12t}-1}{\frac{0.068}{12}}\right]$$

$$10,000 = 300\left[\frac{(1.005667)^{12t}-1}{0.005667}\right]$$

$$33.333333 = \frac{(1.005667)^{12t} - 1}{0.005667}$$

$$0.188900 = (1.005667)^{12t} - 1$$

$$1.188900 = (1.005667)^{12t}$$

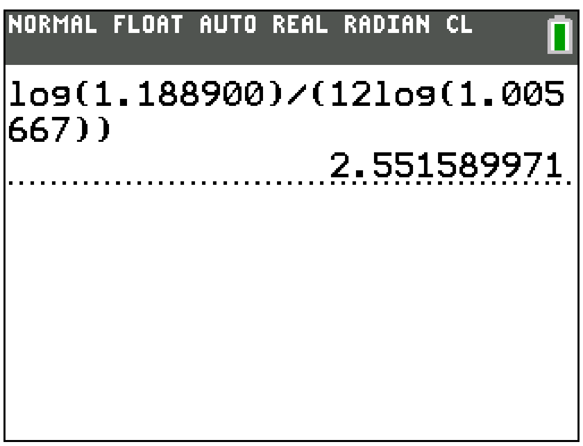
To solve for time you have to take the logarithm (log) of both sides. You can then use the "Power Rule" of logs, which states $\log x^r = r \log x$, when $x > 0$, as stated in section 4.3.

$$\log 1.188900 = \log (1.005667)^{12t}$$

$$\log 1.188900 = 12t \log (1.005667)$$

$$t = \frac{\log 1.188900}{12 \log 1.005667}$$

Figure 5.4.3: Calculation for t for Example 5.4.3



$t = 2.55$

It will take Sara about 2.6 years to save up the $10,000.

Example 5.4.4: Savings Plan—Two-Part Savings Problem

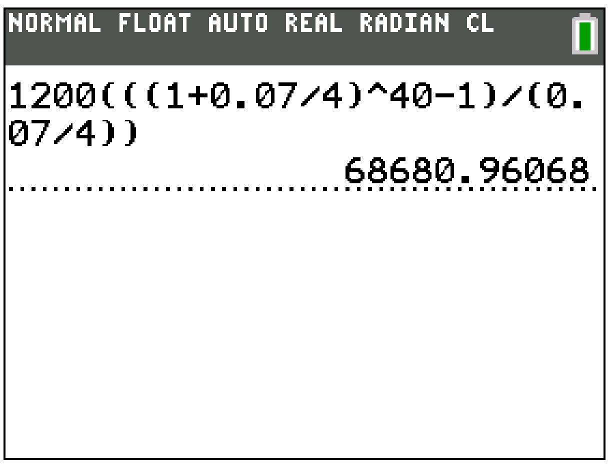
At the end of each quarter a 50-year old woman puts $1200 in a retirement account that pays 7% interest compounded quarterly. When she reaches age 60, she withdraws the entire amount and places it into a mutual fund that pays 9% interest compounded monthly. From then on she deposits $300 in the mutual fund at the end of each month. How much is in the account when she reaches age 65?

First, she deposits \$1200 quarterly at 7% for 10 years.
PMT = \$1200, r = 0.07, t = 10, n = 4

$$F = PMT\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

$$F = 1200\left[\frac{\left(1+\frac{0.07}{4}\right)^{4\cdot 10}-1}{\frac{0.07}{4}}\right]$$

Figure 5.4.4: Calculation for F for Part One of Example 5.4.4



$$F = 68{,}680.96$$

Second, she puts this lump sum plus \$300 a month for 5 years at 9%. Think of the lump sum and the new monthly deposits as separate things. The lump sum just sits there earning interest so use the compound interest formula. The monthly payments are a new payment plan, so use the savings plan formula again.

Total = (lump sum + interest) + (new deposits + interest)

$$68{,}680.96\left(1+\frac{0.09}{12}\right)^{12\cdot 5} + 300\left[\frac{\left(1+\frac{0.09}{12}\right)^{12\cdot 5}-1}{\frac{0.09}{12}}\right]$$

Figure 5.4.5: Calculation for the Lump Sum Plus Interest

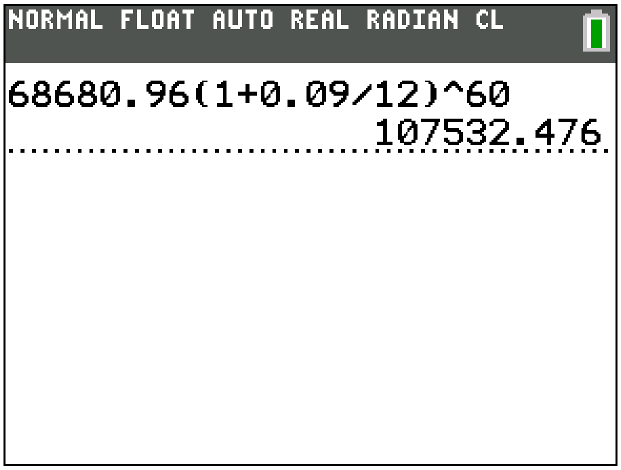
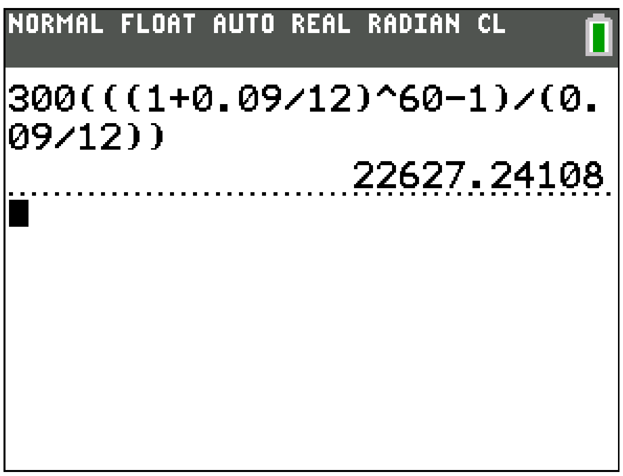


Figure 5.4.6: Calculation for the New Deposits Plus Interest



$Total = 107,532.48 + 22,627.24 = 130,159.72$

She will have $130,159.72 when she reaches age 65.

Section 5.5: Loans

It is a good idea to try to save up money to buy large items or find 0% interest deals so you are not paying interest. However, this is not always possible, especially when buying a house or car. That is when it is important to understand how much interest you will be charged on your loan.

Loan Payment Formula: $P = PMT\left[\dfrac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right]$

where,
P = Present value (Principal)
PMT = Payment
r = Annual percentage rate (APR) changed to a decimal
t = Number of years
n = Number of payments made per year

Example 5.5.1: Loan Payment Formula:

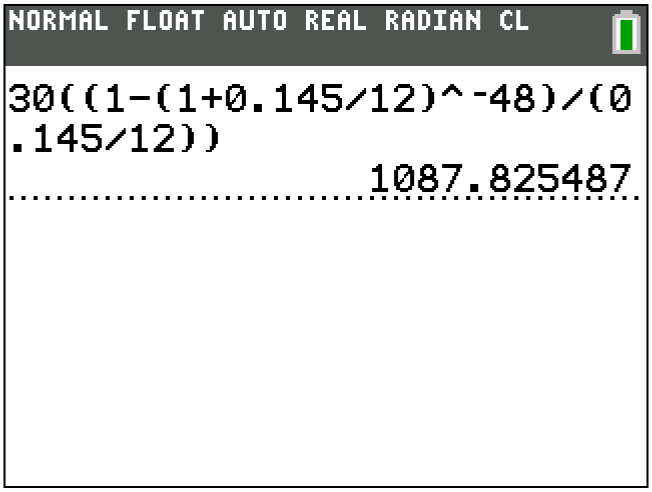
Ed buys an iPad from a rent-to-own business with a credit plan with payments of \$30 a month for four years at 14.5% APR compounded monthly. If Ed had bought the iPad from Best Buy or Amazon it would have cost \$500. What is the price that Ed paid for his iPad at the rent-to-own business? How much interest did he pay?

PMT = \$30, r = 0.145, t = 4, n = 12

$$P = PMT\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right]$$

$$P = 30\left[\frac{1-\left(1+\frac{0.145}{12}\right)^{-12\cdot 4}}{\frac{0.145}{12}}\right]$$

Figure 5.5.1: Calculation of *P* for Example 5.5.1



$P = 1087.83$

The price Ed paid for the iPad was $1,087.83. That's a lot more that $500!

Also, the total amount he paid over the course of the loan was $\$30 \times 12 \times 4 = \$1440$. Therefore, the total amount of interest he paid over the course of the loan was $1440 - $1087.83 = $352.17.

For some problems, you will have to find the payment instead of the present value. In that case, it is helpful to just solve the loan payment formula for *PMT*. Since *PMT* is multiplied by a fraction, to solve for *PMT*, you can just multiply both sides of the formula by that fraction. You should just think of the loan payment formula in two different forms, one solving for present value, *P*, and one solving for payment, *PMT*.

$$P = PMT\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right]$$

$$P \bullet \left[\frac{\frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}\right] = PMT\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right] \bullet \left[\frac{\frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}\right]$$

$$PMT = P \bullet \left[\frac{\frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}} \right]$$

Example 5.5.2: Loan Formula—Finding Payment

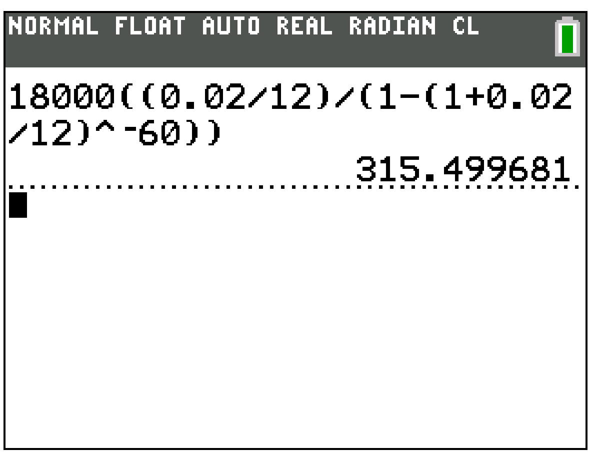
Jack goes to a car dealer to buy a new car for \$18,000 at 2% APR with a five-year loan. The dealer quotes him a monthly payment of \$425. What should the monthly payment on this loan be?

P = \$18,000, r = 0.02, n = 12, t = 5

$$PMT = P \bullet \left[\frac{\frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}} \right]$$

$$PMT = 18,000 \bullet \left[\frac{\frac{0.02}{12}}{1-\left(1+\frac{0.02}{12}\right)^{-12 \cdot 5}} \right]$$

Figure 5.5.2: Calculation for PMT for Example 5.5.2



$$PMT = 315.50$$

Jack should have a monthly payment of \$315.50, not \$425.

Now, let's find out how much the dealer is trying to get Jack to pay for the car.
$\$425 \times 12 \times 5 = \$25,500$

The dealer is trying to sell Jack the car for a total of \$25,500 with principal and interest. What should the total principal and interest be with the \$315.50 monthly payment?
$\$315.50 \times 12 \times 5 = \$18,930$

Therefore, the dealer is trying to get Jack to pay \$25,500 - \$18,930 = \$6,570 in additional principal and interest charges. This means that the quoted rate of 2% APR is not accurate, or the quoted price of \$18,000 is not accurate, or both.

Example 5.5.3: Loan Formula—Mortgage

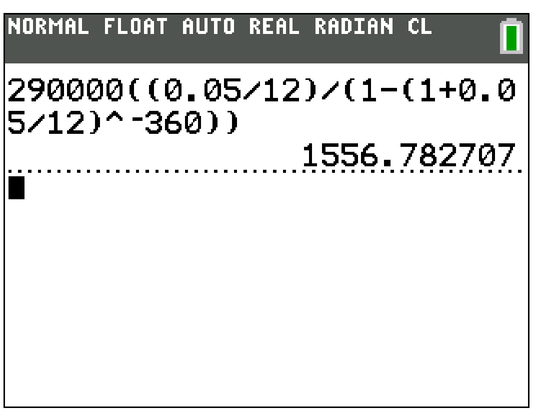
Morgan is going to buy a house for \$290,000 with a 30-year mortgage at 5% APR. What is the monthly payment for this house?

P = \$290,000, r = 0.05, n = 12, t = 30

$$PMT = P \bullet \left[\frac{\frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}} \right]$$

$$PMT = 290,000 \bullet \left[\frac{\frac{0.05}{12}}{1 - \left(1 + \frac{0.05}{12}\right)^{-12 \cdot 30}} \right]$$

Figure 5.5.3: Calculation for PMT for Example 5.5.3



$PMT = 1556.78$

The monthly payment for this mortgage should be \$1556.78.

It is also very interesting to figure out how much Morgan will end up paying overall to buy this house. It is rather easy to calculate this

$\$1556.78 \times 12 \times 30 = \$560,440.80$

Therefore, Morgan will pay \$560,440.80 in principal and interest which means that Morgan will pay \$290,000 for the principal of the loan and \$560,440.80 - \$290,000 = \$270,440.80 in interest. This is enough to buy another comparable home. Interest charges add up quickly.

Example 5.5.4: Loan Formula—Refinance Mortgage

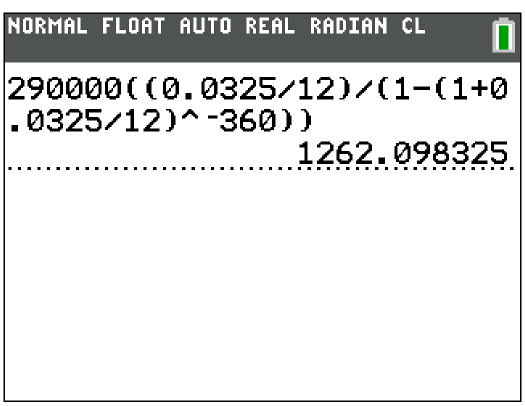
If Morgan refinanced the \$290,000 at 3.25% APR what would her monthly payments be?

P = \$290,000, r = 0.0325, n = 12, t = 30

$$PMT = P \bullet \left[\frac{\frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}} \right]$$

$$PMT = 290,000 \bullet \left[\frac{\frac{0.0325}{12}}{1 - \left(1 + \frac{0.0325}{12}\right)^{-12 \cdot 30}} \right]$$

Figure 5.5.4: Calculation for PMT for Example 5.5.4



$PMT = 1262.10$

If Morgan refinanced the mortgage at 3.25% APR, the monthly payment would now be $1262.10 instead of $1556.78.

How much money would Morgan save over the life of the loan at the new payment amount?

$$\$1262.10 \times 12 \times 30 = \$454,356$$

Then, subtract $560,440.80 - $454,356 = $106,084.80.

Morgan would save $106,084.80 in interest because of refinancing the loan at 3.25% APR.

Example 5.5.5: Loan Formula—Mortgage Comparison

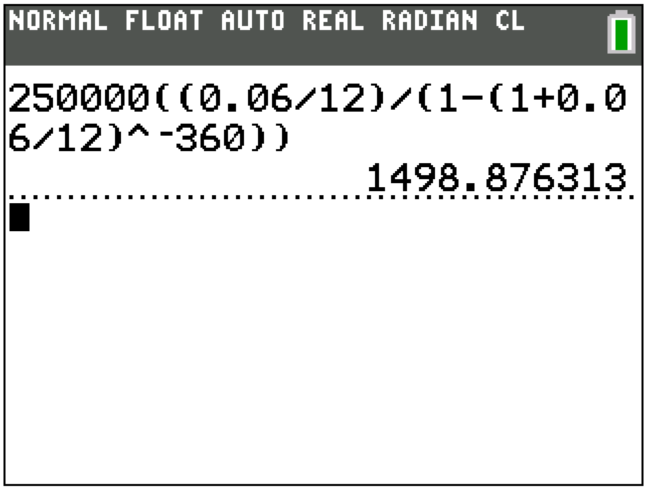
With a fixed rate mortgage, you are guaranteed that the interest rate will not change over the life of the loan. Suppose you need $250,000 to buy a new home. The mortgage company offers you two choices: a 30-year loan with an APR of 6% or a 15-year loan with an APR of 5.5%. Compare your monthly payments and total loan cost to decide which loan you should take. Assume no difference in closing costs.

Option 1: First calculate the monthly payment:

$$PMT = P \bullet \left[\frac{\frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}} \right]$$

$$PMT = 250000 \bullet \left[\frac{\frac{0.06}{12}}{1 - \left(1 + \frac{0.06}{12}\right)^{-12 \cdot 30}} \right]$$

Figure 5.5.5: Calculation for PMT for Example 5.5.5, Option 1



$PMT = 1498.88$

The monthly payment for a 30-year loan at 6% interest is $1498.88.

Now calculate the total cost of the loan over the 30 years:

$\$1498.88 \times 12 \times 30 = \$539,596.80$

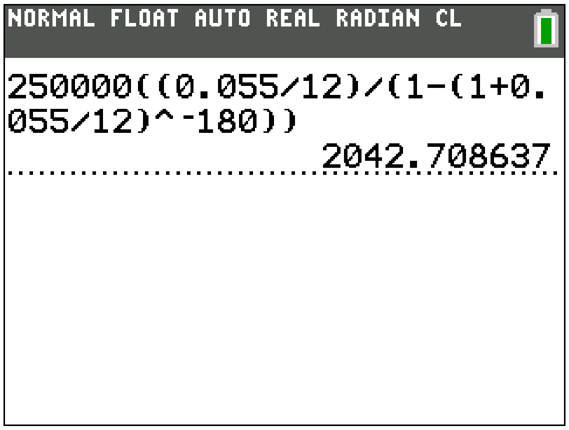
The monthly payments are $1498.88 and the total cost of the loan is $539,596.80.

Option 2: First calculate the monthly payment:

$$PMT = P \bullet \left[\frac{\frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}} \right]$$

$$PMT = 250000 \bullet \left[\frac{\frac{0.055}{12}}{1 - \left(1 + \frac{0.055}{12}\right)^{-12 \cdot 15}} \right]$$

Figure 5.5.6: Calculate PMT for Example 5.5.5, Option 2



$PMT = 2042.71$

The monthly payment for a 15-year loan at 5.5% interest is $2042.71.

Now calculate the total cost of the loan over the 15 years:

$\$2042.71 \times 12 \times 15 = \$367,687.80$

The monthly payments are $2042.71 and the total cost of the loan is $367,687.80.

Therefore, the monthly payments are higher with the 15-year loan, but you spend a lot less money overall.

Chapter 5 Homework

1. Find a budget template or make up your own in Microsoft Excel. Then create a monthly budget that tracks every dollar you earn and where that money goes.

2. If you make $38,000 per year and want to save 15% of your income, how much should you save every month?

3. Suzy got a U.S. Treasury Bond for $8,000 at 5.2% annual simple interest. Create a table showing how much money Suzy will have each year for seven years. Graph this data and identify the type of growth that is shown.

4. Referring to Problem #3, how much would Suzy's Bond be worth after 20 years?

5. Geneva wants to save $12,000 to buy a new car. She just received an $8,000 bonus and plans to invest it in an account earning 7% annual simple interest. How long will she need to leave her money in the account to accumulate the $12,000 she needs?

6. Suppose you take out a payday loan for $400 that charges $13 for every $100 loaned. The term of the loan is 15 days. Find the APR charged on this loan.

7. Sue got a student loan for $12,000 at 5.4% annual simple interest. How much does she owe after one year? How much interest will she pay?

8. If I put $1500 into my savings account and earned $180 of interest at 4% annual simple interest, how long was my money in the bank?

9. Derek invested $1000. What would that money grow to in 18 months at a 5.55% annual simple interest rate?

10. You borrow \$500 for a trip at 11% annual simple interest for two years.
    a. Find the interest you will pay on the loan.
    b. How much will you have to pay the bank at the end of the two years?

11. Jewel deposited \$4000 into an account that earns 8% APR compounded annually. Create a table showing how much money Jewel will have each year for seven years. Graph this data and identify the type of growth that is shown.

12. Amira deposited \$1,000 into a savings account earning 4.6% APR compounded quarterly. How much will she have in her account after 15 years?

13. Matt invested \$1,000,000 into an account earning 5.5% APR compounded monthly. What will his balance be after two years?

14. Matt invested \$1,000,000 into an account earning 5.5% APR compounded continuously. What will his balance be after two years?

15. Find the Annual Percentage Yield (APY) for an investment account with:
    a. 8.2% APR compounded monthly
    b. 8.2% APR compounded daily
    c. 8.2% APR compounded continuously

16. A bank quotes you an APR of 4.3% for a home loan. The interest is compounded monthly. What is the APY?

17. Suppose you need \$1230 to purchase a new T.V. in three years. If the interest rate of a savings account is compounded monthly at 3.8% APR, how much do you need to deposit in the savings account today?

18. What was the principal for a continuously compounded account earning 3.8% APR for 15 years that now has a balance of \$2,500,000?

19. You have saved change throughout the year and now have $712. You can choose from two bank offers for investing this money. The first is 5.4% APR compounded continuously for seven years. The second is 6% APR compounded quarterly for six years. Which account will yield the most money? What is the dollar amount difference between the accounts at the end of their terms?

20. You deposit $25,000 in an account that earns 5.2% APR compounded semiannually. Find the balance in the account at the end of five years, at the end of 10 years, and at the end of 20 years.

21. You gave your friend a short term three-year loan of $35,000 at 2.5% compounded annually. What will be your total return?

22. Isaac is saving his $50 monthly allowance by putting it into an account earning 4.5% APR per year. How much money will he have at the end of five years? Ten years?

23. Sue wants to save up $2500 to buy a new laptop. She wants to pay in cash so is making monthly deposits into a savings account earning 3.8% APR. How much do her monthly payments need to be to be able to save up the $2500 in two years?

24. Business Enterprises is depositing $450 a month into an account earning 7.2% APR to save up the $15,000 they need to expand. How long will it take them to save up the $15,000 they need?

25. Dan has $200 a month he can deposit into an account earning 3.8% APR. How long will it take him to save up the $12,000 he needs?

26. At age 30, Suzy starts an IRA to save for retirement. She deposits $100 at the end of each month. If she can count on an APR of 6%, how much will she have when she retires 35 years later at age 65?

27. You want to save \$100,000 in 18 years for a college fund for your child by making regular, monthly deposits. Assuming an APR of 7%, calculate how much you should deposit monthly. How much comes from the actual deposits and how much from interest?

28. You would like to have \$35,000 to spend on a new car in five years. You open a savings account with an APR of 4%. How much must you deposit each quarter to reach this goal?

29. At the age of 35 you decide to start investing for retirement. You put away \$2000 in a retirement account that pays 6.5% APR compounded monthly. When you reach age 55, you withdraw the entire amount and place it in a new savings account that pays 8% APR compounded monthly. From then on you deposit \$400 in the new savings account at the end of each month. How much is in your account when you reach age 65?

30. Gene bought a washing machine for his rental property with a credit plan of \$35 a month for three years at 12.5% APR. What was the purchase price of the washing machine? How much interest will Gene have paid at the end of the three years?

31. You go to a car dealer and ask to buy a new car listed at \$23,000 at 1.9% APR with a five-year loan. The dealer quotes you a monthly payment of \$475. What should the monthly payment on this loan be?

32. Gina buys her first house for \$230,000 at 5.5% APR with a 30-year mortgage. Find her monthly mortgage payment. How much principal and interest will she end up paying for her house?

33. If Gina (in problem 32) refinanced the \$230,000 at 3.8% APR what would her monthly payments be? How much principal and interest would she end up paying for her house?

34. Your bank offers you two choices for your new home loan of $320,000. Your choices are a 30-year loan with an APR of 4.25% or a 15-year loan with an APR of 3.8%. Compare your monthly payments and total loan cost to decide which loan you should take.

35. Lou purchases a home for $575,000. He makes a down payment of 15% and finances the remaining amount with a 30-year mortgage with an annual percentage rate of 5.25%. Find his monthly mortgage payment. How much principal and interest will he end up paying for his house?

36. You have decided to refinance your home mortgage to a 15-year loan at 4.0% APR. The outstanding balance on your loan is $150,000. Under your current loan, your monthly mortgage payment is $1610, which you must continue to pay for the next 20 years if you do not refinance.
    a. What is the new monthly payment if you refinance?
    b. How much will you save by refinancing?
    c. How much interest will you pay on this new loan?

Chapter 6: Graph Theory

Graph theory deals with routing and network problems and if it is possible to find a "best" route, whether that means the least expensive, least amount of time or the least distance. Some examples of routing problems are routes covered by postal workers, UPS drivers, police officers, garbage disposal personnel, water meter readers, census takers, tour buses, etc. Some examples of network problems are telephone networks, railway systems, canals, roads, pipelines, and computer chips.

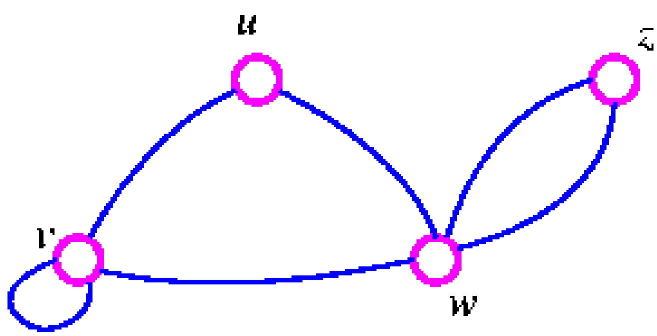
Section 6.1: Graph Theory

There are several definitions that are important to understand before delving into Graph Theory. They are:

- A graph is a picture of dots called **vertices** and lines called **edges**.
- An edge that starts and ends at the same vertex is called a **loop.**
- If there are two or more edges directly connecting the same two vertices, then these edges are called **multiple edges**.
- If there is a way to get from one vertex of a graph to all the other vertices of the graph, then the graph is **connected.**
- If there is even one vertex of a graph that cannot be reached from every other vertex, then the graph is **disconnected.**

Example 6.1.1: Graph Example 1

Figure 6.1.1: Graph 1



In the above graph, the **vertices** are U, V, W, and Z and the **edges** are UV, VV, VW, UW, WZ_1, and WZ_2.

This is a **connected** graph. VV is a loop. WZ_1, and WZ_2 are **multiple edges**.

Example 6.1.2: Graph Example 2

Figure 6.1.2: Graph 2

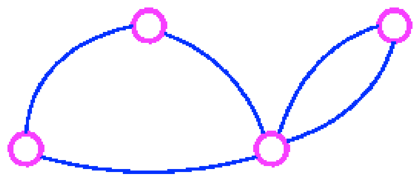
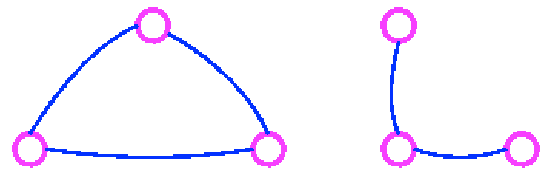


Figure 6.1.3: Graph 3



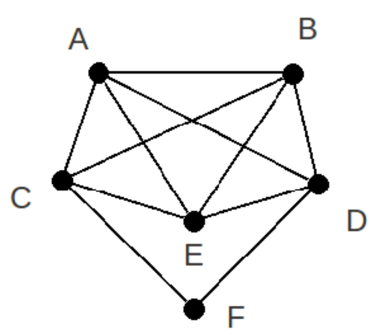
The graph in Figure 6.1.2 is connected while the graph in Figure 6.1.3 is disconnected.

Graph Concepts and Terminology:

Order of a Network: the number of vertices in the entire network or graph
Adjacent Vertices: two vertices that are connected by an edge
Adjacent Edges: two edges that share a common vertex
Degree of a Vertex: the number of edges at that vertex
Path: a sequence of vertices with each vertex adjacent to the next one that starts and ends at different vertices and travels over any edge only once
Circuit: a path that starts and ends at the same vertex
Bridge: an edge such that if it were removed from a connected graph, the graph would become disconnected

Example 6.1.3: Graph Terminology

Figure 6.1.4: Graph 4



In the above graph the following is true:

Vertex A is adjacent to vertex B, vertex C, vertex D, and vertex E.

Vertex F is adjacent to vertex C, and vertex D.

Edge DF is adjacent to edge BD, edge AD, edge CF, and edge DE.

The degrees of the vertices:

A	4
B	4
C	4
D	4
E	4
F	2

Here are some paths in the above graph: (there are many more than listed)

A,B,D

A,B,C,E

F,D,E,B,C

Here are some circuits in the above graph: (there are many more than listed)

B,A,D,B

B,C,F,D,B

F,C, E, D, F

The above graph does not have any bridges.

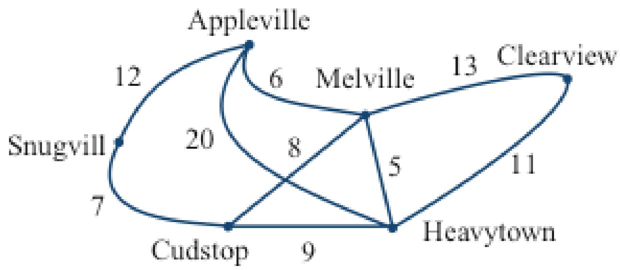
Section 6.2: Networks

A network is a connection of vertices through edges. The internet is an example of a network with computers as the vertices and the connections between these computers as edges.

Spanning Subgraph: a graph that joins all of the vertices of a more complex graph, but does not create a circuit

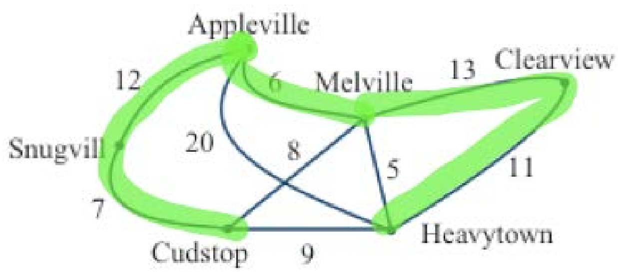
Example 6.2.1: Spanning Subgraph

Figure 6.2.1: Map of Connecting Towns



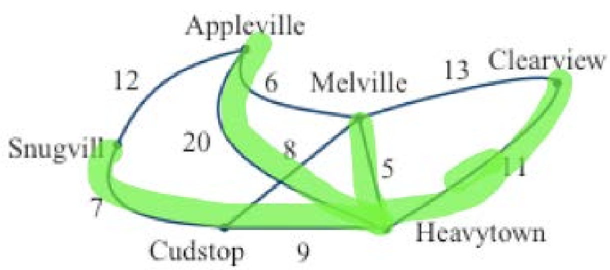
This is a graph showing how six cities are linked by roads. This graph has many spanning subgraphs but two examples are shown below.

Figure 6.2.2: Spanning Subgraph 1



This graph spans all of the cities (vertices) of the original graph, but does not contain any circuits.

Figure 6.2.3: Spanning Subgraph 2



This graph spans all of the cities (vertices) of the original graph, but does not contain any circuits.

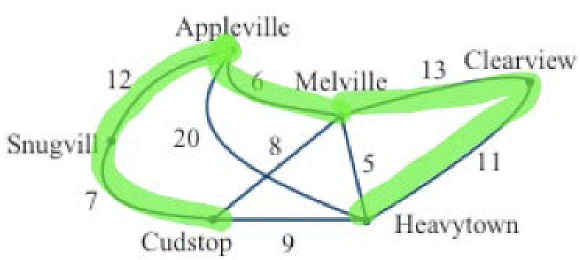
Tree: A tree is a graph that is connected and has no circuits. Therefore, a spanning subgraph is a tree and the examples of spanning subgraphs in Example 6.2.1 above are also trees.

Properties of Trees:

1. If a graph is a tree, there is one and only one path joining any two vertices. Conversely, if there is one and only one path joining any two vertices of a graph, the graph must be a tree.
2. In a tree, every edge is a bridge. Conversely, if every edge of a connected graph is a bridge, then the graph must be a tree.
3. A tree with N vertices must have N-1 edges.
4. A connected graph with N vertices and N-1 edges must be a tree.

Example 6.2.2: Tree Properties

Figure 6.2.2: Spanning Subgraph 1



Consider the spanning subgraph highlighted in green shown in Figure 6.2.2.

a. Tree Property 1
Look at the vertices Appleville and Heavytown. Since the graph is a tree, there is only one path joining these two cities. Also, since there is only one path between any two cities on the whole graph, then the graph must be a tree.

b. Tree Property 2
Since the graph is a tree, notice that every edge of the graph is a bridge, which is an edge such that if it were removed the graph would become disconnected.

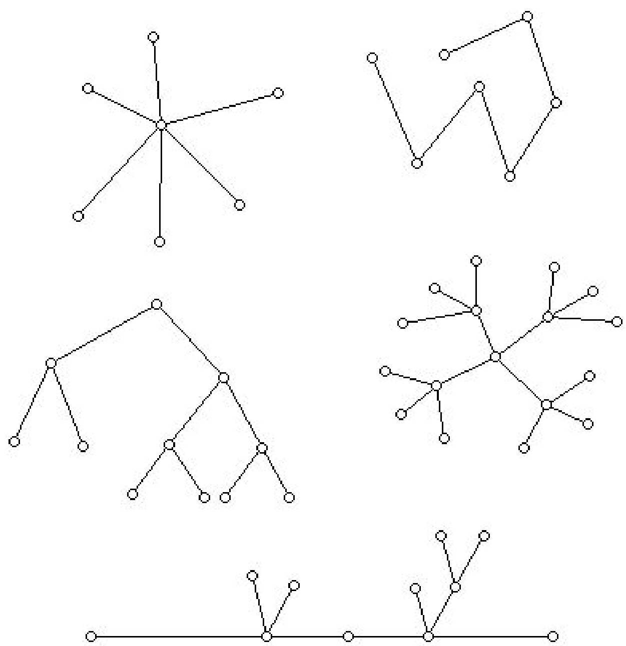
c. Tree Property 3
Since the graph is a tree and it has six vertices, it must have N – 1 or six – 1 = five edges.

d. Tree Property 4
Since the graph is connected and has six vertices and five edges, it must be a tree.

Example 6.2.3: More Examples of Trees:

All of the graphs shown below are trees and they all satisfy the tree properties.

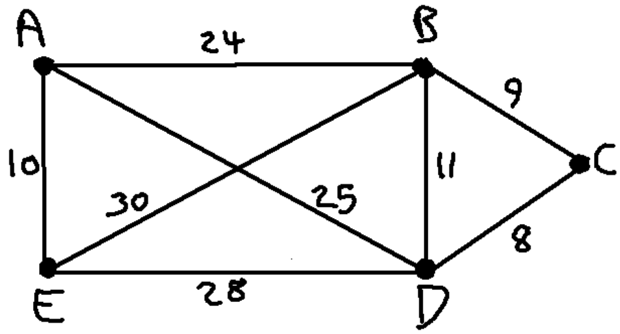
Figure 6.2.4: More Examples of Trees



Minimum Spanning Tree: A minimum spanning tree is the tree that spans all of the vertices in a problem with the least cost (or time, or distance).

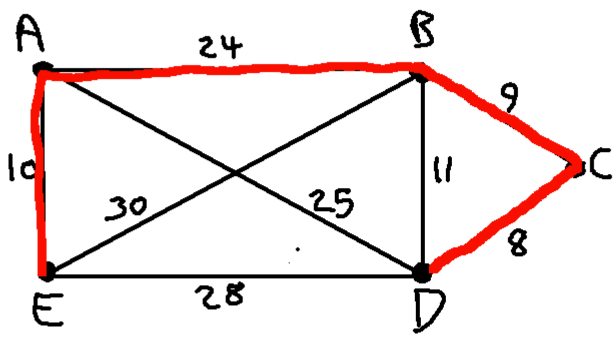
Example 6.2.4: Minimum Spanning Tree

Figure 6.2.5: Weighted Graph 1



The above is a weighted graph where the numbers on each edge represent the cost of each edge. We want to find the minimum spanning tree of this graph so that we can find a network that will reach all vertices for the least total cost.

Figure 6.2.6: Minimum Spanning Tree for Weighted Graph 1



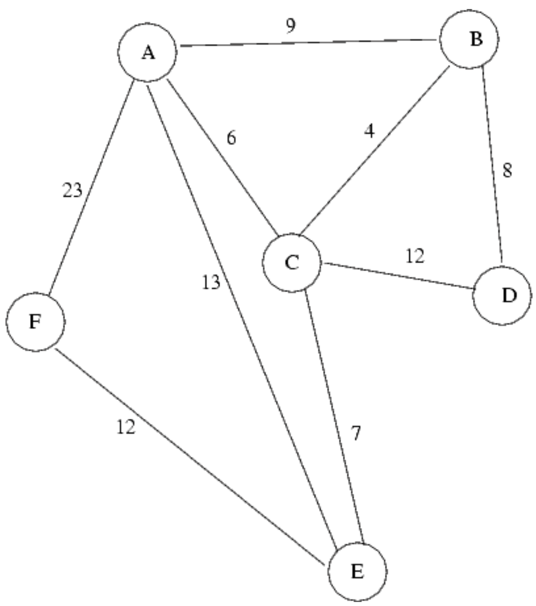
This is the minimum spanning tree for the graph with a total cost of 51.

Kruskal's Algorithm: Since some graphs are much more complicated than the previous example, we can use Kruskal's Algorithm to always be able to find the minimum spanning tree for any graph.

1. Find the cheapest link in the graph. If there is more than one, pick one at random. Mark it in red.
2. Find the next cheapest link in the graph. If there is more than one, pick one at random. Mark it in red.
3. Continue doing this as long as the next cheapest link does not create a red circuit.
4. You are done when the red edges span every vertex of the graph without any circuits. The red edges are the MST (minimum spanning tree).

Example 6.2.5: Using Kruskal's Algorithm

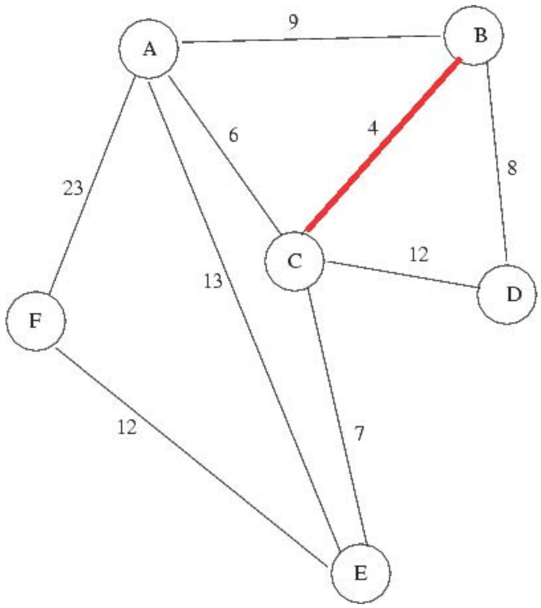
Figure 6.2.7: Weighted Graph 2



Suppose that it is desired to install a new fiber optic cable network between the six cities (A, B, C, D, E, and F) shown above for the least total cost. Also, suppose that the fiber optic cable can only be installed along the roadways shown above. The weighted graph above shows the cost (in millions of dollars) of installing the fiber optic cable along each roadway. We want to find the minimum spanning tree for this graph using Kruskal's Algorithm.

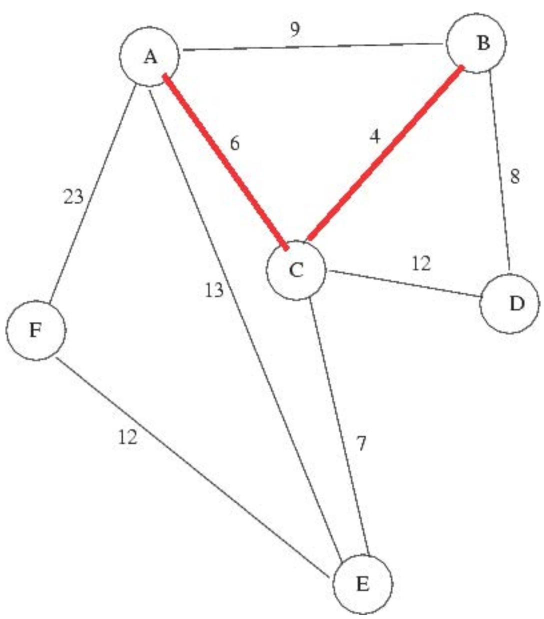
Step 1: Find the cheapest link of the whole graph and mark it in red. The cheapest link is between B and C with a cost of four million dollars.

Figure 6.2.8: Kruskal's Algorithm Step 1



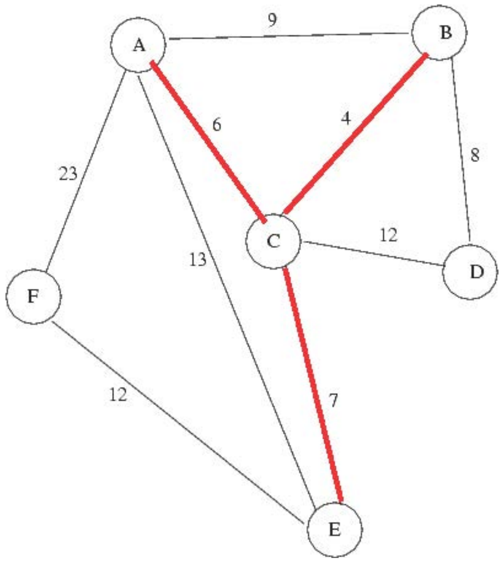
Step 2: Find the next cheapest link of the whole graph and mark it in red. The next cheapest link is between A and C with a cost of six million dollars.

Figure 6.2.9: Kruskal's Algorithm Step 2



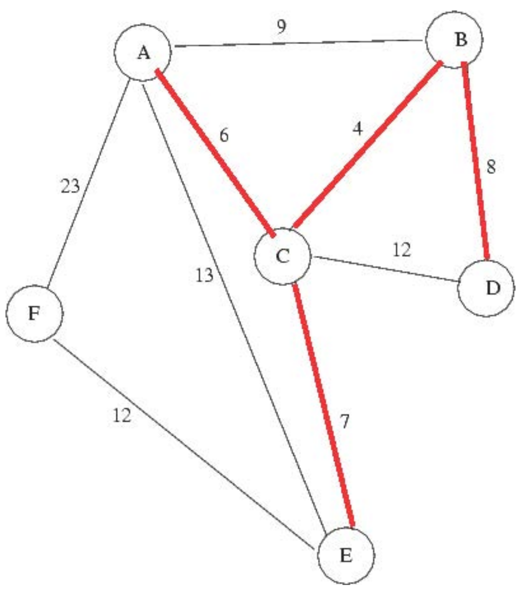
Step 3: Find the next cheapest link of the whole graph and mark it in red as long as it does not create a red circuit. The next cheapest link is between C and E with a cost of seven million dollars.

Figure 6.2.10: Kruskal's Algorithm Step 3



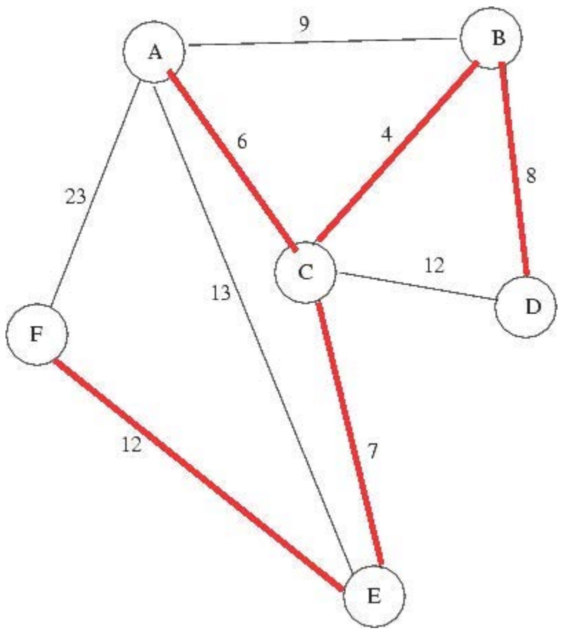
Step 4: Find the next cheapest link of the whole graph and mark it in red as long as it does not create a red circuit. The next cheapest link is between B and D with a cost of eight million dollars.

Figure 6.2.11: Kruskal's Algorithm Step 4



Step 5: Find the next cheapest link of the whole graph and mark it in red as long as it does not create a red circuit. The next cheapest link is between A and B with a cost of nine million dollars, but that would create a red circuit so we cannot use it. Therefore, the next cheapest link after that is between E and F with a cost of 12 million dollars, which we are able to use. We cannot use the link between C and D which also has a cost of 12 million dollars because it would create a red circuit.

Figure 6.2.12: Kruskal's Algorithm Step 5



This was the last step and we now have the minimum spanning tree for the weighted graph with a total cost of $37,000,000.

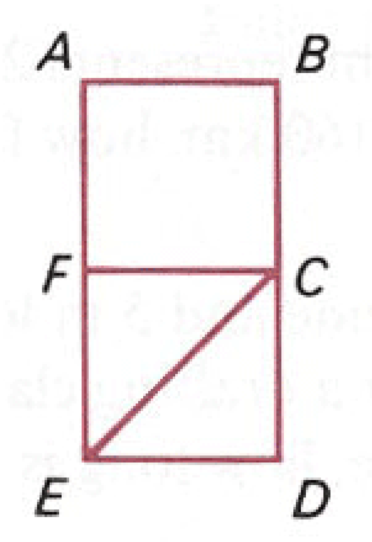
Section 6.3: Euler Circuits

Leonhard Euler first discussed and used Euler paths and circuits in 1736. Rather than finding a minimum spanning tree that visits every vertex of a graph, an Euler path or circuit can be used to find a way to visit every edge of a graph once and only once. This would be useful for checking parking meters along the streets of a city, patrolling the streets of a city, or delivering mail.

Euler Path: a path that travels through *every* edge of a connected graph once and only once and starts and ends at different vertices

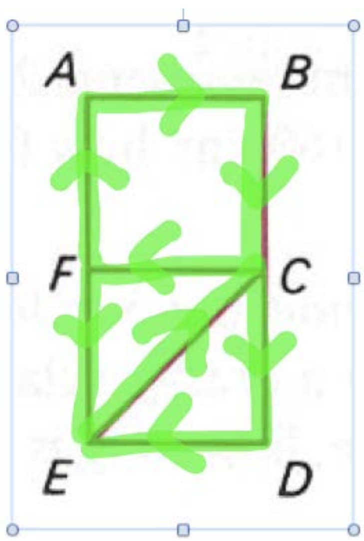
Example 6.3.1: Euler Path

Figure 6.3.1: Euler Path Example



One Euler path for the above graph is F, A, B, C, F, E, C, D, E as shown below.

Figure 6.3.2: Euler Path

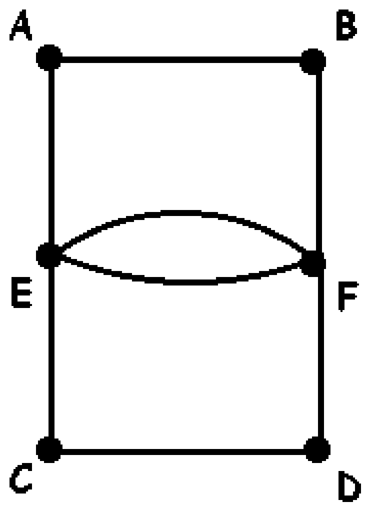


This Euler path travels every edge once and only once and starts and ends at different vertices. This graph cannot have an Euler circuit since no Euler path can start and end at the same vertex without crossing over at least one edge more than once.

Euler Circuit: an Euler path that starts and ends at the same vertex

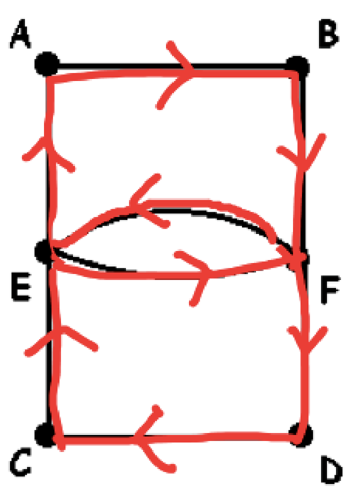
Example 6.3.2: Euler Circuit

Figure 6.3.3: Euler Circuit Example



One Euler circuit for the above graph is E, A, B, F, E, F, D, C, E as shown below.

Figure 6.3.4: Euler Circuit



This Euler path travels every edge once and only once and starts and ends at the same vertex. Therefore, it is also an Euler circuit.

Euler's Theorems:

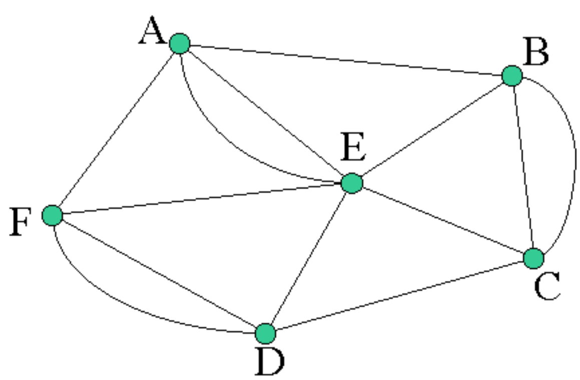
Euler's Theorem 1: If a graph has any vertices of odd degree, then it cannot have an Euler circuit. If a graph is connected and every vertex has an even degree, then it has at least one Euler circuit (usually more).
Euler's Theorem 2: If a graph has more than two vertices of odd degree, then it cannot have an Euler path. If a graph is connected and has exactly two vertices of odd degree, then it has at least one Euler path (usually more). Any such path must start at one of the odd-degree vertices and end at the other one.
Euler's Theorem 3: The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore must be an even number). Therefore, the number of vertices of odd degree must be even.

Finding Euler Circuits:

1. Be sure that every vertex in the network has even degree.
2. Begin the Euler circuit at any vertex in the network.
3. As you choose edges, never use an edge that is the only connection to a part of the network that you have not already visited.
4. Label the edges in the order that you travel them and continue this until you have travelled along every edge exactly once and you end up at the starting vertex.

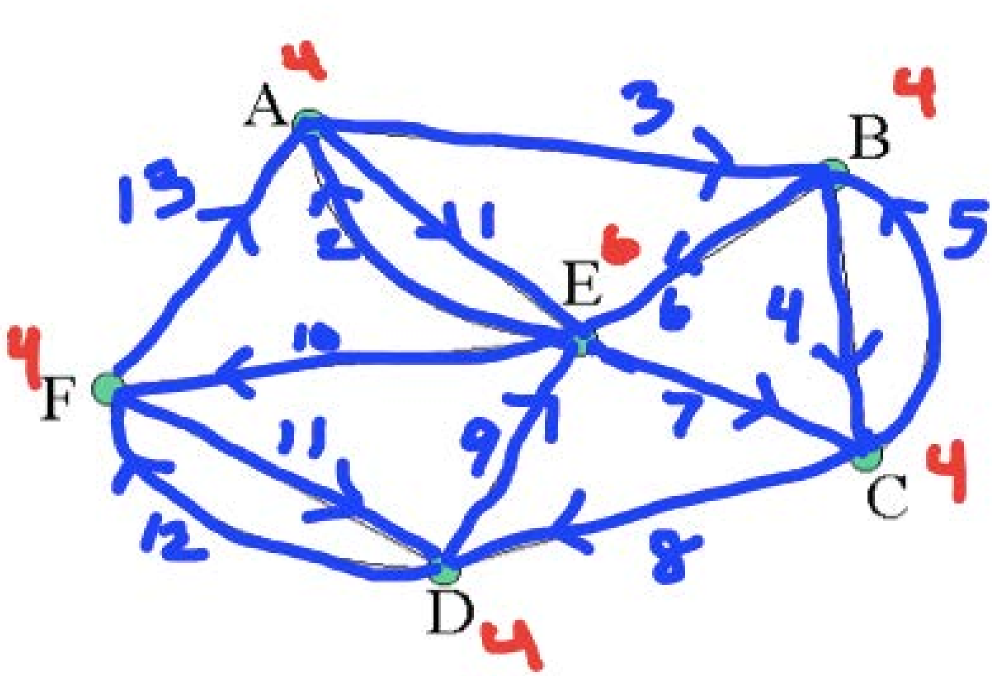
Example 6.3.3: Finding an Euler Circuit

Figure 6.3.5: Graph for Finding an Euler Circuit



The graph shown above has an Euler circuit since each vertex in the entire graph is even degree. Thus, start at one even vertex, travel over each vertex once and only once, and end at the starting point. One example of an Euler circuit for this graph is A, E, A, B, C, B, E, C, D, E, F, D, F, A. This is a circuit that travels over every edge once and only once and starts and ends in the same place. There are other Euler circuits for this graph. This is just one example.

Figure 6.3.6: Euler Circuit



The degree of each vertex is labeled in red. The ordering of the edges of the circuit is labeled in blue and the direction of the circuit is shown with the blue arrows.

Section 6.4 Hamiltonian Circuits

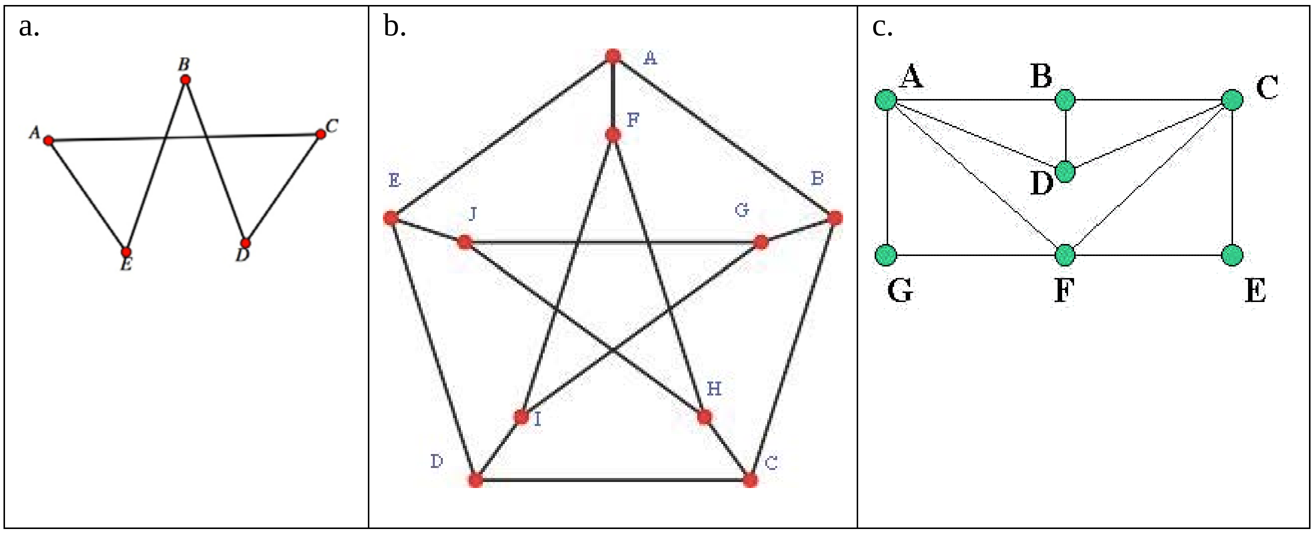
The Traveling Salesman Problem (TSP) is any problem where you must visit every vertex of a weighted graph once and only once, and then end up back at the starting vertex. Examples of TSP situations are package deliveries, fabricating circuit boards, scheduling jobs on a machine and running errands around town.

Hamilton Circuit: a circuit that must pass through each vertex of a graph once and only once

Hamilton Path: a path that must pass through each vertex of a graph once and only once

Example 6.4.1: Hamilton Path:

Figure 6.4.1: Examples of Hamilton Paths



Not all graphs have a Hamilton circuit or path. There is no way to tell just by looking at a graph if it has a Hamilton circuit or path like you can with an Euler circuit or path. You must do trial and error to determine this. By the way if a graph has a Hamilton circuit then it has a Hamilton path. Just do not go back to home.

Graph a. has a Hamilton circuit (one example is ACDBEA)

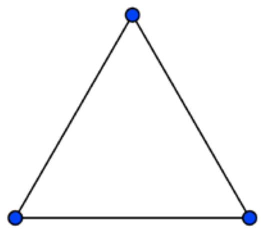
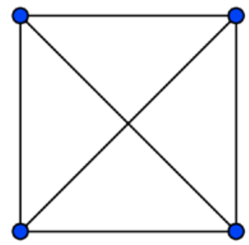
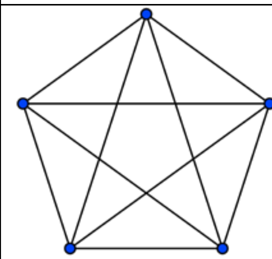
Graph b. has no Hamilton circuits, though it has a Hamilton path (one example is ABCDEJGIFH)

Graph c. has a Hamilton circuit (one example is AGFECDBA)

Complete Graph: A complete graph is a graph with N vertices in which every pair of vertices is joined by exactly one edge. The symbol used to denote a complete graph is K_N.

Example 6.4.2: Complete Graphs

Figure 6.4.2: Complete Graphs for N = 2, 3, 4, and 5

a. K_2	b. K_3	c. K_4	d. K_5
			
two vertices and one edge	three vertices and three edges	four vertices and six edges	five vertices and ten edges

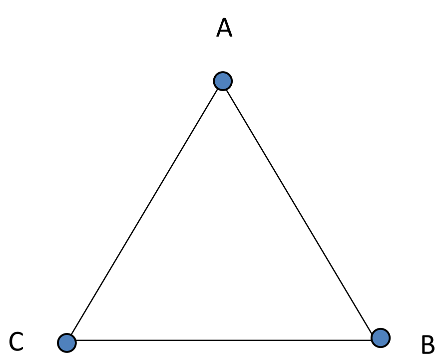
In each complete graph shown above, there is exactly one edge connecting each pair of vertices. There are no loops or multiple edges in complete graphs. Complete graphs do have Hamilton circuits.

Reference Point: the starting point of a Hamilton circuit

Example 6.4.3: Reference Point in a Complete Graph

Many Hamilton circuits in a complete graph are the same circuit with different starting points. For example, in the graph K_3, shown below in Figure 6.4.3, ABCA is the same circuit as BCAB, just with a different starting point (reference point). We will typically assume that the reference point is A.

Figure 6.4.3: K_3



Number of Hamilton Circuits: A complete graph with N vertices is (N-1)! Hamilton circuits. Since half of the circuits are mirror images of the other half, there are actually only half this many unique circuits.

Example 6.4.4: Number of Hamilton Circuits

How many Hamilton circuits does a graph with five vertices have?

$(N - 1)! = (5 - 1)! = 4! = 4*3*2*1 = 24$ Hamilton circuits.

How to solve a Traveling Salesman Problem (TSP):

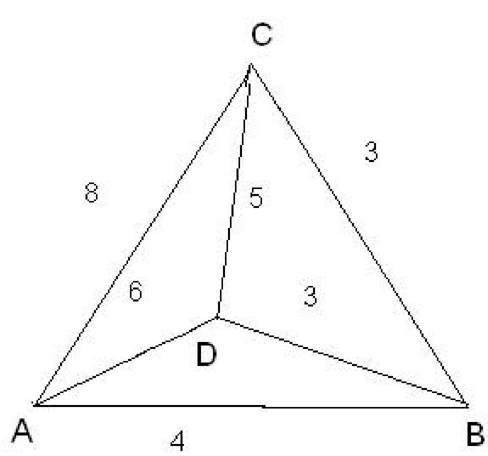
A traveling salesman problem is a problem where you imagine that a traveling salesman goes on a business trip. He starts in his home city (A) and then needs to travel to several different cities to sell his wares (the other cities are B, C, D, etc.). To solve a TSP, you need to find the cheapest way for the traveling salesman to start at home, A, travel to the other cities, and then return home to A at the end of the trip. This is simply finding the Hamilton circuit in a complete graph that has the smallest overall weight. There are several different algorithms that can be used to solve this type of problem.

A. Brute Force Algorithm

1. List all possible Hamilton circuits of the graph.
2. For each circuit find its total weight.
3. The circuit with the least total weight is the optimal Hamilton circuit.

Example 6.4.5: Brute Force Algorithm:

Figure 6.4.4: Complete Graph for Brute Force Algorithm



Suppose a delivery person needs to deliver packages to three locations and return to the home office A. Using the graph shown above in Figure 6.4.4, find the shortest route if the weights on the graph represent distance in miles.

Recall the way to find out how many Hamilton circuits this complete graph has. The complete graph above has four vertices, so the number of Hamilton circuits is:

$(N-1)! = (4-1)! = 3! = 3*2*1 = 6$ Hamilton circuits.

However, three of those Hamilton circuits are the same circuit going the opposite direction (the mirror image).

Hamilton Circuit	**Mirror Image**	**Total Weight (Miles)**
ABCDA	ADCBA	18
ABDCA	ACDBA	20
ACBDA	ADBCA	20

The solution is ABCDA (or ADCBA) with total weight of 18 mi. This is the optimal solution.

B. Nearest-Neighbor Algorithm:

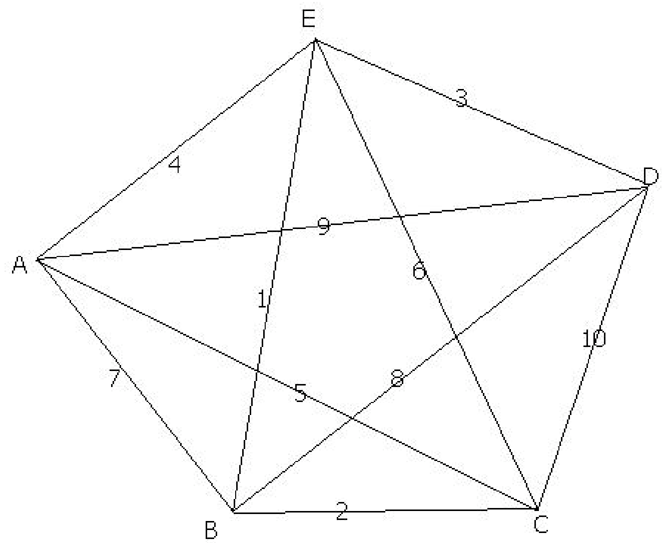
1. Pick a vertex as the starting point.
2. From the starting point go to the vertex with an edge with the smallest weight. If there is more than one choice, choose at random.
3. Continue building the circuit, one vertex at a time from among the vertices that have not been visited yet.
4. From the last vertex, return to the starting point.

Example 6.4.6: Nearest-Neighbor Algorithm

A delivery person needs to deliver packages to four locations and return to the home office A as shown in Figure 6.4.5 below. Find the shortest route if the weights represent distances in miles.

Starting at A, E is the nearest neighbor since it has the least weight, so go to E. From E, B is the nearest neighbor so go to B. From B, C is the nearest neighbor so go to C. From C, the first nearest neighbor is B, but you just came from there. The next nearest neighbor is A, but you do not want to go there yet because that is the starting point. The next nearest neighbor is E, but you already went there. So go to D. From D, go to A since all other vertices have been visited.

Figure 6.4.5: Complete Graph for Nearest-Neighbor Algorithm



The solution is AEBCDA with a total weight of 26 miles. This is not the optimal solution, but it is close and it was a very efficient method.

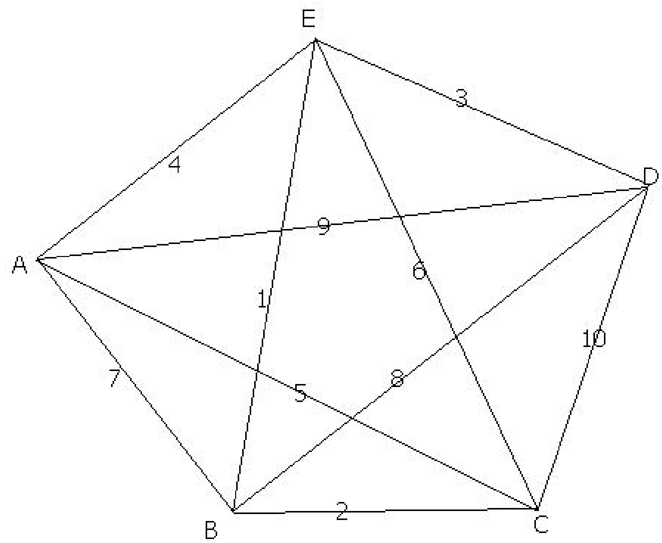
C. Repetitive Nearest-Neighbor Algorithm:

1. Let X be any vertex. Apply the Nearest-Neighbor Algorithm using X as the starting vertex and calculate the total cost of the circuit obtained.
2. Repeat the process using each of the other vertices of the graph as the starting vertex.
3. Of the Hamilton circuits obtained, keep the best one. If there is a designated starting vertex, rewrite this circuit with that vertex as the reference point.

Example 6.4.7: Repetitive Nearest-Neighbor Algorithm

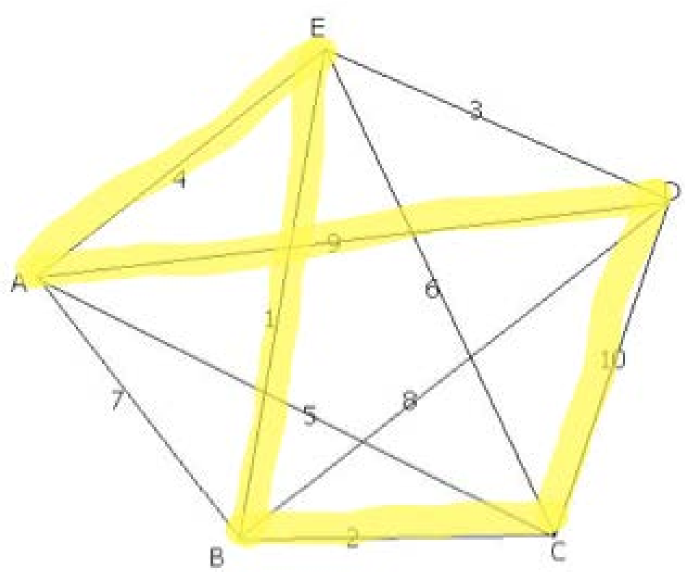
Suppose a delivery person needs to deliver packages to four locations and return to the home office A. Find the shortest route if the weights on the graph represent distances in kilometers.

Figure 6.4.6: Complete Graph for Repetitive Nearest-Neighbor Algorithm



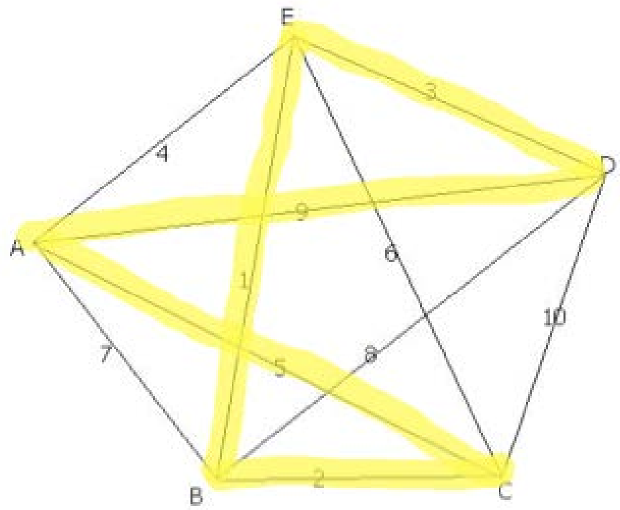
Starting at A, the solution is AEBCDA with total weight of 26 miles as we found in Example 6.4.6. See this solution below in Figure 6.4.7.

Figure 6.4.7: Starting at A



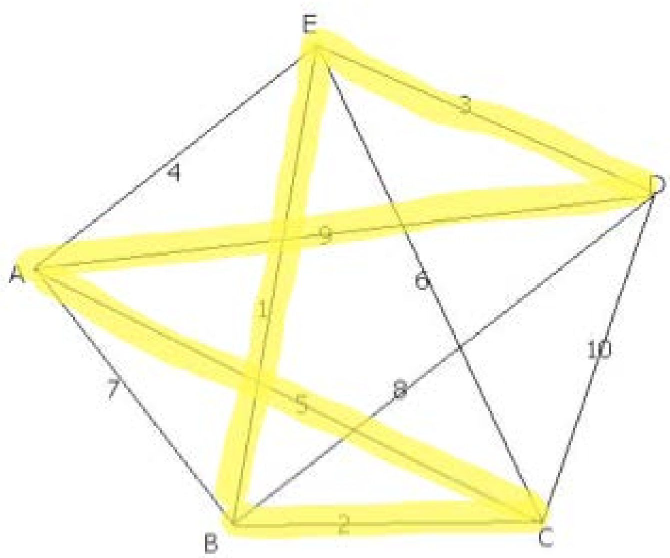
Starting at B, the solution is BEDACB with total weight of 20 miles.

Figure 6.4.8: Starting at B



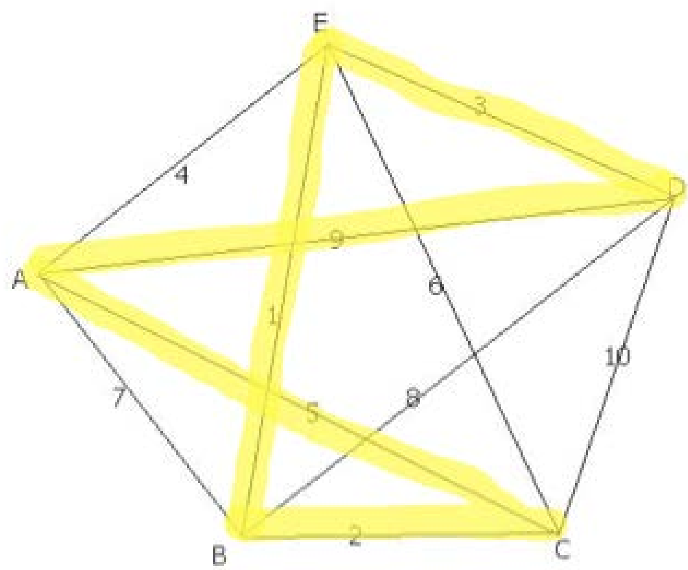
Starting at C, the solution is CBEDAC with total weight of 20 miles.

Figure 6.4.9: Starting at C



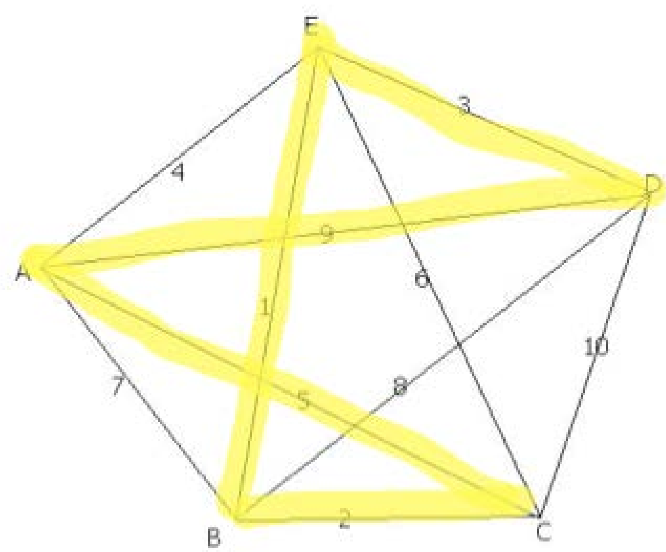
Starting at D, the solution is DEBCAD with total weight of 20 miles.

Figure 6.4.10: Starting at D



Starting at E, solution is EBCADE with total weight of 20 miles.

Figure 6.4.11: Starting at E



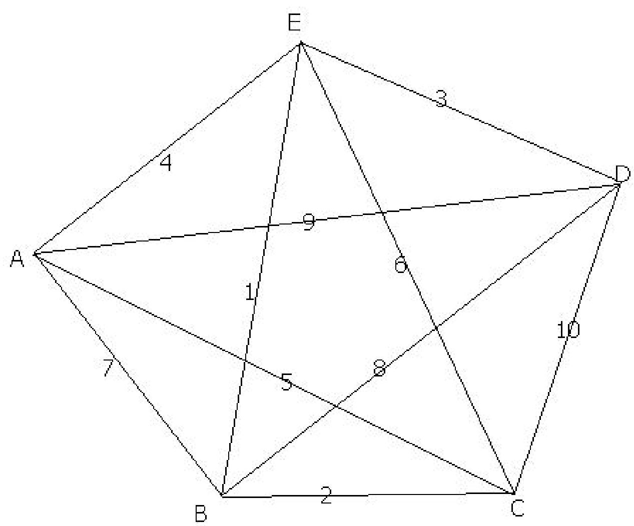
Now, you can compare all of the solutions to see which one has the lowest overall weight. The solution is any of the circuits starting at B, C, D, or E since they all have the same weight of 20 miles. Now that you know the best solution using this method, you can rewrite the circuit starting with any vertex. Since the home office in this example is A, let's rewrite the solutions starting with A. Thus, the solution is ACBEDA or ADEBCA.

D. Cheapest-Link Algorithm

1. Pick the link with the smallest weight first (if there is a tie, randomly pick one). Mark the corresponding edge in red.
2. Pick the next cheapest link and mark the corresponding edge in red.
3. Continue picking the cheapest link available. Mark the corresponding edge in red except when a) it closes a circuit or b) it results in three edges coming out of a single vertex.
4. When there are no more vertices to link, close the red circuit.

Example 6.4.8: Cheapest-Link Algorithm

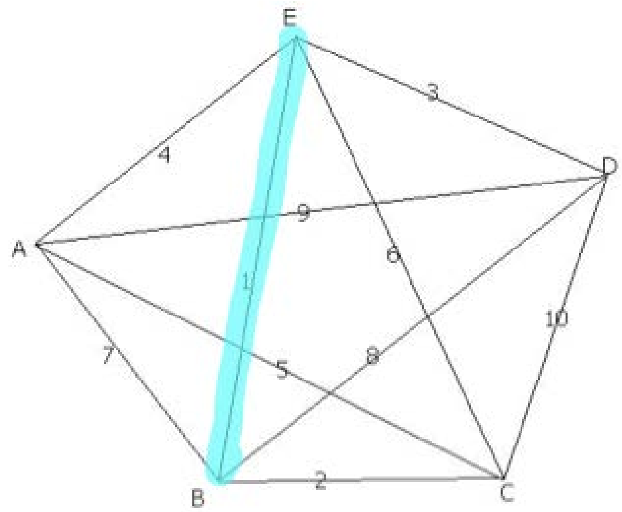
Figure 6.4.12: Complete Graph for Cheapest-Link Algorithm



Suppose a delivery person needs to deliver packages to four locations and return to the home office A. Find the shortest route if the weights represent distances in miles.

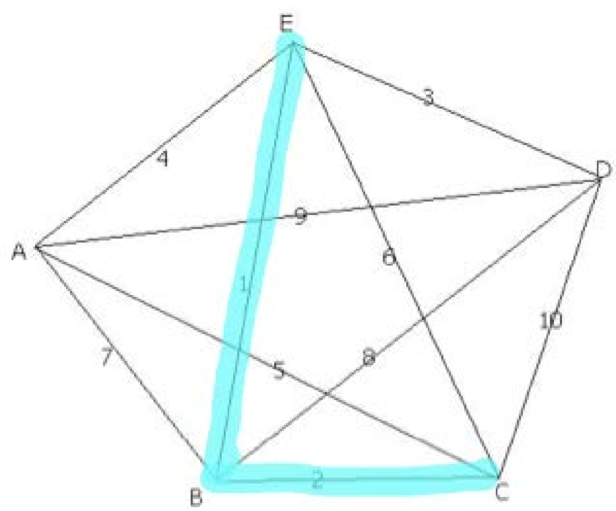
Step 1: Find the cheapest link of the graph and mark it in blue. The cheapest link is between B and E with a weight of one mile.

Figure 6.4.13: Step 1



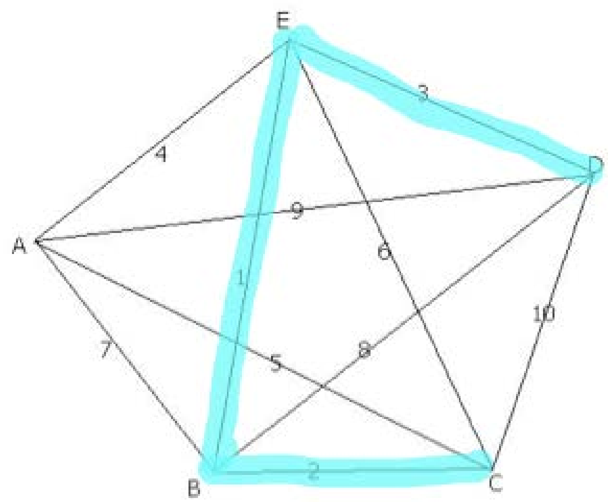
Step 2: Find the next cheapest link of the graph and mark it in blue. The next cheapest link is between B and C with a weight of two miles.

Figure 6.4.14: Step 2



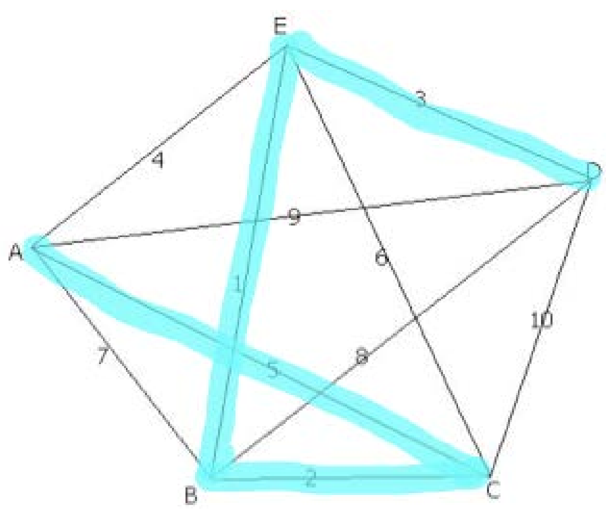
Step 3: Find the next cheapest link of the graph and mark it in blue provided it does not make a circuit or it is not a third edge coming out of a single vertex. The next cheapest link is between D and E with a weight of three miles.

Figure 6.4.15: Step 3



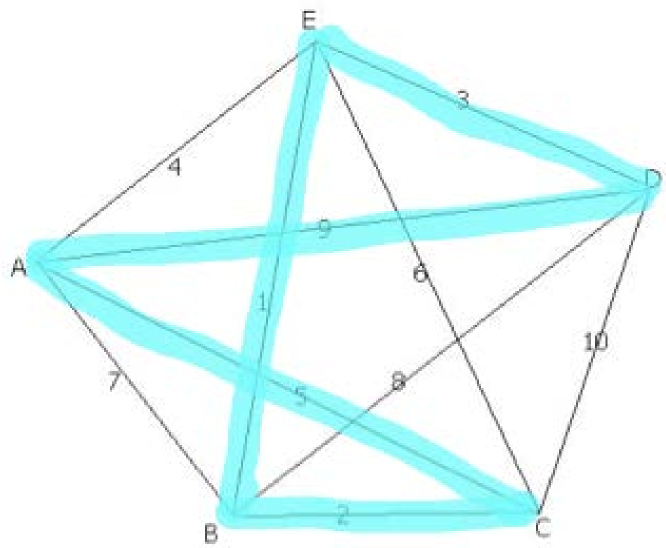
Step 4: Find the next cheapest link of the graph and mark it in blue provided it does not make a circuit or it is not a third edge coming out of a single vertex. The next cheapest link is between A and E with a weight of four miles, but it would be a third edge coming out of a single vertex. The next cheapest link is between A and C with a weight of five miles. Mark it in blue.

Figure 6.4.16: Step 4



Step 5: Since all vertices have been visited, close the circuit with edge DA to get back to the home office, A. This is the only edge we could close the circuit with because AB creates three edges coming out of vertex B and BD also created three edges coming out of vertex B.

Figure 6.4.17: Step 5



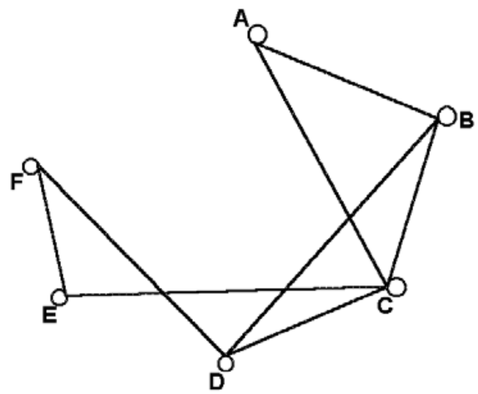
The solution is ACBEDA or ADEBCA with total weight of 20 miles.

Efficient Algorithm: an algorithm for which the number of steps needed to carry it out grows in proportion to the size of the input to the problem.

Approximate Algorithm: any algorithm for which the number of steps needed to carry it out grows in proportion to the size of the input to the problem. There is no known algorithm that is efficient and produces the optimal solution.

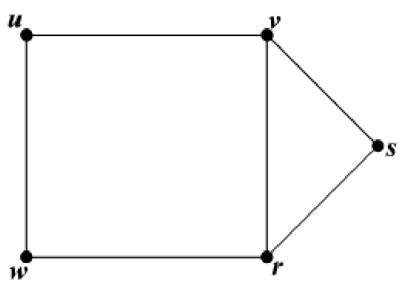
Chapter 6 Homework

1. Answer the following questions based on the graph below.



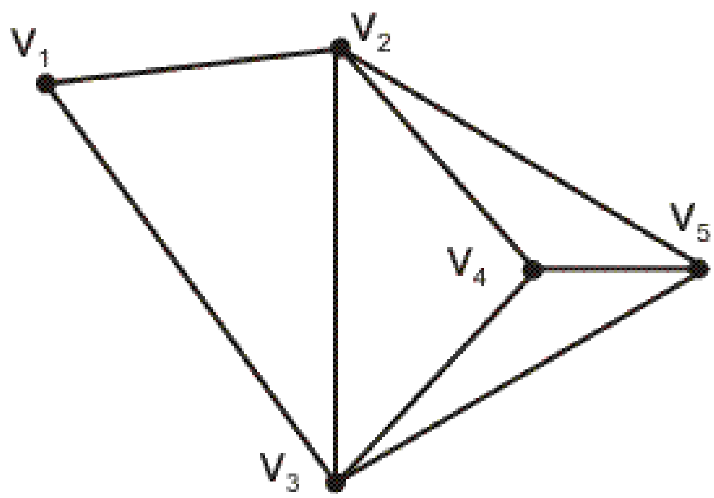
   a. What are the vertices?
   b. Is this graph connected?
   c. What is the degree of vertex C?
   d. Edge FE is adjacent to which edges?
   e. Does this graph have any bridges?

2. Answer the following questions based on the graph below.

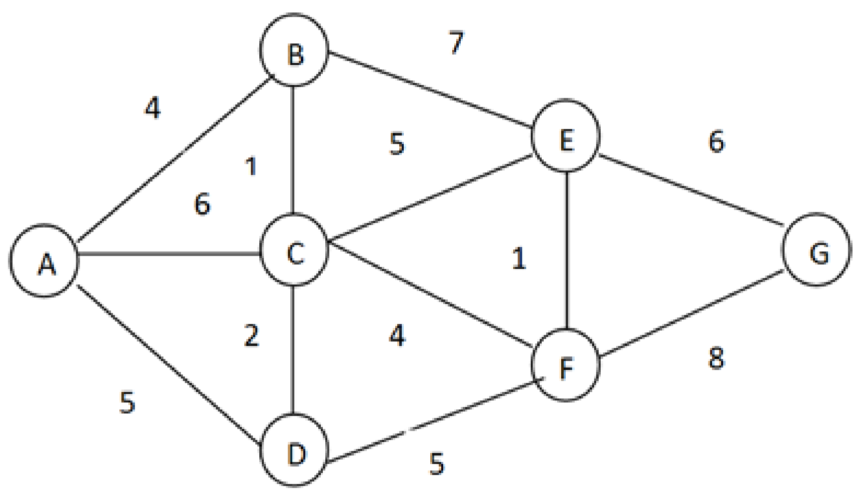


   a. What are the vertices?
   b. What is the degree of vertex *u*?
   c. What is the degree of vertex *s*?
   d. What is one circuit in the graph?

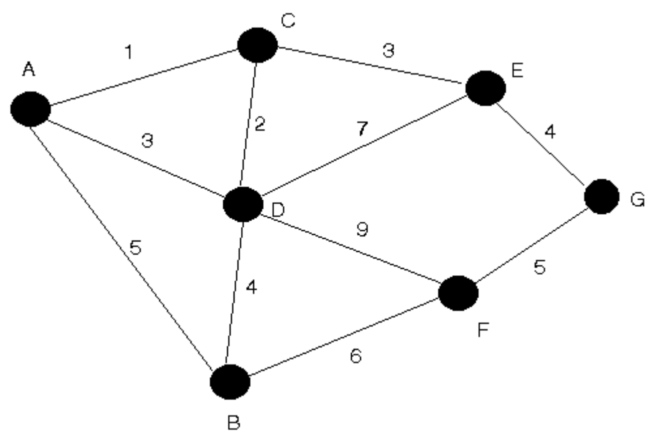
3. Draw a spanning subgraph in the graph below.



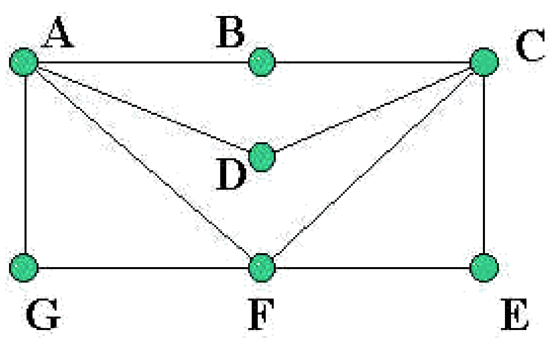
4. Find the minimum spanning tree in the graph below using Kruskal's Algorithm.



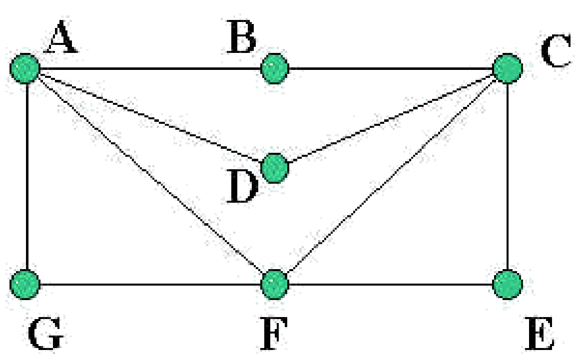
5. Find the minimum spanning tree in the graph below using Kruskal's Algorithm.



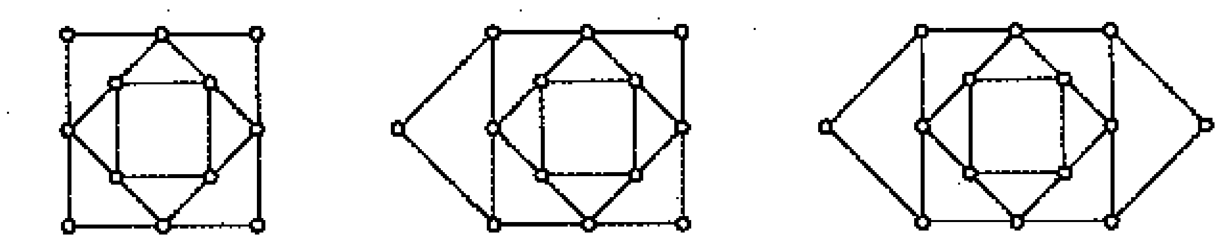
6. Find an Euler Path in the graph below.



7. Find an Euler circuit in the graph below.

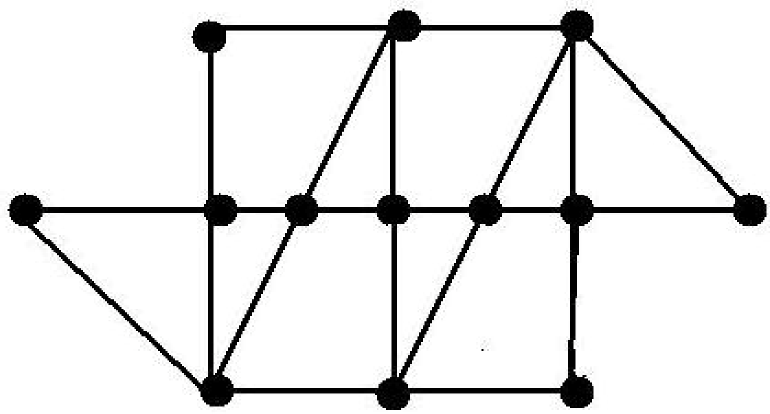


8. Which graphs below have Euler Paths? Which graphs have Euler circuits?

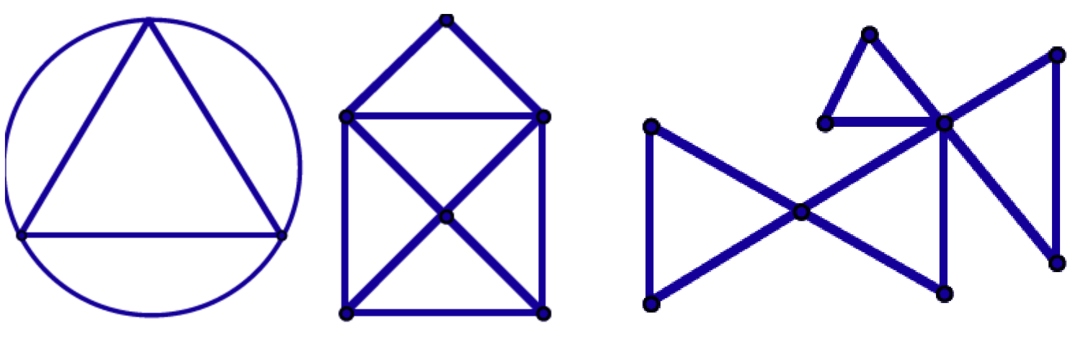


(a) (b) (c)

9. Highlight an Euler circuit in the graph below.

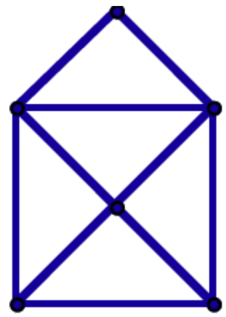


10. For each of the graphs below, write the degree of each vertex next to each vertex.

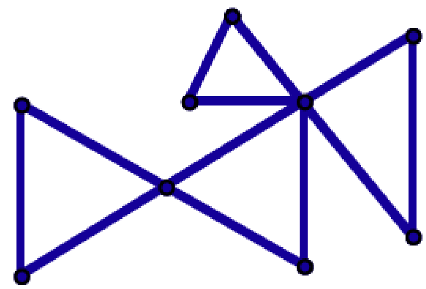


11. Circle whether each of the graphs below has an Euler circuit, an Euler path, or neither.

a. Euler circuit
b. Euler path
c. Neither



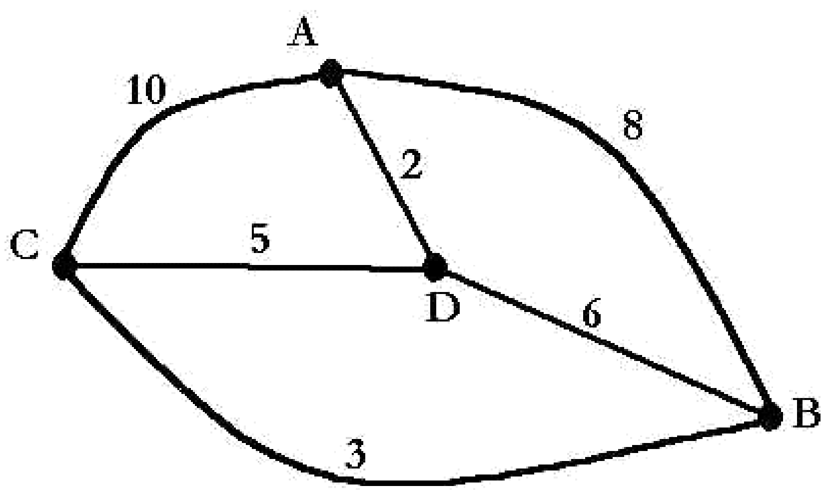
a. Euler circuit
b. Euler path
c. Neither



a. Euler circuit
b. Euler path
c. Neither

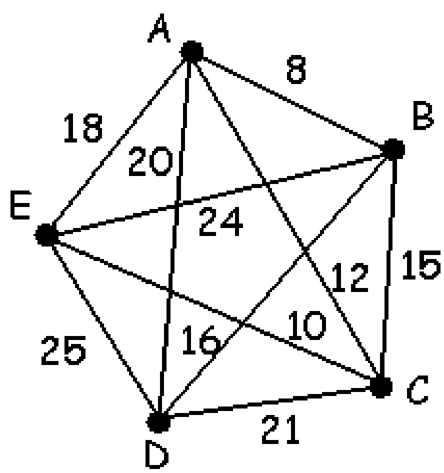
12. How many Hamilton circuits does a complete graph with 6 vertices have?

13. Suppose you need to start at A, visit all three vertices, and return to the starting point A. Using the Brute Force Algorithm, find the shortest route if the weights represent distances in miles.

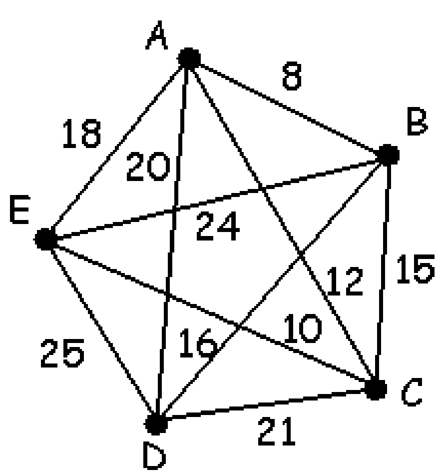


14. If a graph is connected and ___________________, the graph will have an Euler circuit.
    a. the graph has an even number of vertices
    b. the graph has an even number of edges
    c. the graph has all vertices of even degree
    d. the graph has only two odd vertices

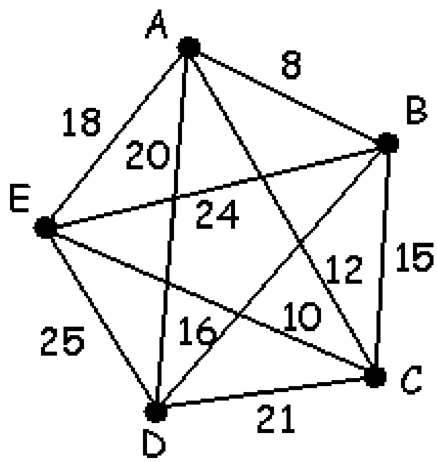
15. Starting at vertex A, use the Nearest-Neighbor Algorithm to find the shortest route if the weights represent distances in miles.



16. Find a Hamilton circuit using the Repetitive Nearest-Neighbor Algorithm.



17. Find a Hamilton circuit using the Cheapest-Link Algorithm.



18. Which is a circuit that traverses each edge of the graph exactly once?
    A. Euler circuit b. Hamilton circuit c. Minimum Spanning Tree

19. Which is a circuit that traverses each vertex of the graph exactly once?
    A. Euler circuit b. Hamilton circuit c. Minimum Spanning Tree

20. For each situation, would you find an Euler circuit or a Hamilton Circuit?
    a. The department of Public Works must inspect all streets in the city to remove dangerous debris.
    b. Relief food supplies must be delivered to eight emergency shelters located at different sites in a large city.
    c. The Department of Public Works must inspect traffic lights at intersections in the city to determine which are still working.
    d. An insurance claims adjuster must visit 11 homes in various neighborhoods to write reports.

Chapter 7: Voting Systems

Section 7.1: Voting Methods

Every couple of years or so, voters go to the polls to cast ballots for their choices for mayor, governor, senator, president, etc. Then the election officials count the ballots and declare a winner. But how do the election officials determine who the winner is. If there are only two candidates, then there is no problem figuring out the winner. The candidate with more than 50% of the votes wins. This is known as the majority. So the candidate with the majority of the votes is the winner.

Majority Rule: This concept means that the candidate (choice) receiving more than 50% of the vote is the winner.

But what happens if there are three candidates, and no one receives the majority? That depends on where you live. Some places decide that the person with the most votes wins, even if they don't have a majority. There are problems with this, in that someone could be liked by 35% of the people, but is disliked by 65% of the people. So you have a winner that the majority doesn't like. Other places conduct runoff elections where the top two candidates have to run again, and then the winner is chosen from the runoff election. There are some problems with this method. First, it is very costly for the candidates and the election office to hold a second election. Second, you don't know if you will have the same voters voting in the second election, and so the preferences of the voters in the first election may not be taken into account.

So what can be done to have a better election that has someone liked by more voters yet doesn't require a runoff election? A ballot method that can fix this problem is known as a preference ballot.

Preference Ballots: Ballots in which voters choose not only their favorite candidate, but they actually order all of the candidates from their most favorite down to their least favorite.

Note: Preference Ballots are transitive: If a voter prefers choice A to choice B and also prefers choice B to choice C, then the voter must prefer choice A to choice C.

To understand how a preference ballot works and how to determine the winner, we will look at an example.

Example 7.1.1: Preference Ballot for the Candy Election

Suppose an election is held to determine which bag of candy will be opened. The choices (candidates) are Hershey's Miniatures (M), Nestle Crunch (C), and Mars' Snickers (S). Each voter is asked to fill in the following ballot, by marking their first, second, and third place choices.

Figure 7.1.1: Preference Ballot for the Candy Election

Candy	Preference
Crunch	________
Miniatures	________
Snickers	________

Each voter fills out the above ballot with their preferences, and what follows is the results of the election.

Table 7.1.2: Ballots Cast for the Candy Election

Voter	Anne	Bob	Chloe	Dylan	Eli	Fred
1st choice	C	M	C	M	S	S
2nd choice	S	S	M	C	M	M
3rd choice	M	C	S	S	C	C

Voter	George	Hiza	Isha	Jacy	Kalb	Lan
1st choice	S	S	S	M	C	M
2nd choice	M	M	M	C	M	C
3rd choice	C	C	C	S	S	S

Voter	Makya	Nadira	Ochen	Paki	Quinn	Riley
1st choice	S	S	C	C	S	S
2nd choice	M	M	M	M	M	M
3rd choice	C	C	S	S	C	C

Now we must count the ballots. It isn't as simple as just counting how many voters like each candidate. You have to look at how many liked the candidate in first-place, second place, and third place. So there needs to be a better way to organize the results. This is known as a preference schedule.

Preference Schedule: A table used to organize the results of all the preference ballots in an election.

Example 7.1.2: Preference Schedule for the Candy Election

Using the ballots from Example 7.1.1, we can count how many people liked each ordering. Looking at Table 7.1.2, you may notice that three voters (Dylan, Jacy, and Lan) had the order M, then C, then S. Bob is the only voter with the order M, then S, then C. Chloe, Kalb, Ochen, and Paki had the order C, M, S. Anne is the only voter who voted C, S, M. All the other 9 voters selected the order S, M, C. Notice, no voter liked the order S, C, M. We can summarize this information in a table, called the preference schedule.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

Methods of Counting Ballots:

Now that we have organized the ballots, how do we determine the winner? There are several different methods that can be used. The easiest, and most familiar, is the Plurality Method.

Plurality Method: The candidate with the most first-place votes wins the election.

Example 7.1.3: The Winner of the Candy Election—Plurality Method

Using the preference schedule in Table 7.1.3, find the winner using the Plurality Method.

From the preference schedule you can see that four (3 + 1) people choose Hershey's Miniatures as their first choice, five (4 + 1) picked Nestle Crunch as their first choice, and nine picked Snickers as their first choice. So Snickers wins with the most first-place votes, although Snickers does not have the majority of first-place votes.

There is a problem with the Plurality Method. Notice that nine people picked Snickers as their first choice, yet seven chose it as their third choice. Thus, nine people may be happy if the Snickers bag is opened, but seven people will not be happy at all. So let's look at another way to determine the winner.

The Borda Count Method (Point System): Each place on a preference ballot is assigned points. Last place receives one point, next to last place receives two points, and so on. Thus, if there are *N* candidates, then first-place receives *N* points. Now, multiply the point value for each place by the number of voters at the top of the column to find the points each candidate wins in a column. Lastly, total up all the points for each candidate. The candidate with the most points wins.

Example 7.1.4: The Winner of the Candy Election—Borda Count Method

Using the preference schedule in Table 7.1.3, find the winner using the Borda Count Method.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

The third choice receives one point, second choice receives two points, and first choice receives three points. There were three voters who chose the order M, C, S. So M receives 3*3 = 9 points for the first-place, C receives 3*2 = 6 points, and S receives 3*1 = 3 points for those ballots. The same process is conducted for the other columns. The table below summarizes the points that each candy received.

Table 7.1.4: Preference Schedule of the Candy Election with Borda Count Points

Number of voters	3	1	4	1	9
1st choice	M 9	M 3	C 12	C 3	S 27
2nd choice	C 6	S 2	M 8	S 2	M 18
3rd choice	S 3	C 1	S 4	M 1	C 9

Adding up these points gives,
M = 9 + 3 + 8 + 1 + 18 = 39
C = 6 + 1 + 12 + 3 + 9 = 31
S = 3 + 2 + 4 + 2 + 27 = 38

Thus, Hershey's Miniatures wins using the Borda Count Method.

So who is the winner? With one method Snicker's wins and with another method Hershey's Miniatures wins. The problem is that it all depends on which method you use. Therefore, you need to decide which method to use before you run the election.

The Plurality with Elimination Method (Sequential Runoffs): Eliminate the candidate with the least amount of 1st place votes and re-distribute their votes amongst the other candidates. Repeat this process until you find a winner. *Note: At any time during this process if a candidate has a majority of first-place votes, then that candidate is the winner.*

Example 7.1.5: The Winner of the Candy Election—Plurality with Elimination Method

Using the preference schedule in Table 7.1.3, find the winner using the Plurality with Elimination Method.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

This isn't the most exciting example, since there are only three candidates, but the process is the same whether there are three or many more. So look at how many first-place votes there are. M has $3+1=4$, C has $4+1=5$, and S has 9. So M is eliminated from the preference schedule.

Table 7.1.5: Preference Schedule for the Candy Election with M Eliminated

Number of voters	3	1	4	1	9
1st choice	~~M~~	~~M~~	C	C	S
2nd choice	C	S	~~M~~	S	~~M~~
3rd choice	S	C	S	~~M~~	C

So the preference schedule becomes:

Table 7.1.6: Preference Schedule for the Candy Election with M Eliminated

Number of voters	3	1	4	1	9
1st choice	C	S	C	C	S
2nd choice	S	C	S	S	C

And then we can condense it down to:

Table 7.1.7: Preference Schedule for the Candy Election Condensed

Number of voters	8	10
1st choice	C	S
2nd choice	S	C

So C has eight first-place votes, and S has 10. So S wins.

The Method of Pairwise Comparisons: Compare each candidate to the other candidates in one-on-one match-ups. Give the winner of each pairwise comparison a point. The candidate with the most points wins.

Example 7.1.6: The Winner of the Candy Election—Pairwise Comparisons Method

Using the preference schedule in Table 7.1.3, find the winner using the Pairwise Comparisons Method.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

If you only have an election between M and C (the first one-on-one match-up), then M wins the three votes in the first column, the one vote in the second column, and the nine votes in the last column. That means that M has thirteen votes while C has five. So M wins when compared to C. M gets one point.

If you only compare M and S (the next one-on-one match-up), then M wins the first three votes in column one, the next one vote in column two, and the four votes in column three. M has eight votes and S has 10 votes. So S wins compared to M, and S gets one point.

Comparing C to S, C wins the three votes in column one, the four votes in column three, and one vote in column four. C has eight votes while S has 10 votes. So S wins compared to C, and S gets one point.

To summarize, M has one point, and S has two points. Thus, S wins the election using the Method of Pairwise Comparisons.

Table 7.1.8: Summary of One-on-One Match-Ups for the Candy Election

Match-Up 1	**Match-Up 2**	**Match-Up 3**
M vs. C	M vs. S	S vs. C
13 to 5	8 to 10	10 to 8
Winner of Match-Up 1: M	Winner of Match-Up 2: S	Winner of Match-Up 3: S

M: 1
S: 2
C: 0
Thus, S wins the election.

Note: If any one given match-up ends in a tie, then both candidates receive ½ point each for that match-up.

The problem with this method is that many overall elections (not just the one-on-one match-ups) will end in a tie, so you need to have a tie-breaker method designated before beginning the tabulation of the ballots. Another problem is that if there are more than three candidates, the number of pairwise comparisons that need to be analyzed becomes unwieldy. So, how many pairwise comparisons are there?

In Example 7.1.6, there were three one-on-one comparisons when there were three candidates. You may think that means the number of pairwise comparisons is the same as the number of candidates, but that is not correct. Let's see if we can come up with a formula for the number of candidates. Suppose you have four candidates called A, B, C, and D. A is to be matched up with B, C, and D (three comparisons). B is to be compared with C and D, but has already been compared with A (two comparisons). C needs to be compared with D, but has already been compared with A and B (one more comparison). Therefore, the total number of one-on-one match-ups is $3+2+1=6$ comparisons that need to be made with four candidates. What about five or six or more candidates? Looking at five candidates, the first candidate needs to be matched-up with four other candidates, the second candidate needs to be matched-up with three other candidates, the third candidate needs to be matched-up with two other candidates, and the fourth candidate needs to only be matched-up with the last candidate for one more match-up. Thus, the total is $4+3+2+1=10$ pairwise comparisons when there are five candidates.

Now, for six candidates, you would have $5+4+3+2+1=15$ pairwise comparisons to do. Continuing this pattern, if you have N candidates then there are $(N-1)+(N-2)+\cdots+3+2+1$ pairwise comparisons. For small numbers of candidates, it isn't hard to add these numbers up, but for large numbers of candidates there is a shortcut for adding the numbers together. It turns out that the following formula is true:

$(N-1)+(N-2)+\cdots+3+2+1=\frac{N(N-1)}{2}$. Thus, for 10 candidates, there are $9+8+7+6+5+4+3+2+1=\frac{10(10-1)}{2}=\frac{10(9)}{2}=45$ pairwise comparisons. So you can see that in this method, the number of pairwise comparisons to do can get large quite quickly.

Now that we have reviewed four different voting methods, how do you decide which method to use? One question to ask is which method is the fairest? Unfortunately, there is no completely fair method. This is based on Arrow's Impossibility Theorem.

Arrow's Impossibility Theorem: No voting system can satisfy all four fairness criteria in all cases.

This brings up the question, what are the four fairness criteria? They are guidelines that people use to help decide which voting method would be best to use under certain circumstances. They are the Majority Criterion, Condorcet Criterion, Monotonicity Criterion, and Independence of Irrelevant Alternatives Criterion.

Fairness Criteria:
The Majority Criterion (Criterion 1): If a candidate receives a majority of the 1st-place votes in an election, then that candidate should be the winner of the election.
The Condorcet Criterion (Criterion 2): If there is a candidate that in a head-to-head comparison is preferred by the voters over every other candidate, then that candidate should be the winner of the election. This candidate is known as the Condorcet candidate.
The Monotonicity Criterion (Criterion 3): If candidate X is a winner of an election and, in a re-election, the only changes in the ballots are changes that favor X, then X should remain a winner of the election.
The Independence of Irrelevant Alternatives Criterion (Criterion 4): If candidate X is a winner of an election and one (or more) of the other candidates is removed and the ballots recounted, then X should still be a winner of the election.

Example 7.1.7: Condorcet Criterion Violated

Suppose you have a vacation club trying to figure out where it wants to spend next year's vacation. The choices are Hawaii (H), Anaheim (A), or Orlando (O). The preference schedule for this election is shown below in Table 7.1.9.

Table 7.1.9: Preference Schedule of Vacation Election

Number of voters	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Using the Plurality Method, A has four first-place votes, O has three first-place votes, and H has three first-place votes. So, Anaheim is the winner. However, if you use the Method of Pairwise Comparisons, A beats O (A has seven while O has three), H beats A (H has six while A has four), and H beats O (H has six while O has four). Thus, Hawaii wins all pairwise comparisons against the other candidates, and would win the election. In fact Hawaii is the Condorcet candidate. However, the Plurality Method declared Anaheim the winner, so the Plurality Method violated the Condorcet Criterion.

Example 7.1.8: Monotonicity Criterion Violated

Suppose you have a voting system for a mayor. The resulting preference schedule for this election is shown below in Table 7.1.10.

Table 7.1.10: Preference Schedule of Mayoral Election

Number of voters	37	22	12	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

Using the Plurality with Elimination Method, Adams has 37 first-place votes, Brown has 34, and Carter has 29, so Carter would be eliminated. Carter's votes go to Adams, and Adams wins. Suppose that the results were announced, but then the election officials accidentally destroyed the ballots before they could be certified, so the election must be held again. Wanting to "jump on the bandwagon," 10 of the voters who had originally voted in the order Brown, Adams, Carter; change their vote to the order of Adams, Brown, Carter. No other voting changes are made. Thus, the only voting changes are in favor of Adams. The new preference schedule is shown below in Table 7.1.11.

Table 7.1.11: Preference Schedule of Mayoral Re-election

Number of voters	47	22	2	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

Now using the Plurality with Elimination Method, Adams has 47 first-place votes, Brown has 24, and Carter has 29. This time, Brown is eliminated first instead of Carter. Two of Brown's votes go to Adams and 22 of Brown's votes go to Carter. Now, Adams has 47 + 2 = 49 votes and Carter has 29 + 22 = 51 votes. Carter wins the election. This doesn't make sense since Adams had won the election before, and the only changes that were made to the ballots were in favor of Adams. However, Adams doesn't win the re-election. The reason that this happened is that there was a difference in who was eliminated first, and that caused a difference in how the votes are re-distributed. In this example, the Plurality with Elimination Method violates the Monotonicity Criterion.

Example 7.1.9: Majority Criterion Violated

Suppose a group is planning to have a conference in one of four Arizona cities: Flagstaff, Phoenix, Tucson, or Yuma. The votes for where to hold the conference are summarized in the preference schedule shown below in Table 7.1.12.

Table 7.1.12: Preference Schedule for Conference City

Number of voters	51	25	10	14
1st choice	Flagstaff	Phoenix	Yuma	Tucson
2nd choice	Phoenix	Yuma	Phoenix	Phoenix
3rd choice	Tucson	Tucson	Tucson	Yuma
4th choice	Yuma	Flagstaff	Flagstaff	Flagstaff

If we use the Borda Count Method to determine the winner then the number of Borda points that each candidate receives are shown in Table 7.1.13.

Table 7.1.13: Preference Schedule for Conference City with Borda Points

Number of voters	51	25	10	14
1st choice **4 points**	Flagstaff 204	Phoenix 100	Yuma 40	Tucson 56
2nd choice **3 points**	Phoenix 153	Yuma 75	Phoenix 30	Phoenix 42
3rd choice **2 points**	Tucson 102	Tucson 50	Tucson 20	Yuma 28
4th choice **1 point**	Yuma 51	Flagstaff 25	Flagstaff 10	Flagstaff 14

The totals of all the Borda points for each city are:

Flagstaff: $204+25+10+14=253$ points

Phoenix: $153+100+30+42=325$ points

Yuma: $51+75+40+28=194$ points

Tucson: $102+50+20+56=228$ points

Phoenix wins using the Borda Count Method. However, notice that Flagstaff actually has the majority of first-place votes. There are 100 voters total and 51 voters voted for Flagstaff in first place (51/100 = 51% or a majority of the first-place votes). So, Flagstaff should have won based on the Majority Criterion. This shows how the Borda Count Method can violate the Majority Criterion.

Example 7.1.10: Independence of Irrelevant Alternatives Criterion Violated

A committee is trying to award a scholarship to one of four students: Anna (A), Brian (B), Carlos (C), and Dmitri (D). The votes are shown below.

Table 7.1.14: Preference Schedule for Scholarship

Number of voters	5	5	6	4
1st choice	D	A	C	B
2nd choice	A	C	B	D
3rd choice	C	B	D	A
4th choice	B	D	A	C

Using the Method of Pairwise Comparisons:

A vs B: 10 votes to 10 votes, A gets ½ point and B gets ½ point

A vs C: 14 votes to 6 votes, A gets 1 point

A vs D: 5 votes to 15 votes, D gets 1 point

B vs C: 4 votes to 16 votes, C gets 1 point

B vs D: 15 votes to 5 votes, B gets 1 point

C vs D: 11 votes to 9 votes, C gets 1 point

So A has 1½ points, B has 1 point, C has 2 points, and D has 1 point. So Carlos is awarded the scholarship.

Now suppose it turns out that Dmitri didn't qualify for the scholarship after all. Though it should make no difference, the committee decides to recount the vote. The preference schedule without Dmitri is below.

Table 7.1.15: Preference Schedule for Scholarship with Dmitri Removed

Number of voters	10	6	4
1st choice	A	C	B
2nd choice	C	B	A
3rd choice	B	A	C

Using the Method of Pairwise Comparisons:
A vs B: 10 votes to 10 votes, A gets ½ point and B gets ½ point
A vs C: 14 votes to 6 votes, A gets 1 point
B vs C: 4 votes to 16 votes, C gets 1 point

So A has 1½ points, B has ½ point, and C has 1 point. Now Anna is awarded the scholarship instead of Carlos. This is an example of The Method of Pairwise Comparisons violating the Independence of Irrelevant Alternatives Criterion.

In summary, every one of the fairness criteria can possibly be violated by at least one of the voting methods as shown in Table 7.1.16. However, keep in mind that this does not mean that the voting method in question will violate a criterion in every election. It is just important to know that these violations are possible.

Table 7.1.16: Summary of Violations of Fairness Criteria

	Plurality	**Borda Count**	**Plurality with Elimination**	**Pairwise Comparisons**
Majority Criterion	*	Violation Possible	*	*
Condorcet Criterion	Violation Possible	Violation Possible	Violation Possible	*
Monotonicity Criterion	*	*	Violation Possible	*
Independence of Irrelevant Alternatives Criterion	Violation Possible	Violation Possible	Violation Possible	Violation Possible

* *The indicated voting method does not violate the indicated criterion in any election.*

Insincere Voting:
This is when a voter will not vote for whom they most prefer because they are afraid that the person they are voting for won't win, and they really don't want another candidate to win. So, they may vote for the person whom they think has the best chance of winning over the person they don't want to win. This happens often when there is a third party

candidate running. As an example, if a Democrat, a Republican, and a Libertarian are all running in the same race, and you happen to prefer the Libertarian candidate. However, you are afraid that the Democratic candidate will win if you vote for the Libertarian candidate, so instead you vote for the Republican candidate. You have voted insincerely to your true preference.

Approval Voting:
Since there is no completely fair voting method, people have been trying to come up with new methods over the years. One idea is to have the voters decide whether they approve or disapprove of candidates in an election. This way, the voter can decide that they would be happy with some of the candidates, but would not be happy with the other ones. A possible ballot in this situation is shown in Table 7.1.17:

Table 7.1.17: Approval Voting Ballot

Candidate	**Approve**	**Disapprove**
Smith	X	
Baker		X
James		X
Paulsen	X	

This voter would approve of Smith or Paulsen, but would not approve of Baker or James. In this type of election, the candidate with the most approval votes wins the election.

One issue with approval voting is that it tends to elect the least disliked candidate instead of the best candidate. Another issue is that it can result in insincere voting as described above.
As a reminder, there is no perfect voting method. Arrow proved that there never will be one. So make sure that you determine the method of voting that you will use before you conduct an election.

Section 7.2: Weighted Voting

Voting Power:
There are some types of elections where the voters do not all have the same amount of power. This happens often in the business world where the power that a voter possesses may be based on how many shares of stock he/she owns. In this situation, one voter may control the equivalent of 100 votes where other voters only control 15 or 10 or fewer votes. Therefore, the amount of power that each voter possesses is different. Another example is in how the President of the United States is elected. When a person goes to the

polls and casts a vote for President, he or she is actually electing who will go to the Electoral College and represent that state by casting the actual vote for President. Each state has a certain number of Electoral College votes, which is determined by the number of Senators and number of Representatives in Congress. Some states have more Electoral College votes than others, so some states have more power than others. How do we determine the power that each state possesses?

To figure out power, we need to first define some concepts of a weighted voting system. The individuals or entities that vote are called **players**. The notation for the players is $P_1, P_2, P_3, \ldots, P_N$, where N is the number of players. Each player controls a certain number of votes, which are called the **weight** of that player. The notation for the weights is $w_1, w_2, w_3, \ldots, w_N$, where w_1 is the weight of P_1, w_2 is the weight of P_2, etc. In order for a motion to pass, it must have a minimum number of votes. This minimum is known as the **quota**. The notation for quota is q. The quota must be over half the total weights and cannot be more than total weight. In other words,

$$\frac{w_1 + w_2 + w_3 + \cdots w_N}{2} < q \le w_1 + w_2 + w_3 + \cdots + w_N$$

The way to denote a weighted voting system is $[q : w_1, w_2, w_3, \ldots, w_N]$.

Example 7.2.1: Weighted Voting System

A company has 5 shareholders. Ms. Lee has 30% ownership, Ms. Miller has 25%, Mr. Matic has 22% ownership, Ms. Pierce has 14%, and Mr. Hamilton has 9%. There is a motion to decide where best to invest their savings. The company's by-laws define the quota as 58%. What does this voting system look like?

Treating the percentages of ownership as the votes, the system looks like:
$[58 : 30, 25, 22, 14, 9]$

Example 7.2.2: Valid Weighted Voting System

Which of the following are valid weighted voting systems?

a. $[8 : 5, 4, 4, 3, 2]$

The quota is 8 in this example. The total weight is $5 + 4 + 4 + 3 + 2 = 18$. Half of 18 is 9, so the quota must be $9 < q \le 18$. Since the quota is 8, and 8 is not more than 9, this system is not valid.

b. $[16:6,5,3,1]$

The quota is 16 in this example. The total weight is $6+5+3+1=15$. Half of 15 is 7.5, so the quota must be $7.5 < q \leq 15$. Since the quota is 16, and 16 is more than 15, this system is not valid.

c. $[9:5,4,4,3,1]$

The quota is 9 in this example. The total weight is $5+4+4+3+1=17$. Half of 17 is 8.5, so the quota must be $8.5 < q \leq 17$. Since the quota is 9, and 9 is more than 8.5 and less than 17, this system is valid.

d. $[16:5,4,3,3,1]$

The quota is 16 in this example. The total weight is $5+4+3+3+1=16$. Half of 16 is 8, so the quota must be $8 < q \leq 16$. Since the quota is 16, and 16 is equal to the maximum of the possible values of the quota, this system is valid. In this system, all of the players must vote in favor of a motion in order for the motion to pass.

e. $[9:10,3,2]$

The quota is 9 in this example. The total weight is $10+3+2=15$. Half of 15 is 7.5, so the quota must be $7.5 < q \leq 15$. Since the quota is 9, and 9 is between 7.5 and 15, this system is valid.

f. $[8:5,4,2]$

The quota is 8 in this example. The total weight is $5+4+2=11$. Half of 11 is 5.5, so the quota must be $5.5 < q \leq 11$. Since the quota is 8, and 8 is between 5.5 and 11, the system is valid.

In Example 7.2.2, some of the weighted voting systems are valid systems. Let's examine these for some concepts. In the system $[9:10,3,2]$, player one has a weight of 10. Since the quota is nine, this player can pass any motion it wants to. So, player one holds all the power. A player with all the power that can pass any motion alone is called a **dictator**. In the system $[16:5,4,3,3,1]$, every player has the same amount of power since all players are needed to pass a motion. That also means that any player can stop a motion from passing. A player that can stop a motion from passing is said to have **veto power**. In the system $[8:5,4,2]$, player three has a weight of two. Players one and two can join together and pass any motion without player three, and player three doesn't have enough weight to join with either player one or player two to pass a motion. So player three has no power. A player who has no power is called a **dummy**.

Example 7.2.3: Dictator, Veto Power, or Dummy?

In the weighted voting system $[57:23,21,16,12]$, are any of the players a dictator or a dummy or do any have veto power.

Since no player has a weight higher than or the same as the quota, then there is no dictator. If players one and two join together, they can't pass a motion without player three, so player three has veto power. Under the same logic, players one and two also have veto power. Player four cannot join with any players to pass a motion, so player four's votes do not matter. Thus, player four is a dummy.

Now that we have an understanding of some of the basic concepts, how do we quantify how much power each player has? There are two different methods. One is called the Banzhaf Power Index and the other is the Shapely-Shubik Power Index. We will look at each of these indices separately.

Banzhaf Power Index:

A coalition is a set of players that join forces to vote together. If there are three players P_1, P_2 and P_3 then the coalitions would be:

$\{P_1\},\{P_2\},\{P_3\},\{P_1,P_2\},\{P_1,P_3\},\{P_2,P_3\},\{P_1,P_2,P_3\}$.

Not all of these coalitions are winning coalitions. To find out if a coalition is winning or not look at the sum of the weights in each coalition and then compare that sum to the quota. If the sum is the quota or more, then the coalition is a winning coalition.

Example 7.2.4: Coalitions with Weights

In the weighted voting system $[17:12,7,3]$, the weight of each coalition and whether it wins or loses is in the table below.

Table 7.2.1: Coalition Listing

Coalition	Weight	Win or Lose?
$\{P_1\}$	12	Lose
$\{P_2\}$	7	Lose
$\{P_3\}$	3	Lose
$\{P_1, P_2\}$	19	Win
$\{P_1, P_3\}$	15	Lose
$\{P_2, P_3\}$	10	Lose
$\{P_1, P_2, P_3\}$	22	Win

In each of the winning coalitions you will notice that there may be a player or players that if they were to leave the coalition, the coalition would become a losing coalition. If there is such a player or players, they are known as the **critical player(s)** in that coalition.

Example 7.2.5: Critical Players

In the weighted voting system $[17:12,7,3]$, determine which player(s) are critical player(s). Note that we have already determined which coalitions are winning coalitions for this weighted voting system in Example 7.2.4. Thus, when we continue on to determine the critical player(s), we only need to list the winning coalitions.

Table 7.2.2: Winning Coalitions and Critical Players

Coalition	Weight	Win or Lose?	Critical Player
$\{P_1, P_2\}$	19	Win	P_1, P_2
$\{P_1, P_2, P_3\}$	22	Win	P_1, P_2

Notice, player one and player two are both critical players two times and player three is never a critical player.

Banzhaf Power Index:

The Banzhaf power index is one measure of the power of the players in a weighted voting system. In this index, a player's power is determined by the ratio of the number of times that player is critical to the total number of times any and all players are critical.

Banzhaf Power Index for Player $P_i = \dfrac{B_i}{T}$

where B_i = number of times player P_i is critical

and T = total number of times all players are critical

Example 7.2.6: Banzhaf Power Index

In the weighted voting system $[17:12,7,3]$, determine the Banzhaf power index for each player.

Using table 7.2.2, Player one is critical two times, Player two is critical two times, and Player three is never critical. So $T = 4$, $B_1 = 2$, $B_2 = 2$, and $B_3 = 0$. Thus:

Banzhaf power index of P_1 is $\frac{2}{4} = \frac{1}{2} = 0.5 = 50\%$

Banzhaf power index of P_2 is $\frac{2}{4} = \frac{1}{2} = 0.5 = 50\%$

Banzhaf power index of P_3 is $\frac{0}{4} = 0 = 0\%$

So players one and two each have 50% of the power. This means that they have equal power, even though player one has five more votes than player two. Also, player three has 0% of the power and so player three is a dummy.

How many coalitions are there? From the last few examples, we know that if there are three players in a weighted voting system, then there are seven possible coalitions. How about when there are four players?

Table 7.2.3: Coalitions with Four Players

1 Player	2 Players	3 Players	4 Players
$\{P_1\},\{P_2\},\{P_3\},\{P_4\}$	$\{P_1,P_2\},\{P_1,P_3\},\{P_1,P_4\}$ $\{P_2,P_3\},\{P_2,P_4\},\{P_3,P_4\}$	$\{P_1,P_2,P_3\},\{P_1,P_2,P_4\}$ $\{P_1,P_3,P_4\},\{P_2,P_3,P_4\}$	$\{P_1,P_2,P_3,P_4\}$

So when there are four players, it turns out that there are 15 coalitions. When there are five players, there are 31 coalitions (there are too many to list, so take my word for it). It doesn't look like there is a pattern to the number of coalitions, until you realize that 7, 15, and 31 are all one less than a power of two. In fact, seven is one less than 2^3, 15 is one less than 2^4, and 31 is one less than 2^5. So it appears that the number of coalitions for N players is $2^N - 1$.

Example 7.2.7: Banzhaf Power Index

Example 7.2.1 had the weighted voting system of $[58:30,25,22,14,9]$. Find the Banzhaf power index for each player.

Since there are five players, there are 31 coalitions. This is too many to write out, but if we are careful, we can just write out the winning coalitions. No player can win alone, so we can ignore all of the coalitions with one player. Also, no two-player coalition can win either. So we can start with the three player coalitions.

Table 7.2.4: Winning Coalitions and Critical Players

Winning Coalition	Critical Player
$\{P_1,P_2,P_3\}$	P_1,P_2,P_3
$\{P_1,P_2,P_4\}$	P_1,P_2,P_4
$\{P_1,P_2,P_5\}$	P_1,P_2,P_5
$\{P_1,P_3,P_4\}$	P_1,P_3,P_4
$\{P_1,P_3,P_5\}$	P_1,P_3,P_5
$\{P_2,P_3,P_4\}$	P_2,P_3,P_4
$\{P_1,P_2,P_3,P_4\}$	
$\{P_1,P_2,P_3,P_5\}$	P_1
$\{P_1,P_2,P_4,P_5\}$	P_1,P_2
$\{P_1,P_3,P_4,P_5\}$	P_1,P_3
$\{P_2,P_3,P_4,P_5\}$	P_2,P_3,P_4
$\{P_1,P_2,P_3,P_4,P_5\}$	

So player one is critical eight times, player two is critical six times, player three is critical six times, player four is critical four times, and player five is critical two times. Thus, the total number of times any player is critical is $T = 26$.

Banzhaf power index for P_1 $= \frac{8}{26} = \frac{4}{13}$ = 0.308 = 30.8%

Banzhaf power index for P_2 = $\frac{6}{26} = \frac{3}{13}$ = 0.231 = 23.1%

Banzhaf power index for P_3 = $\frac{6}{26} = \frac{3}{13}$ = 0.231 = 23.1%

Banzhaf power index for P_4 = $\frac{4}{26}=\frac{2}{13}$ = 0.154 = 15.4%

Banzhaf power index for P_5 = $\frac{2}{26}=\frac{1}{13}$ = 0.077 = 7.7%

Every player has some power. Player one has the most power with 30.8% of the power. No one has veto power, since no player is in every winning coalition.

Shapely-Shubik Power Index:
Shapely-Shubik takes a different approach to calculating the power. Instead of just looking at which players can form coalitions, Shapely-Shubik decided that all players form a coalition together, but the order that players join a coalition is important. This is called a **sequential coalition**. Instead of looking at a player leaving a coalition, this method examines what happens when a player joins a coalition. If when a player joins the coalition, the coalition changes from a losing to a winning coalition, then that player is known as a **pivotal player**. Now we count up how many times each player is pivotal, and then divide by the number of sequential coalitions. Note, that in reality when coalitions are formed for passing a motion, not all players will join the coalition. The sequential coalition is used only to figure out the power each player possess.

As an example, suppose you have the weighted voting system of $[17:12,7,3]$. One of the sequential coalitions is $\langle P_1,P_2,P_3\rangle$ which means that P_1 joins the coalition first, followed by P_2 joining the coalition, and finally, P_3 joins the coalition. When player one joins the coalition, the coalition is a losing coalition with only 12 votes. Then, when player two joins, the coalition now has enough votes to win (12 + 7 = 19 votes). Player three joining doesn't change the coalition's winning status so it is irrelevant. Thus, player two is the pivotal player for this coalition. Another sequential coalition is $\langle P_1,P_3,P_2\rangle$. When player one joins the coalition, the coalition is a losing coalition with only 12 votes. Then player three joins but the coalition is still a losing coalition with only 15 votes. Then player two joins and the coalition is now a winning coalition with 22 votes. So player two is the pivotal player for this coalition as well.

How many sequential coalitions are there for N players? Let's look at three players first. The sequential coalitions for three players (P_1, P_2, P_3) are:
$\langle P_1,P_2,P_3\rangle,\langle P_1,P_3,P_2\rangle,\langle P_2,P_1,P_3\rangle,\langle P_2,P_3,P_1\rangle,\langle P_3,P_1,P_2\rangle,\langle P_3,P_2,P_1\rangle$.

Note: The difference in notation: We use $\{\ \}$ *for coalitions and* $\langle\ \rangle$ *sequential coalitions.*

So there are six sequential coalitions for three players. Can we come up with a mathematical formula for the number of sequential coalitions? For the first player in the sequential coalition, there are 3 players to choose from. Once you choose one for the first spot, then there are only 2 players to choose from for the second spot. The third spot will only have one player to put in that spot. Notice, 3*2*1 = 6. It looks like if you have N players, then you can find the number of sequential coalitions by multiplying $N(N-1)(N-2)\cdots(3)(2)(1)$. This expression is called a *N* factorial, and is denoted by *N!*.

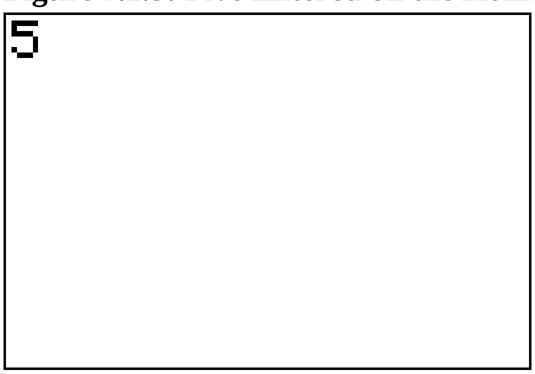
Most calculators have a factorial button. The process for finding a factorial on the TI-83/84 is demonstrated in the following example.

Example 7.2.8: Finding a Factorial on the TI-83/84 Calculator

Find 5! on the TI-83/84 Calculator.

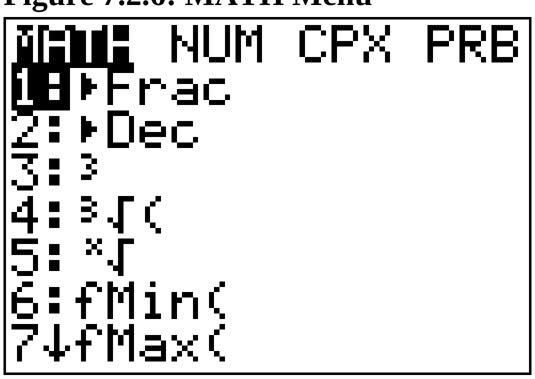
First, note that $5! = 5*4*3*2*1$, which is easy to do without the special button on the calculator, be we will use it anyway. First, input the number five on the home screen of the calculator.

Figure 7.2.5: Five Entered on the Home Screen



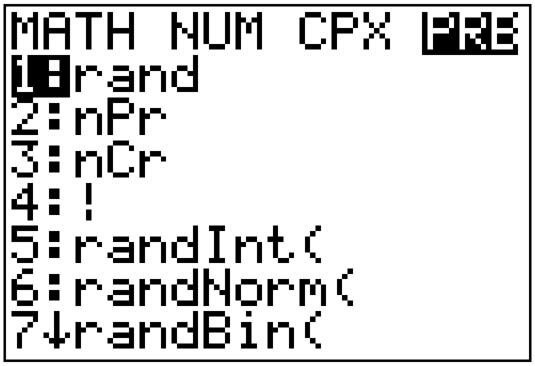
Then press the MATH button. You will see the following:

Figure 7.2.6: MATH Menu



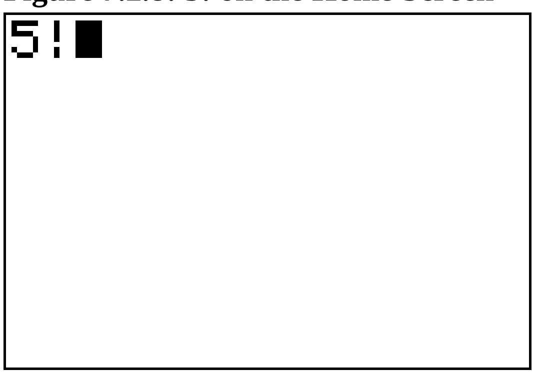
Now press the right arrow key to move over to the abbreviation PRB, which stands for probability.

Figure 7.2.7: PRB Menu



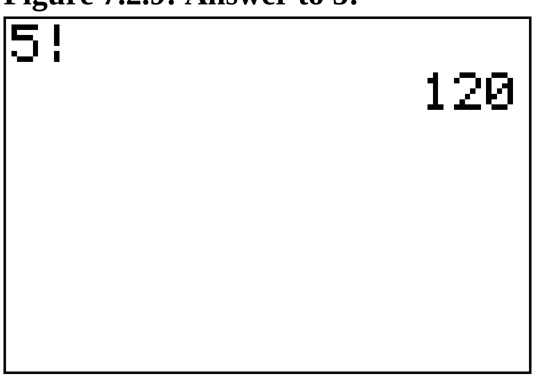
Number 4:! is the factorial button. Either arrow down to the number four and press ENTER, or just press the four button. This will put the ! next to your five on the home screen.

Figure 7.2.8: 5! on the Home Screen



Now press ENTER and you will see the result.

Figure 7.2.9: Answer to 5!



Notice that 5! is a very large number. So if you have 5 players in the weighted voting system, you will need to list 120 sequential coalitions. This is quite large, so most calculations using the Shapely-Shubik power index are done with a computer.

Now we have the concepts for calculating the Shapely-Shubik power index.

> **Shapely-Shubik Power Index** for Player $P_i = \dfrac{S_i}{N!}$
>
> where S_i is how often the player is pivotal
>
> N is the number of players and *N!* is the number of sequential coalitions

Example 7.2.9: Shapely-Shubik Power Index

In the weighted voting system $[17:12,7,3]$, determine the Shapely-Shubik power index for each player.

First list every sequential coalition. Then determine which player is pivotal in each sequential coalition. There are 3! = 6 sequential coalitions.

Table 7.2.10: Sequential Coalitions and Pivotal Players

Sequential coalition	Pivotal player
$\langle P_1, P_2, P_3 \rangle$	P_2
$\langle P_1, P_3, P_2 \rangle$	P_2
$\langle P_2, P_1, P_3 \rangle$	P_1
$\langle P_2, P_3, P_1 \rangle$	P_1
$\langle P_3, P_1, P_2 \rangle$	P_2
$\langle P_3, P_2, P_1 \rangle$	P_1

So, $S_1 = 3$, $S_2 = 3$, and $S_3 = 0$.

Shapely-Shubik power index for $P_1 = \dfrac{3}{6} = \dfrac{1}{2}$ = 0.5 = 50%

Shapely-Shubik power index for $P_2 = \dfrac{3}{6} = \dfrac{1}{2}$ = 0.5 = 50%

Shapely-Shubik power index for $P_3 = \dfrac{0}{6} = 0$ = 0%

This is the same answer as the Banzhaf power index. The two methods will not usually produce the same exact answer, but their answers will be close to the same value. Notice that player three is a dummy using both indices.

Example 7.2.10: Calculating the Power

For the voting system $[7:6,4,2]$, find:

a. The Banzhaf power index for each player

The first thing to do is list all of the coalitions and determine which ones are winning and which ones are losing. Then determine the critical player(s) in each winning coalition.

Table 7.2.11 Coalitions and Critical Players

Coalition	Weight	Win or Lose?	Critical Player
$\{P_1\}$	6	Lose	
$\{P_2\}$	4	Lose	
$\{P_3\}$	2	Lose	
$\{P_1,P_2\}$	10	Win	P_1, P_2
$\{P_1,P_3\}$	8	Win	P_1, P_3
$\{P_2,P_3\}$	6	Lose	
$\{P_1,P_2,P_3\}$	12	Win	P_1

So, $B_1=3,\ B_2=1,\ B_3=1,\ T=3+1+1=5$

Banzhaf power index of $P_1 = \frac{3}{5}$ = 0.6 = 60%

Banzhaf power index of $P_2 = \frac{1}{5}$ = 0.2 = 20%

Banzhaf power index of $P_3 = \frac{1}{5}$ = 0.2 = 20%

b. The Shapely-Shubik power index for each player

The first thing to do is list all of the sequential coalitions, and then determine the pivotal player in each sequential coalition.

Table 7.2.12: Sequential Coalitions and Pivotal Players

Sequential Coalition	Pivotal Player
$\langle P_1, P_2, P_3 \rangle$	P_2
$\langle P_1, P_3, P_2 \rangle$	P_3
$\langle P_2, P_1, P_3 \rangle$	P_1
$\langle P_2, P_3, P_1 \rangle$	P_1
$\langle P_3, P_1, P_2 \rangle$	P_1
$\langle P_3, P_2, P_1 \rangle$	P_1

So $S_1 = 4,\ \mathrm{S}_2 = 1,\ \mathrm{S}_3 = 1,\ 3! = 6$

Shapely-Shubik power index of $P_1 = \frac{4}{6} = \frac{2}{3}$ = 0.667 = 66.7%

Shapely-Shubik power index of $P_2 = \frac{1}{6}$ = 0.167 = 16.7%

Shapely-Shubik power index of $P_3 = \frac{1}{6}$ = 0.167 = 16.7%

Notice the two indices give slightly different results for the power distribution, but they are close to the same values.

Chapter 7 Homework

1. An organization recently made a decision about which company to use to redesign its website and host its members' information. The Board of Directors will vote using preference ballots ranking their first choice to last choice of the following companies: Allied Web Design (A), Ingenuity Incorporated (I), and Yeehaw Web Trends (Y). The individual ballots are shown below. Create a preference schedule summarizing these results.
   AIY, YIA, YAI, AIY, YIA, IAY, IYA, IAY, YAI, YIA, AYI, YIA, YAI

2. A group needs to decide where their next conference will be held. The choices are Kansas City (K), Lafayette (L), and Minneapolis (M). The individual ballots are shown below. Create a preference schedule summarizing these results.
   KLM, LMK, MLK, LMK, MKL, KLM, KML, LMK, MKL, MKL, MLK, MLK

3. A book club holds a vote to figure out what book they should read next. They are picking from three different books. The books are labeled A, B, and C, and the preference schedule for the vote is below.

Number of voters	12	9	8	5	10
1st choice	A	B	B	C	C
2nd choice	C	A	C	A	B
3rd choice	B	C	A	B	A

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

4. An election is held for a new vice president at a college. There are three candidates (A, B, C), and the faculty rank which candidate they like the most. The preference ballot is below.

Number of voters	8	10	12	9	4	1
1st choice	A	A	B	B	C	C
2nd choice	B	C	A	C	A	B
3rd choice	C	B	C	A	B	A

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

5. A city election for a city council seat was held between 4 candidates, Martorana (M), Jervey (J), Riddell (R), and Hanrahan (H). The preference schedule for this election is below.

Number of voters	60	73	84	25	110
1st choice	M	M	H	J	J
2nd choice	R	H	R	R	M
3rd choice	H	R	M	M	R
4th choice	J	J	J	H	H

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

6. A local advocacy group asks members of the community to vote on which project they want the group to put its efforts behind. The projects are green spaces (G), city energy code (E), water conservation (W), and promoting local business (P). The preference schedule for this vote is below.

Number of voters	12	57	23	34	13	18	22	39
1^{st} choice	W	W	G	G	E	E	P	P
2^{nd} choice	G	P	W	E	G	P	W	E
3^{rd} choice	E	E	E	W	P	W	G	W
4^{th} choice	P	G	P	P	W	G	E	G

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

7. An election is held and candidate A wins. A mistake was found that showed that candidate C was not qualified to run in the election. The candidate was removed, and the election office determined the winner with candidate C removed. Now candidate D wins. What fairness criterion was violated?

8. An election is held and candidate C wins. Before the election is certified the ballots are misplaced. Another election is held, and the only change in the ballots was that more people put C as their first choice. When the winner is determined, candidate A now wins. What fairness criterion was violated?

9. An election is held and candidate B has the majority of first-place votes. However, candidate B does not win the election. What fairness criterion was violated?

10. An election is held and candidate D is favored in a head-to-head comparison to every other candidate. However, D does not win the election. What is D called, and what fairness criterion was violated?

11. Consider the weighted voting system $[15:7,6,3,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

12. Consider the weighted voting system $[23:10,3,2,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

13. Consider the weighted voting system $[9:10,3,2,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

14. Consider the weighted voting system $[16:9,6,3,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

15. Consider the weighted voting system $[q:7,6,4,1]$.
    a. What is the minimum value of the quota q?
    b. What is the maximum value of the quota q?
    c. What is the quota q if a motion can only pass with 2/3's of the vote?
    d. What is the quota q if a motion can only pass with more than 2/3's of the vote?

16. Consider the weighted voting system $[q : 25, 20, 15, 15, 6]$.
    a. What is the minimum value of the quota q?
    b. What is the maximum value of the quota q?
    c. What is the quota q if a motion can only pass with 2/3's of the vote?
    d. What is the quota q if a motion can only pass with more than 2/3's of the vote?

17. Consider the weighted voting system $[12 : 13, 5, 4, 1]$. Are any players dictators? Explain.

18. Consider the weighted voting system $[16 : 12, 2, 2, 1]$. Do any players have veto power? Explain.

19. Consider the weighted voting system $[24 : 19, 16, 12]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapely-Shubik power index for each player.
    c. Are any players a dummy?

20. Consider the weighted voting system $[54 : 42, 13, 12]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapely-Shubik power index for each player.
    c. Are any players a dummy?

21. Consider the weighted voting system $[13 : 7, 6, 2]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapely-Shubik power index for each player.
    c. Are any players a dummy?

22. Consider the weighted voting system $[15 : 16, 12, 1]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapely-Shubik power index for each player.
    c. Are any players a dummy?

23. Consider the weighted voting system $[16:12,2,2,1]$. Find the Banzhaf power index for each player?

24. Consider the weighted voting system $[16:9,6,3,1]$. Find the Banzhaf power index for each player?

25. The United Nations (UN) Security Council consists of five permanent members (United States, Russian Federation, the United Kingdom, France, and China) and 10 non-permanent members elected for two-year terms by the General Assembly. The five permanent members have veto power, and a resolution cannot pass without nine members voting for it. Set up the weighted voting system for the UN.

Chapter 8: Fair Division

Many students believe that mathematics hasn't changed in several hundred years. However, the field of fair division is relatively young. Some of the methods discussed in this chapter were developed after the 1940s. This is an open field of study in mathematics. The methods we will look at do not always give the best possible answer but they are the best methods we have at this point in time.

Fair division tries to divide something in an equitable way. It can be used to divide up an estate, a jewelry collection, or a piece of land among heirs. Fair division can also be used to split up the assets of a business when a partnership is being dissolved. It can even be used by roommates to divide up the cleaning chores when the cleaning deposit is on the line.

Section 8.1: Basic Concepts of Fair Division

How do we divide items or collections of items among 2 or more people so that every person feels he/she received a fair share: Different people may assign a different value to the same item. A "fair share" to one person may not be the same as a "fair share" to another person. The methods in this chapter will guarantee that everyone gets a "fair share" but it might not be the "fair share" he/she wanted.

People often refer to fair division as a game. It has players and rules just like a game. The set of goods to be divided is called S. The players P_1, P_2, P_3,…, P_n are the parties entitled to a share of S. Each player must be able to assign a value to the set S or any subset of the set S.

In a **continuous** fair-division game the set S is divisible in an infinite number of ways, and shares can be increased or decreased by arbitrarily small amounts. Typical examples of continuous fair-division games involve the division of land, cake, pizza, etc… A fair-division game is **discrete** when the set S is made up of objects that are indivisible like paintings, houses, cars, boats, jewelry, etc. A pizza can be cut into slices of almost any size but a painting cannot be cut into pieces. To make the problems simple to think about, we will use cakes or pizzas for continuous examples and collections of small candies for discrete examples. A **mixed** fair-division game is one in which some of the components are continuous and some are discrete and is not covered in this book.

The method we use to divide a cake or pizza can be used to divide a piece of land or to divide the rights of access to mine the ocean floor (between countries). The method we use to divide a mixed bag of Halloween candy can be used to divide a large jewelry collection. This book will not get to all of them but we can also use fair division ideas to decide which transplant patient gets a liver when it becomes available, or to help two companies merge (who is CEO, who has layoffs, which name to use, etc.).

Rules: In order for the division of S to be fair, the players in the game must be willing participants and accept the rules of the game as binding.

- The players must act rationally according to their system of beliefs.
- The rules of mathematics apply when assigning values to the objects in S.
- Only the players are involved in the game, there are no outsiders such as lawyers.

If the players follow the rules the game will end after a finite number of moves by the players and result in a division of S.

Assumptions: We must assume the following:

- The players all play fair.
- They have no prior information about the likes or dislikes of the other players.
- They do not assign values in a way to manipulate the game.
- All players have equal rights in sharing the set S. In other words, if there are three players, each player is entitled to at least 1/3 of S.

If these assumptions are not met, the division may not be totally fair.

What is a fair share?

The basic idea in a continuous fair division game is that S is divided into pieces. Each player assigns a value to each piece of S. Based on these values a player decides which pieces he/she considers a fair share. Since each player is entitled to at least a proportional share of S, it is easy to determine what is considered a fair share.

Example 8.1.1: Determining Fair Shares

Four players Abby, Betty, Christy, and Debbie are to divide a cake S. The cake has been sliced into four pieces, S_1, S_2, S_3, and S_4 (not necessarily the same size or with the same amount of frosting). Each of the players has assigned a value to each piece of cake as shown in the following table.

Table 8.1.1: Values of Each Piece of Cake

Player/Piece	S_1	S_2	S_3	S_4
Abby	10%	50%	30%	10%
Betty	30%	30%	10%	30%
Christy	40%	20%	20%	20%
Debbie	25%	25%	25%	25%

Which of the pieces would each player consider a fair share?

Since there are four players, each player is entitled to at least $\frac{100\%}{4} = 25\%$ of S.

The fair shares are highlighted in the following table.

Table 8.1.2: Fair Shares for Each Player

Player/Piece	S_1	S_2	S_3	S_4
Abby	10%	50%	30%	10%
Betty	30%	30%	10%	30%
Christy	40%	20%	20%	20%
Debbie	25%	25%	25%	25%

Find the Value of a Piece of S:

How did the players in Example 8.1.1 determine the value of each piece: Part of the value is emotional and part is mathematical. Consider three heirs who must split up an island consisting of a stretch of beach, a small mountain, and a large meadow. A person who really likes the beach may be willing to settle for a smaller piece of the island to get the beach.

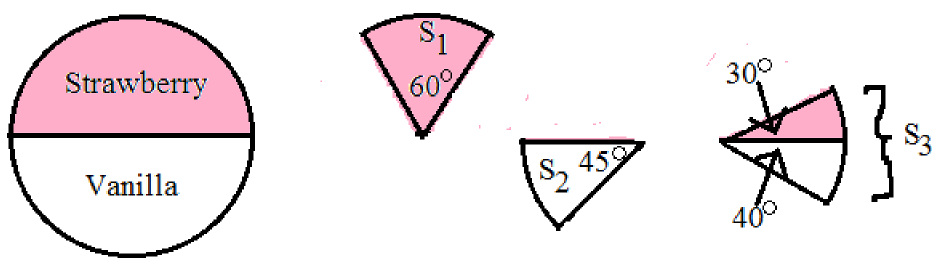
For the examples in this chapter, all cakes will be drawn in two-dimensions. The height of the cake is not relevant to the problem as long as the cake is uniform in height.

It will be helpful to remember that there are 360° in a full circle.

Example 8.1.2: The Value of a Slice of S, #1

Fred buys a half strawberry, half vanilla cake for $9. He loves vanilla cake but only likes strawberry cake so he values the vanilla half of the cake twice as much as the strawberry cake. Find the values of slices S_1, S_2, and S_3 in the following figure.

Figure 8.1.3: Half Strawberry/Half Vanilla Cake with Three Slices



Find the value of each half of the cake using algebra.

Let x = the value of the strawberry half of the cake.

Then 2x = the value of the vanilla half of the cake.

The total value of the cake is $9.

$$\begin{aligned} x + 2x &= \$9 \\ 3x &= \$9 \\ x &= \$3 \end{aligned}$$

The strawberry half is worth $3 and the vanilla half is worth $6.

a. Value of S_1:

60° is 1/3 of 180° (the strawberry half of the cake) so the slice is worth 1/3 of $3.

$$\frac{60^\circ}{180^\circ}(\$3) = \$1$$

b. Value of S_2:

45° is 1/4 of 180° (the vanilla half of the cake) so the slice is worth 1/4 of $6.

$$\frac{45^\circ}{180^\circ}(\$6) = \$1.50$$

c. Value of S_3:

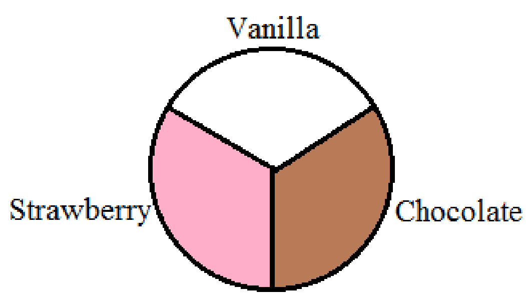
Since S_3 is part strawberry and part vanilla we must find the value of each part separately and then add them together.

$$\frac{30^\circ}{180^\circ}(\$3) + \frac{40^\circ}{180^\circ}(\$6) = \$1.83$$

Example 8.1.3: The Value of a Slice of S, #2

Tom and Fred were given a cake worth \$12 that is equal parts strawberry, vanilla and chocolate. Tom likes vanilla and strawberry the same but does not like chocolate at all. Fred will eat vanilla but likes strawberry twice as much as vanilla and likes chocolate three times as much as vanilla.

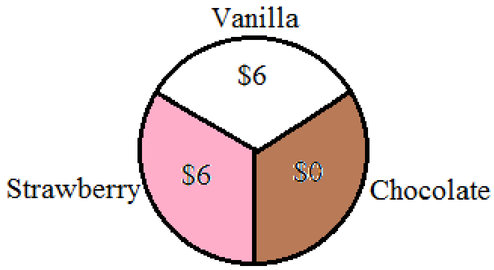
Figure 8.1.4: Three-Flavored Cake



a. How should Tom divide the cake into two pieces so that each piece is a fair share to him?

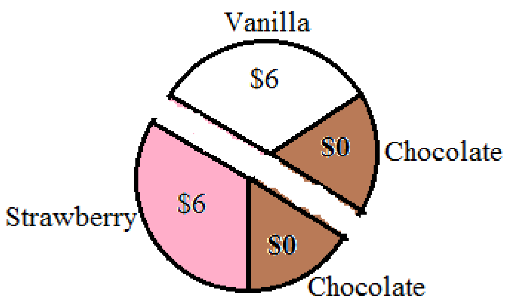
The cake is worth \$12 so a fair share is $\frac{\$12}{2} = \$6$. Tom must cut the cake into two pieces. They do not have to be the same size but they do have to have the same value of \$6 in Tom's eyes. Tom does not like chocolate so all the value of the cake is in the strawberry and vanilla parts. Also, Tom likes vanilla and strawberry equally well so each of these parts of the cakes are worth \$6 in his eyes.

Figure 8.1.5: How Tom Sees the Cake



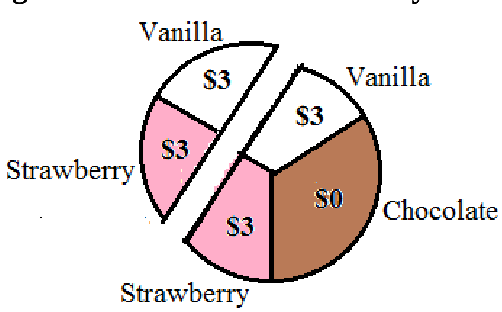
The easiest division is to cut the cake so that one slice contains all the strawberry part and half the chocolate part and the other slice contains all the vanilla part and the other half of the chocolate part.

Figure 8.1.6: Tom's Two Pieces



Note that Tom could slice the cake along a chord of the circle as shown below. The mathematics involved in that type of cut is beyond the scope of this course

Figure 8.1.7: Another Possibility for Tom's Two Pieces

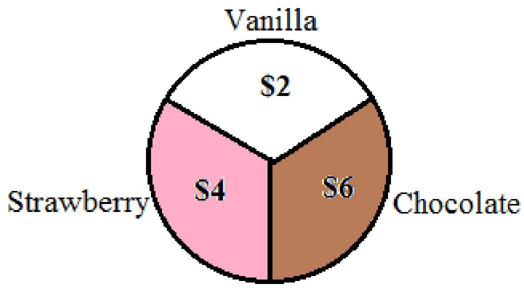


b. How should Fred divide the cake into two pieces so that each piece is a fair share to him?
Let x = value of the vanilla part to Fred.
Then 2x = value of the strawberry part.

Also, 3x = value of the chocolate part.

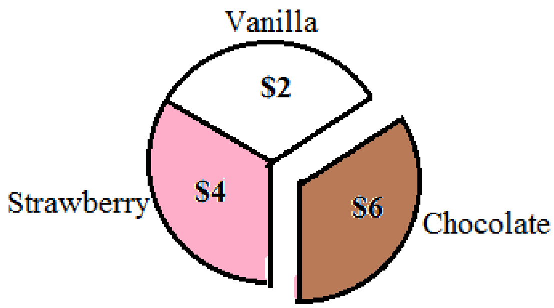
$$x + 2x + 3x = 12$$
$$6x = 12$$
$$x = 2$$
$$2x = 4$$
$$3x = 6$$

Figure 8.1.8: How Fred Sees the Cake



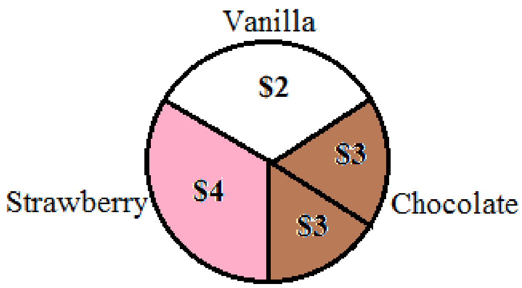
Fred needs to cut the cake into two pieces so that each piece has a value of $6. An obvious choice would be to make the chocolate part one of the pieces and both the vanilla and strawberry parts the other piece. As you can see in Figure 8.1.9 each of the pieces is worth a total of $6.

Figure 8.1.9: First Possibility for Fred's Two Pieces



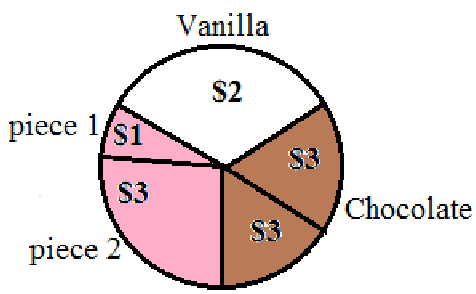
c. Find another way for Fred to divide the cake.
   There are many possibilities. Let's assume he wants to cut the chocolate part in half since he likes chocolate the most. This will guarantee that Fred gets some of the chocolate part regardless of which piece he gets in the game. The chocolate part is worth $6 to Fred so each half of the chocolate piece would be worth $3.

Figure 8.1.10: Dividing the Chocolate Part in Half



Fred needs to add another $3 worth of cake to each of the chocolate pieces. It is easy to see that Fred needs to cut the cake somewhere in the strawberry part so that the strawberry part is cut into two pieces. Let's call the smaller piece of the strawberry part "piece 1" and the larger piece of the strawberry part "piece 2." Piece 1 needs to be worth $1 and piece 2 needs to be worth $3 so that each slice of the cake ends up with a total of $6.

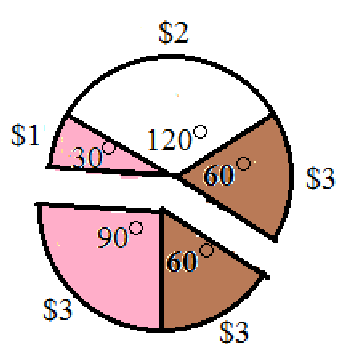
Figure 8.1.11: Dividing the Strawberry and Vanilla Parts



The strawberry part of the whole cake is 1/3 of the cake or $\frac{1}{3}\left(360^\circ\right)=120^\circ$.

Piece 2 is worth $3 of the $4 for the strawberry part so it is ¾ of the 120˚ or 90˚. Piece 1 would be 120˚ - 90˚ = 30˚.

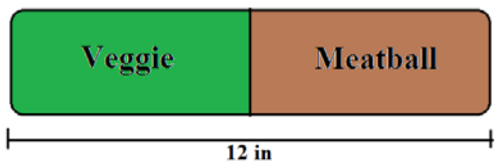
Figure 8.1.12: Second Possibility for Fred's Two Pieces



Example 8.1.4: The Value of a Slice of S, #3

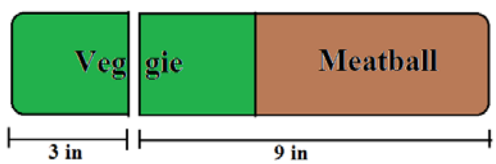
George and Ted wish to split a 12-inch sandwich worth $9. Half the sandwich is vegetarian and half the sandwich is meatball. George does not eat meat at all. Ted likes the meatball part twice as much as vegetarian part.

Figure 8.1.13: Sub Sandwich



a. How should George divide the sandwich so that each piece is a fair share to him?
George is a vegetarian so all the value is in the vegetarian half of the sandwich. He should divide the vegetarian part of the sandwich in half. One piece will be all of the meatball part plus three inches of the vegetarian part. The second piece will just be three inches of the vegetarian part.

Figure 8.1.14: How George Should Cut the Sandwich



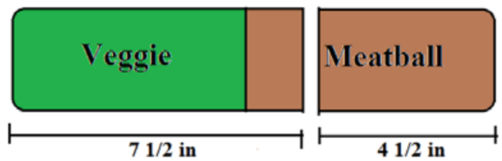
b. How should Ted divide the sandwich so that each piece is a fair share to him?
Let x be the value of vegetarian part of the sandwich.
Then 2x is the value of the meatball part of the sandwich.

$$x + 2x = \$9$$
$$3x = \$9$$
$$x = \$3$$

Ted sees the meatball part with a value of $6 and the vegetarian part with a value of $3. A fair share to Ted would be $4.50, half the value of the sandwich. Ted must divide the sandwich somewhere in the meatball part.

$$\frac{\$4.50}{\$6.00} = \frac{3}{4}, \quad \frac{3}{4}(6") = 4\tfrac{1}{2} \text{ inches}$$

Figure 8.1.15: How Ted Should Cut the Sandwich



Section 8.2: Continuous Methods 1: Divider/Chooser and Lone Divider Methods

The Divider/Chooser method and the Lone Divider method are two fairly simple methods for dividing a continuous set S. They can be used to split up a cake or to split up a piece of land. The Lone Divider method works for three or more players but works best with only three or four players. The Divider/Chooser method is a special case of the Lone Divider method for only two players.

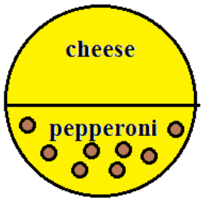
Divider/Chooser Method:

If you have siblings you probably used the Divider/Chooser method for fair division as a kid. Remember when Mom told one child to break the candy bar in half and then the other child got to choose which half to take: That was the Divider/Chooser method. It is a very simple method for dividing a single continuous item between two players. Simply put, one player cuts and the other player chooses. Call the object to be divided S. The divider is the player who cuts the object S. The divider is forced to cut the object S in a way that he/she would be satisfied with either piece as a fair share. The chooser then picks the piece that he/she considers a fair share. Once the chooser picks a piece the divider gets the remaining piece. The divider always gets exactly half the value of S. The chooser sometimes ends up with more than half of the value of S. This sounds contradictory but remember that each player has his/her own value system.

Example 8.2.1: Divider/Chooser Method with a Pizza

> Bill and Ted want to divide a pizza that is half cheese and half pepperoni. Bill likes cheese pizza but not pepperoni and Ted likes all pizza equally.

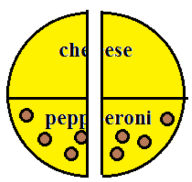
Figure 8.2.1: Half Cheese and Half Pepperoni Pizza



a. If Bill cuts and Ted chooses, describe the fair division.

Bill likes cheese but not pepperoni so he sees all the value of the pizza in the cheese part. He cuts the pizza in a way that half of the cheese part ends up in each piece. The most obvious way to do this is to cut it in half vertically. You might think that Bill would choose half the pizza as the cheese side and half as the pepperoni side in the hopes he would end up with the entire cheese side. However, that would not be a division that results in two equal halves in his eyes. In other words, he could end up with the entire pepperoni side which he does not like.

Figure 8.2.2: Bill's Division of the Pizza



Since Ted likes all pizza equally and both parts are the same it does not matter which piece Ted chooses. Let's say he chooses the piece on the right.

Ted is happy because he got half of the pizza, a fair share in his value system.

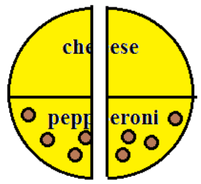
Bill is happy because he got half of the cheese part of the pizza, half of the value (or a fair share) in his value system.

b. If Ted cuts and Bill chooses, describe three different fair divisions.

Remember that one of our assumptions is that Ted does not know that Bill only likes the cheese part of the pizza. Since Ted likes all pizza equally, he should cut the pizza in half in terms of the volume (for our two-dimensional pizza, cut it in half in terms of the area).

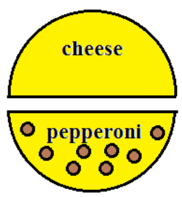
i. Ted could cut the pizza in half vertically just like Bill did in part (a). It would not matter which piece Bill chose since both pieces are the same.

Figure 8.2.3: Ted's Division of the Pizza, #1



ii. Ted could cut the pizza in half horizontally so that one piece was all cheese and the other piece was all pepperoni.

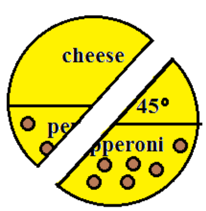
Figure 8.2.4: Ted's Division of the Pizza, #2



Bill would choose the cheese half and Ted would get the pepperoni half. Bill is happy because he gets 100% of the value of the pizza in his value system. Ted is happy because he gets 50% of the value of the pizza in his value system.

iii. Ted could cut the pizza at an angle so that each piece is part pepperoni and part cheese.

Figure 8.2.5: Ted's Division of the Pizza, #3



Since Bill only likes the cheese part, he should choose the piece on the left with the 75% of the cheese part of the pizza. Bill is happy because he gets 75% of the value of the pizza in his value system.

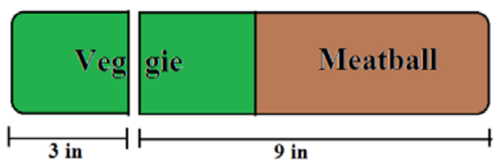
Ted is happy because he gets 50% of the value of the pizza in his value system.

Example 8.2.2: Divider/Chooser Method with a Sub Sandwich (Example 8.1.4 Continued)

In Example 8.1.4, George and Ted want to split a 12-inch sandwich worth $9. Half the sandwich is vegetarian and half the sandwich is meatball. George does not eat meat at all. Ted likes the meatball part twice as much as vegetarian part. We already figured out how each player should cut the sandwich.

a. If George cuts which piece should Ted choose?

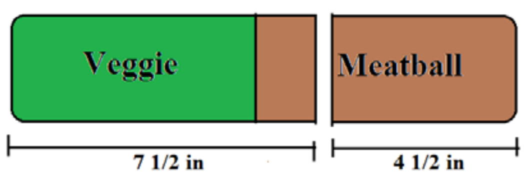
Figure 8.2.6: George's Division of the Sandwich



Ted sees the meatball part with a value of $6 and the vegetarian part with a value of $3. Half of the vegetarian part would be worth $1.50 to him. The larger part of the sandwich would have a value of $6.00 + $1.50 = $7.50 and the smaller part of the sandwich would have a value of $1.50. He should choose the larger part of the sandwich.

b. If Ted cuts which piece should George choose?

Figure 8.2.7: Ted's Division of the Sandwich



George does not eat meat so the smaller all meatball piece is worth $0 to him. The larger piece contains all the vegetarian part of the sandwich so it contains all the value to him. George should choose the larger piece which is worth $9 to him.

Note that in Example 8.2.2, part (a), Ted's piece was worth $7.50 to him and in part (b) George's piece was worth $9 to him. In both situations, the chooser ends up with more than a fair share. The divider always gets exactly a fair share. Given the choice, it is always better to be the chooser than the divider.

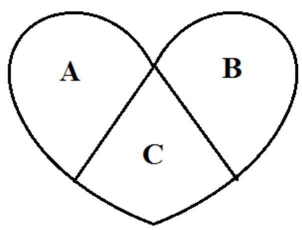
Lone Divider Method:

The Divider/Chooser method only works for two players. For more than two players we can use a method called the Lone Divider method. The basic idea is that a divider cuts the object into pieces. The rest of the players, called choosers, bid on the pieces they feel are fair shares. Each chooser is given a piece he/she considers a fair share with the remaining piece going to the divider. As we saw in the Divider/Chooser method, the divider always gets exactly a fair share but the choosers may get more than a fair share.

Example 8.2.3: Lone Divider Method, Basic Example

Three cousins, Russ, Sam, and Tom want to divide a heart-shaped cake. They draw straws to choose a divider and Russ is chosen. Russ must divide the cake into three pieces. Each piece must be a fair share in his value system. Assume Russ divides the cake as shown in the following figure.

Figure 8.2.8: Russ's Division of the Cake



Sam and Tom now bid on each piece of the cake. They privately and independently determine a value for each piece of the cake according to their value system.

Sam sees the value of the cake as: Piece A – 40%, piece B – 30%, and piece C – 30%.

Tom sees the value of the cake as: Piece A – 35%, piece B – 35%, and piece C – 30%.

Since there are three players, a fair share would be 1/3 or 33.3%.

Each player writes down which pieces they would consider a fair share of the cake. These are called the bids.

Sam would bid {A} and Tom would bid {A, B}.

Neither Sam nor Tom consider piece C to be a fair share so piece C goes to Russ, the divider.

Sam only considers piece A to be a fair share so give Sam piece A.

Tom would be satisfied with either piece A or B. Since piece A was given to Sam, Tom gets piece B.

Notice that Sam believes his piece is worth 40% of the value and Tom believes his piece is worth 35% of the value so both of them got more than a fair share. The divider Russ got a piece worth exactly 33.3% or a fair share in his opinion. The divider always receives exactly a fair share using this method.

Summary of the Lone Divider Method:

1. The n players use a random method to choose a divider. The other n-1 players are all choosers.
2. The divider divides the object S into n pieces of equal value in his/her value system.
3. Each of the choosers assigns a value to each piece of the object and submits his/her bid. The bid is a list of the pieces the player would consider a fair share.
4. The pieces are allocated using the bids. Sometimes, in the case of a tie, two pieces must be combined and divided again to satisfy all players.

Example 8.2.4: Lone Divider Method with a Cake, No Standoff

A cake is to be divided between four players, Ian, Jack, Kent, and Larry. The players draw straws and Ian is chosen to be the divider. Ian divides the cake into four pieces, S_1, S_2, S_3, and S_4. Each of these pieces would be a fair share to Ian. The other three players assign values to each piece as summarized in Table 8.2.9.

Table 8.2.9: Players' Valuation of Shares

	S_1	S_2	S_3	S_4
Ian	25%	25%	25%	25%
Jack	40%	30%	20%	10%
Kent	15%	35%	35%	15%
Larry	40%	20%	20%	20%

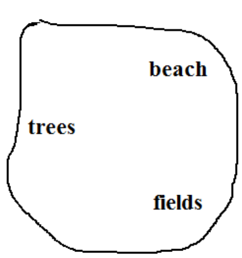
Since there are four players a fair share is 25% of the cake. The three choosers submit their bids as follows:

Jack: $\{S_1, S_2\}$, Kent: $\{S_2, S_3\}$, and Larry: $\{S_1\}$

The distribution is fairly straightforward. Larry gets S_1 since it is the only piece he considers a fair share. With S_1 taken Jack will get S_2, his only remaining possible fair share. With S_2 taken Kent will get S_3, his only remaining possible fair share. That leaves S_4 for the divider Ian.

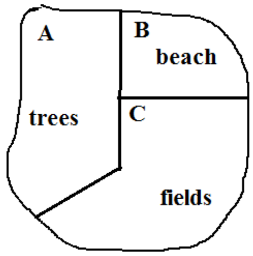
Example 8.2.5: Lone Divider Method with a Piece of Land, Simple Standoff

Figure 8.2.10: A Map of the Land



Amy, Bob and Carly want to divide a piece of land using the lone-divider method. They draw straws and Bob is chosen as the divider. Bob draws lines on the map to divide the land into three pieces of equal value according to his value system.

Figure 8.2.11: Bob's Division of the Land



Amy and Carly bid on the pieces of land that they would consider fair shares. Both of them like the beach and the fields but not the trees so their bids are Amy: {B, C} and Carly: {B, C}.

Since neither Amy nor Carly want piece A with the trees, that piece will go to the divider Bob.

Both Amy and Carly would be happy with either of the remaining pieces. A simple way to allocate the pieces is to toss a coin to see who gets piece B with the beach. The other player would get piece C with the fields.

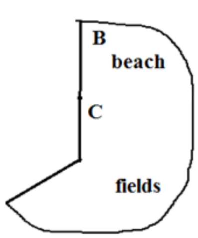
Example 8.2.6: Lone Divider Method with a Piece of Land, More Complicated Standoff

Let's look at the land in Example 8.2.5 again. This time let's assume the bids are Amy: {B} and Carly: {B}.

Since both Amy and Carly want the same piece of land we have a standoff. Neither of the women want pieces A and C so give one of them to the divider Bob. Toss a coin to choose which piece he gets. Let's assume the toss results in Bob getting piece A.

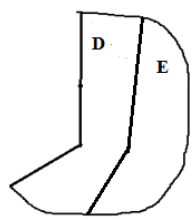
To resolve the standoff we combine pieces B and C to make one large piece.

Figure 8.2.12: Combining Pieces B and C



We now have one piece of land to be divided equally between two players. Amy and Carly can use the Divider/Chooser method to the finish the division. Toss a coin to determine the divider. Assume Amy is chosen to divide and divides the land as shown in Figure 8.2.13.

Figure 8.2.13: Amy's Division of the Piece



Let's assume that Carly picks piece E, leaving piece D for Amy, to complete the fair division.

You can see from the previous examples that sometimes the lone divider method is very straight forward and other times it can be more complicated. Imagine how complicated the method could become with 10 players. Regardless of the number of players or how complicated the division is, one fact remains. The choosers always get at least a fair share while the divider only gets an exact fair share. It is better to be a chooser than the divider.

Section 8.3: Continuous Methods 2: Lone Chooser and Last Diminisher Methods

The Lone Chooser method, like the Lone Divider method, is an extension of the Divider/Chooser method. The Lone Chooser method for three players involves two dividers and one chooser. It can be extended to N players with N-1 dividers and one chooser. We will focus on the three player Lone Chooser method in this book.

The Last Diminisher is a very different method from the Divider/Chooser methods we discuss in this book. In a sense, everyone is a divider and everyone is a chooser. The Last Diminisher method works well when many players must divide a continuous object like a cake or a piece of land.

Lone Chooser Method:

In the Lone Chooser method for three players, there are two dividers and one chooser. The basic idea is that the two dividers use the Divider/Chooser method to divide the object into two pieces. At this point each of the dividers believes that he/she has at least half the value of the object. Next each divider divides his/her piece into three smaller pieces for a total of six pieces. The chooser then picks one piece from each of the dividers' pieces leaving all three players with two pieces each.

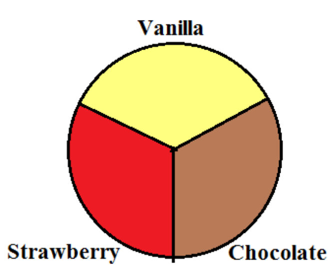
Summary of the Lone Chooser method:

1. Randomly choose one player to be the chooser, C. The other two players are dividers, D_1 and D_2.
2. The dividers D_1 and D_2 use the Divider/Chooser method to divide the object into two pieces.
3. Each of the dividers D_1 and D_2 subdivide his/her piece into three pieces of equal value. Use the same ideas as we used in the Divider/Chooser method for determining the value of each piece.
4. The chooser assigns a value to each of the six pieces according to his/her value system. The chooser then picks the piece with the greatest value from each of the dividers. Each of the dividers keep his/her other two pieces.

Example 8.3.1: Lone Chooser Method for Three Players

Fred, Gloria, and Harvey wish to split a three-flavored cake worth $36 that is one-third chocolate, one-third strawberry, and one-third vanilla. Fred does not like chocolate, but likes strawberry and vanilla equally well. Gloria likes chocolate twice as much as vanilla and likes strawberry three times as much as vanilla. Harvey likes chocolate and strawberry equally but likes vanilla twice as much as chocolate or strawberry. Use the Lone-Chooser method to find a fair division for the cake.

Figure 8.3.1: Three-Flavored Cake



First let's figure out how each player values each of the pieces of the cake.

Fred sees the chocolate part of the cake as having a value of $0 since he does not like chocolate. All of the value of the cake is in the strawberry and vanilla parts. Since he likes them equally well, the strawberry part and the vanilla part are both worth $36/2=$18.

For Gloria, let x = the value of the vanilla part of the cake. Then the chocolate part is worth 2x and the strawberry part is worth 3x.

$$\begin{aligned} x+2x+3x &= 36 \\ 6x &= 36 \\ x &= 6 \end{aligned}$$

To Gloria the vanilla part is worth \$6, the chocolate part is worth \$12 and the chocolate part is worth \$18.

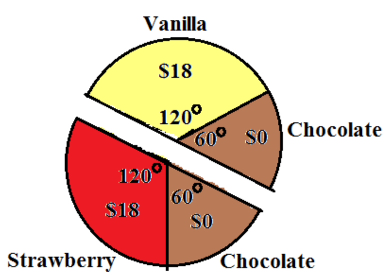
For Harvey, let y = the value of the chocolate part of the cake. Then the value of the strawberry part is also y and the value of the vanilla part is 2y.

$$\begin{aligned} y+y+2y &= 36 \\ 4y &= 36 \\ y &= 9 \end{aligned}$$

To Harvey the chocolate and strawberry parts are each worth \$9 and the vanilla part is worth \$18.

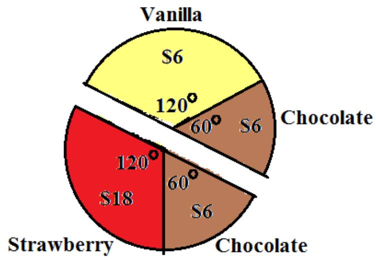
Assume the players draw cards to pick the chooser and Henry wins. Then Fred and Gloria must first do the Divider/Chooser method on the cake. They draw cards again and Fred is chosen as the divider. He needs to divide the cake into two pieces each worth \$18. There are many possible ways for Fred to cut the cake. Let's assume he cuts it as shown in figure 8.3.2.

Figure 8.3.2: How Fred Cuts the Cake



Before Gloria can choose a piece, she must find the value of each of Fred's pieces in her value system.

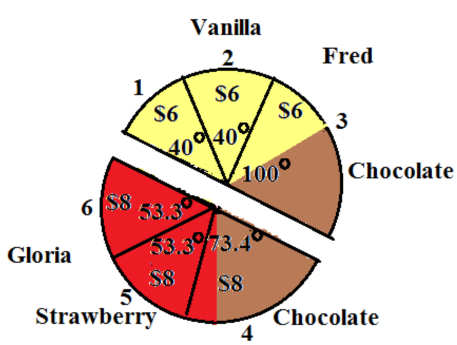
Figure 8.3.3: How Gloria Sees the Cake



Gloria should choose the strawberry and chocolate piece since it has the highest value to her.

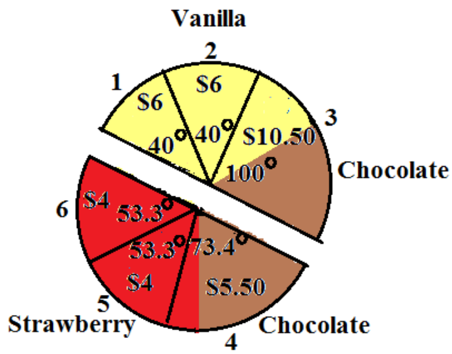
Now Fred and Gloria independently divide their pieces into three pieces of equal value in their respective value systems. Remember that each flavor of cake takes up 120°. Since Fred sees the chocolate as having no value, he needs to divide the 120° of vanilla into three equal pieces of 40°, each worth \$6. One of the pieces will also include the 60° of chocolate. Gloria's piece is worth a total of \$24 so she needs to divide it into three pieces each worth \$8. Start with the Strawberry part. $\frac{\$8}{\$18}\left(120^\circ\right)=53.3^\circ$. She cuts two pieces of strawberry that have an angle of 53.3°. The remaining amount of strawberry is $120^\circ-2\left(53.3^\circ\right)=13.4^\circ$. Combining the small strawberry piece with the chocolate piece gives a larger piece with an angle of 73.4° and a value of $\frac{13.4^\circ}{120^\circ}(\$18)+\$6=\$8.00$. The pieces are numbered for convenience.

Figure 8.3.4: The Subdivisions of Each Piece



At this point, Harvey joins the game. He assigns a value to each piece according to his value system.

Figure 8.3.5: How Harvey Sees the Pieces



Pieces 1 & 2: $\dfrac{40^\circ}{120^\circ}(\$18)=\$6.00$

Piece 3: $\dfrac{40^\circ}{120^\circ}(\$18)+\dfrac{60^\circ}{120^\circ}(\$9)=\$10.50$

Piece 4: $\dfrac{60^\circ}{120^\circ}(\$9)+\dfrac{13.4^\circ}{120^\circ}(\$9)=\$5.50$

Pieces 5 & 6: $\dfrac{53.3^\circ}{120^\circ}(\$9)=\$4.00$

Harvey should pick the piece from each divider that he sees as having the greatest value. Fred and Gloria each keep the two pieces from their parts that Harvey does not pick. That gives each of the players two of the six pieces.

Now let's look at the final division. Harvey gets pieces 3 and 4 for a total value of $16.00, more than a fair share to him. Gloria gets pieces 5 and 6 for a total value of $16, more than a fair share to her. The original divider Fred gets pieces 1 and 2 for a total value of $12, exactly a fair share to him.

Last Diminisher Method:

The Last Diminisher method can be used to divide a continuous object among many players. The concept is fairly simple. A player cuts a piece of the object. Each of the other players gets to decide if the piece is a fair share or too big. Any player that thinks the piece is too big can cut it smaller (diminish it). The key to the method is that the last person to cut/diminish a piece has to keep it.

Summary of the Last Diminisher method:

1. The players use a random method to choose an order. The players continue in the same order throughout the game. Call the n players, in order, P_1, P_2, P_3, …, P_n.
2. Player 1 cuts a piece he/she considers a fair share. In order, each of the remaining players either passes (says the piece is a fair share and P_1 can have it) or diminishes the piece. The last player to diminish the piece keeps the piece and leaves the game. If no one diminishes the piece, P_1 keeps the piece and leaves the game.
3. The lowest numbered player still in the game cuts a piece of the object. Each of the remaining players can either pass or diminish the piece. The last player to cut/diminish the piece keeps it and leaves the game.
4. Repeat step three until only two players remain. These players use the Divider/Chooser method to finish the game.

Example 8.3.2: Last Diminisher Method, #1

Suppose six players want to divide a piece of land using the Last Diminisher method. They draw cards to choose an order. Assume the players in order are denoted P_1, P_2, P_3, P_4, P_5, and P_6.

In round one, P_1 cuts a piece by drawing lines on a map of the land. Assume players P_2 through P_6 pass on the piece. Since P_1 is the last player to cut or diminish the piece, P_1 keeps the piece and leaves the game.

In round two, players P_2 through P_6 remain in the game. P_2 is the lowest numbered player so P_2 cuts a piece of the land. Assume that P_3 and P_4 pass on the piece. P_5 thinks that the piece is more than a fair share so P_5 diminishes the piece by redrawing the lines on the map to make the piece smaller. Assume P_6 passes. Since P_5 is the last player to diminish the piece, P_5 keeps the piece and leaves the game.

In round three, players P_2, P_3, P_4, and P_6 remain in the game. P_2 is still the lowest numbered player so P_2 cuts a piece of the land. This time assume that P_3 diminishes, P_4 passes and P_6 diminishes the piece. P_6 keeps the piece and leaves the game.

In round four, players P_2, P_3, and P_4 remain in the game. Once again P_2 cuts a piece. Assume both P_3 and P_4 pass on the piece. P_2 keeps the piece and leaves the game.

For round five, since only P_3 and P_4 are left, they do Divider/Chooser on the remaining land. P_3 cuts and P_4 chooses.

Example 8.3.3: Last Diminisher Method, #2

Eight players want to divide a piece of land using the Last Diminisher method. They draw straws to determine an order. Assume the players in order are denoted by P_1, P_2, P_3, P_4, P_5, P_6, P_7, and P_8. P_3 and P_5 are the only diminishers in round one. No one diminishes in rounds two, three and six. P_8 is the only diminisher in round four. Both P_4 and P_6 diminish in round five. Describe the fair division round by round.

Round one: P_1 cuts a piece, P_2 passes, P_3 diminishes, P_4 passes, P_5 diminishes, P_6, P_7, and P_8 pass. P_5 is the last diminisher so P_5 keeps the piece and leaves the game.

Round two: P_1 cuts a piece and everyone else passes so P_1 keeps the piece and leaves the game.

Round three: P_2 is now the lowest numbered player so P_2 cuts a piece. Everyone else passes so P_2 keeps the piece and leaves the game.

Round four: P_3 is now the lowest numbered player so P_3 cuts a piece. P_4, P_6, and P_7 pass. P_8 diminishes the piece making P_8 the last diminisher so P_8 keeps the piece and leaves the game.

Round five: P_3 is still the lowest numbered player so P_3 cuts a piece. P_4 and P_6 both diminish the piece but P_7 passes. P_6 is the last diminisher so P_6 keeps the piece and leaves the game.

Round six: P_3 cuts a piece again and everyone else passes, so P_3 keeps the piece and leaves the game.

Round seven: P_4 and P_7 are the only players left so they use the Divider/Chooser method to divide the remaining land. P_4 divides and P_7 chooses.

Section 8.4: Discrete Methods: Sealed Bids and Markers

There are two more fair division methods that deal with discrete objects. It two heirs have to split a house they cannot just cut the house in half. Instead we have to figure out a way to keep the house intact and still have both heirs feel like they received a fair share. The method of sealed bids is used for dividing up a small number of objects not necessarily similar in value. If there are many objects similar in value, like a jewelry collection, the method of markers can be used to find a fair division.

Method of Sealed Bids

The method of sealed bids can be used to split up an estate among a small number of heirs. A nice feature of this method is that every player in the game ends up with more than a fair share (in their own eyes). The method can also be used when business partners wish to dissolve a partnership in an equitable way or roommates want to divide up a large list of chores.

We make the following assumptions in the method of sealed bids.

1. The players are the only ones involved in the game and are willing to accept the outcome.
2. The players have no prior knowledge of the other players' preferences so they do not try to manipulate the game. If this assumption is not met the game might not produce a fair division.
3. The players are not emotionally/irrationally attached to any of the items. The players will settle for any of the items or cash as long as it is a fair share. For example, no one would say "I want the house and I will do anything to get it."

The easiest way to explain the method is to work through an example. An easy way to keep the steps neat and organized is to do the steps in one big table, working from the top to the bottom.

Example 8.4.1: Method of Sealed Bids, #1

Three heirs, Alice, Betty and Charles inherit an estate consisting of a house, a painting and a tractor. They decide to use the method of sealed bids to divide the estate among themselves.

1. The players each submit a list of bids for the items. The bid is the value that a player would assign to the item. The bids are done privately and independently. The bids are usually listed in a table.

Table 8.4.1: Initial Bids

	Alice	Betty	Charles
House	$280,000	$275,000	$300,000
Painting	$75,000	$70,000	$72,000
Tractor	$56,000	$60,000	$63,000

2. For each item, the player with the highest bid wins the item. The winning bids are highlighted in the table.

Table 8.4.2: Winning Bids

	Alice	Betty	Charles
House	$280,000	$275,000	$300,000
Painting	$75,000	$70,000	$72,000
Tractor	$56,000	$60,000	$63,000

3. For each player find the sum of his/her bids. This amount is what the player thinks the whole estate is worth. For three players, each player is entitled to one third of the estate. Divide each sum by three to get a fair share for each player. Remember that each player sees the values differently so the fair shares will not be the same.

Table 8.4.3: Total Bids and Fair Shares

	Alice	Betty	Charles
House	$280,000	$275,000	$300,000
Painting	$75,000	$70,000	$72,000
Tractor	$56,000	$60,000	$63,000
Total Bids	$411,000	$405,000	$435,000
Fair Share	$137,000	$135,000	$145,000

4. Each player either gets more than his/her fair or less than his/her fair share when the items are awarded. Find the difference between the fair share and the items awarded for each player. If a player was awarded more than his/her fair share, the player owes the difference to the estate. If a player was awarded less than his/her fair share, the estate owes the player the difference.
   Alice: $137,000 - $75,000 = $62,000. The estate owes Alice $62,000.
   Betty: $135,000 - $0 = $135,000. The estate owes Betty $135,000.
   Charles: $145,000 – ($300,000 + $63,000) = -$218,000. Charles owes the estate $218,000.

Table 8.4.4: Owed to Estate and Estate Owes

	Alice	Betty	Charles
House	$280,000	$275,000	$300,000
Painting	$75,000	$70,000	$72,000
Tractor	$56,000	$60,000	$63,000
Total Bids	$411,000	$405,000	$435,000
Fair Share	$137,000	$135,000	$145,000
Owed to Estate			$218,000
Estate Owes	$62,000	$135,000	

5. At this point in the game, there is always some extra money in the estate called the surplus. To find the surplus, we find the difference between all the money owed to the estate and all the money the estate owes.
   $218,000 – ($62,000 + $135,000) = $21,000.
   Divide this surplus evenly between the three players.
   Share of surplus = $\frac{\$21,000}{3} = \$7000$

Table 8.4.5: Share of Surplus

	Alice	Betty	Charles
House	$280,000	$275,000	$300,000
Painting	$75,000	$70,000	$72,000
Tractor	$56,000	$60,000	$63,000
Total Bids	$411,000	$405,000	$435,000
Fair Share	$137,000	$135,000	$145,000
Owed to Estate			$218,000
Estate Owes	$62,000	$135,000	
Share of Surplus	$7,000	$7,000	$7,000

6. Finish the problem by combining the share of the surplus to either the amount owed to the estate or the amount the estate owes. Include the items awarded in the final share as well as any money.
   Alice: $62,000 + $7,000 = $69,000
   Betty: $135,000 + $7,000 = $142,000
   Charles: -$218,000 + $7,000 = -$211,000

Table 8.4.6: Final Share

	Alice	Betty	Charles
House	$280,000	$275,000	$300,000
Painting	$75,000	$70,000	$72,000
Tractor	$56,000	$60,000	$63,000
Total Bids	$411,000	$405,000	$435,000
Fair Share	$137,000	$135,000	$145,000
Owed to Estate			$218,000
Estate Owes	$62,000	$135,000	
Share of Surplus	$7,000	$7,000	$7,000
Final Share	Gets painting and $69,000 cash	Gets $142,000 cash	Gets house and tractor and pays $211,000

Alice gets the painting and $69,000 cash. Betty gets $142,000 cash. Charles gets the house and the tractor and pays $211,000 to the estate.

Now, we find the value of the final settlement for each of the three heirs in this example. Remember that each player has their own value system in this game so fair shares are not the same amount.

Alice: Painting worth $75,000 and $69,000 cash for a total of $144,000. This is $7,000 more than her fair share of $137,000.

Betty: $142,000 cash. This is $7,000 more than her fair share of $135,000.

Charles: House worth $300,000, tractor worth $63,000 and pays $211,000 for a total share of $152,000. This is $7,000 more that his fair share of $145,000.

At the end of the game, each player ends up with more than a fair share. It always works out this way as long as the assumptions are satisfied.

Summary of the Method of Sealed Bids:

1. Each player privately and independently bids on each item. A bid is the amount the player thinks the item is worth.
2. For each item, the player with the highest bid wins the item.
3. For each player find the sum of the bids and divide this sum by the number of players to find the fair share for that player.

4. Find the difference (fair share) – (total of items awarded) for each player. If the difference is negative, the player owes the estate that amount of money. If the difference is positive, the estate owes the player that amount of money.
5. Find the surplus by finding the difference (sum of money owed to the estate) – (sum of money the estate owes). Divide the surplus by the number of players to find the fair share of surplus.
6. Find the final settlement by adding the share of surplus to either the amount owed to the estate or the amount the estate owes. Include any items awarded and any cash owed in the final settlement. The sum of all the cash in the final settlement should be $0.

Example 8.4.2: Method of Sealed Bids, #2

Doug, Edward, Frank and George have inherited some furniture from their great-grandmother's estate and wish to divide the furniture equally among themselves. Use the method of sealed bids to find a fair division of the furniture.

Note: We start with one table and add lines to the bottom as we go through the steps.

1. List the bids in table form.

Table 8.4.7: Initial Bids

	Doug	Edward	Frank	George
Dresser	$280.00	$275.00	$250.00	$300.00
Desk	$480.00	$500.00	$450.00	$475.00
Wardrobe	$775.00	$800.00	$850.00	$800.00
Dining Set	$1,000.00	$800.00	$900.00	$950.00
Poster Bed	$500.00	$650.00	$600.00	$525.00

2. Award each item to the highest bidder.

Table 8.4.8: Winning Bids

	Doug	Edward	Frank	George
Dresser	$280.00	$275.00	$250.00	$300.00
Desk	$480.00	$500.00	$450.00	$475.00
Wardrobe	$775.00	$800.00	$850.00	$800.00
Dining Set	$1,000.00	$800.00	$900.00	$950.00
Poster Bed	$500.00	$650.00	$600.00	$525.00

3. Find the fair share for each player.

Doug:
$\$280.00 + \$480.00 + \$775.00 + \$1000.00 + \$500.00 = \$3035.00$
$\frac{\$3035.00}{4} = \$758.75$
Edward:
$\$275.00 + \$500.00 + \$800.00 + \$800.00 + \$650.00 = \$3025.00$
$\frac{\$3025.00}{4} = \$756.25$
Calculate similarly for Frank and George.

Table 8.4.9: Total Bids and Fair Shares

	Doug	Edward	Frank	George
Dresser	\$280.00	\$275.00	\$250.00	\$300.00
Desk	\$480.00	\$500.00	\$450.00	\$475.00
Wardrobe	\$775.00	\$800.00	\$850.00	\$800.00
Dining Set	\$1,000.00	\$800.00	\$900.00	\$950.00
Poster Bed	\$500.00	\$650.00	\$600.00	\$525.00
Total Bids	\$3,035.00	\$3,025.00	\$3,050.00	\$3,050.00
Fair Share	\$758.75	\$756.25	\$762.50	\$762.50

4. Find the amount owed to the estate or the amount the estate owes.
Doug: $\$758.75 - \$1000.00 = -\$241.25$
Doug owes the estate \$241.25.

Calculate similarly for Edward and Frank.

George: $\$762.50 - \$300.00 = \$462.50$
The estate owes George \$462.50.

Table 8.4.10: Owes to Estate and Estate Owes

	Doug	Edward	Frank	George
Dresser	\$280.00	\$275.00	\$250.00	\$300.00
Desk	\$480.00	\$500.00	\$450.00	\$475.00
Wardrobe	\$775.00	\$800.00	\$850.00	\$800.00
Dining Set	\$1,000.00	\$800.00	\$900.00	\$950.00
Poster Bed	\$500.00	\$650.00	\$600.00	\$525.00
Total Bids	\$3,035.00	\$3,025.00	\$3,050.00	\$3,050.00
Fair Share	\$758.75	\$756.25	\$762.50	\$762.50
Owes to Estate	\$241.25	\$393.75	\$87.50	
Estate Owes				\$462.50

5. Find the share of surplus for each player.

$$\begin{aligned}\text{Surplus} &= (\text{total money owed to estate}) - (\text{total money estate owes})\\ &= (\$241.25 + \$393.75 + \$87.50) - (\$462.50)\\ &= \$260.00\end{aligned}$$

$$\text{Share of surplus} = \frac{\$260.00}{4} = \$65.00$$

Table 8.4.11: Share of Surplus

	Doug	Edward	Frank	George
Dresser	\$280.00	\$275.00	\$250.00	\$300.00
Desk	\$480.00	\$500.00	\$450.00	\$475.00
Wardrobe	\$775.00	\$800.00	\$850.00	\$800.00
Dining Set	\$1,000.00	\$800.00	\$900.00	\$950.00
Poster Bed	\$500.00	\$650.00	\$600.00	\$525.00
Total Bids	\$3,035.00	\$3,025.00	\$3,050.00	\$3,050.00
Fair Share	\$758.75	\$756.25	\$762.50	\$762.50
Owes to Estate	\$241.25	\$393.75	\$87.50	
Estate Owes				\$462.50
Share of Surplus	\$65.00	\$65.00	\$65.00	\$65.00

6. Find the final share for each player.

Doug: $-\$241.25 + \$65.00 = -\$176.25$

Calculate similarly for Edward and Frank.

George: $\$462.50 + \$65.00 = \$527.50$

Table 8.4.12: Final Shares

	Doug	Edward	Frank	George
Dresser	$280.00	$275.00	$250.00	$300.00
Desk	$480.00	$500.00	$450.00	$475.00
Wardrobe	$775.00	$800.00	$850.00	$800.00
Dining Set	$1,000.00	$800.00	$900.00	$950.00
Poster Bed	$500.00	$650.00	$600.00	$525.00
Total Bids	$3,035.00	$3,025.00	$3,050.00	$3,050.00
Fair Share	$758.75	$756.25	$762.50	$762.50
Owes to Estate	$241.25	$393.75	$87.50	
Estate Owes				$462.50
Share of Surplus	$65.00	$65.00	$65.00	$65.00
Final Share	Dining set and pays $176.25	Desk and poster bed and pays $328.75	Wardrobe and pays $22.50	Dresser and gets $527.50

Doug gets the dining set and pays $176.25. Edward gets the desk and poster bed and pays $328.75. Frank gets the wardrobe and pays $22.50. George gets the dresser and $527.50 in cash.

Note that the sum of all the money in the final shares is $0 as it should be. Also note that each player's final share is worth $65.00 more than the fair share in his eyes.

Example 8.4.3: Method of Sealed Bids in Dissolving a Partnership

Jack, Kelly and Lisa are partners in a local coffee shop. The partners wish to dissolve the partnership to pursue other interests. Use the method of sealed bids to find a fair division of the business. Jack bids $450,000, Kelly bids $420,000 and Lisa bids $480,000 for the business.

Make a table similar to the table for dividing up an estate and follow the same set of steps to solve this problem.

Table 8.4.13: Method of Sealed Bids for Dissolving a Partnership

	Jack	Kelly	Lisa
Business	\$450,000	\$420,000	\$480,000
Total Bids	\$450,000	\$420,000	\$480,000
Fair Share	\$150,000	\$140,000	\$160,000
Owes to Business			\$320,000
Business Owes	\$150,000	\$140,000	
Share of Surplus	\$10,000	\$10,000	\$10,000
Final Share	\$160,000 cash	\$150,000 cash	Business and pays \$310,000

Lisa gets the business and pays Jack \$160,000 and Kelly \$150,000.

Method of Markers

The method of markers is used to divide up a collection of many objects of roughly the same value. The heirs could use the method of markers to divide up their grandmother's jewelry collection. The basic idea of the method is to arrange the objects in a line. Then, each player puts markers between the objects dividing the line of objects into distinct parts. Each part is a fair share to that particular player. Based on the placement of the markers, the objects are allotted to the players. If there are n players, each player places $n-1$ markers among the objects. We will use notation A_1 to represent the first marker for player A, A_2 to represent the second marker for player A, and so on.

Many times when the players have done all the steps in the method of markers there are some objects left over. If many objects remain, the players can line them up and do the method of markers again. If only a few objects remain, a common approach is to randomly choose an order for the players, then let each player pick an object until all the objects are gone.

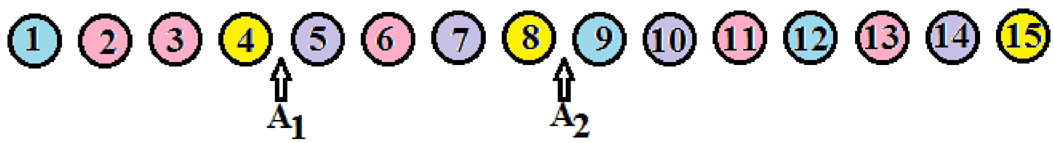
It is interesting to see that one player may only receive one or two objects while another player may receive four or five objects. The number of objects allotted depends on each player's value system. First we will look at the allocation of the pieces after the markers have been placed. Once we understand that, we will look at placing the markers in the correct places for each player.

Example 8.4.4: Method of Markers, #1

Three players Albert (A), Bertrand (B), and Charles (C), wish to divide a collection of 15 objects using the method of markers. Determine the final allocation of objects to each player. Since there are three players, each player uses two markers.

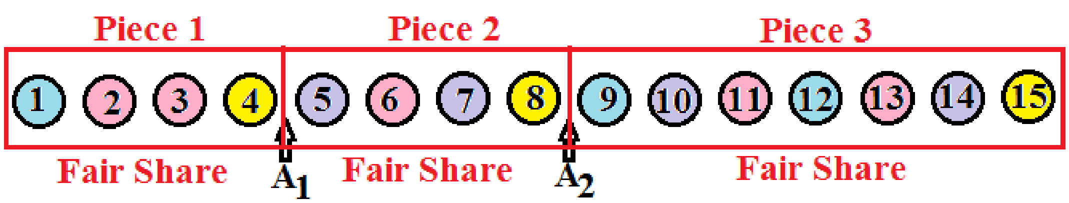
Let's start by looking at the line of objects and Albert's markers.

Figure 8.4.14: Markers for Albert



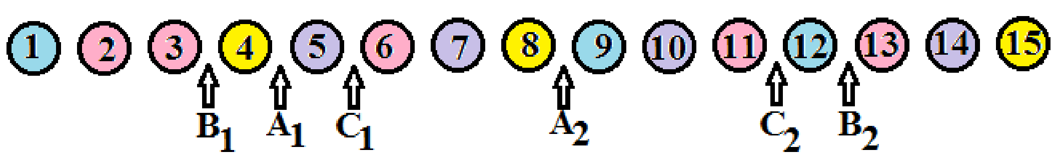
The markers divide the line of objects into three pieces. Each piece of the line is a fair share in Albert's value system. He would be satisfied with any of the three pieces in the final allocation. For now, do not worry about how Albert determined where to place the markers. We will look at that in Example 8.4.6.

Figure 8.4.15: Pieces (Fair Shares) for Albert



Now let's add the markers for Bertrand and Charles.

Figure 8.4.16: Markers for Albert, Bertrand, and Charles



Step 1: As you examine the objects from left to right, find the first marker, B_1. Give Bertrand all the objects from the beginning of the line to the marker B_1. Bertrand removes the rest of his markers and leaves the game for now.

Figure 8.4.17: Allocate the First Fair Share

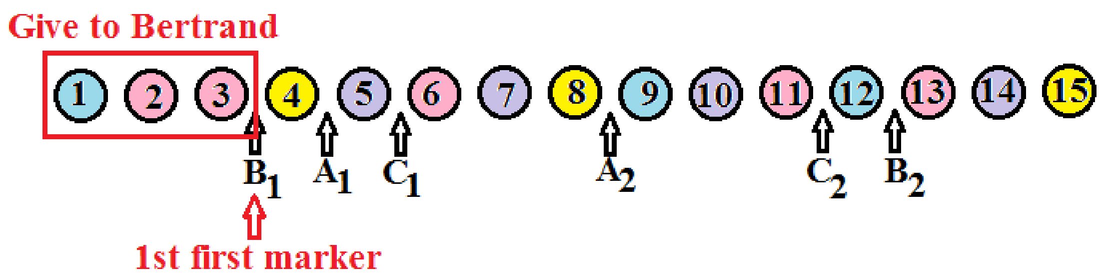
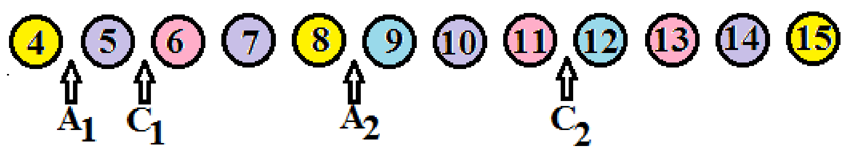


Figure 8.4.18: Remove the Bertrand's Fair Share and His Remaining Markers



Step 2: Now, continuing from left to right, find the first marker out of the second group of markers (A_2 and C_2). The first marker from this group we come across is A_2. Give Albert all the objects from his first marker A_1 to his second marker A_2. Remember that a fair share is from one marker to the next. Object #4 is not part of Albert's fair share since it is before his first marker. Albert removes the rest of his markers and leaves the game for now.

Figure 8.4.19: Allocate the Second Fair Share

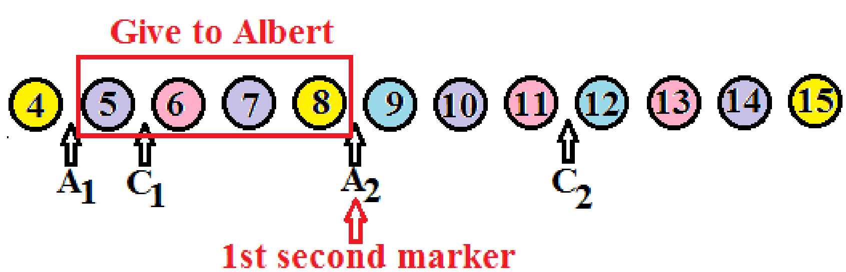
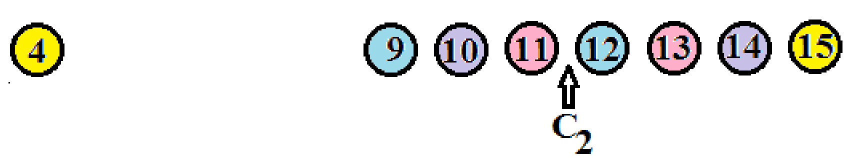
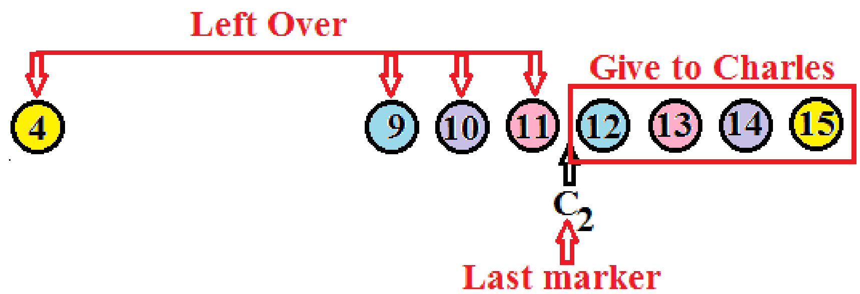


Figure 8.4.20: Remove Albert's Fair Share and His Remaining Markers



Step 3: Charles is the only player left in the game. He considers everything from his second marker to the end of the line to be a fair share so give it to him. Any objects not allocated at this point are left over.

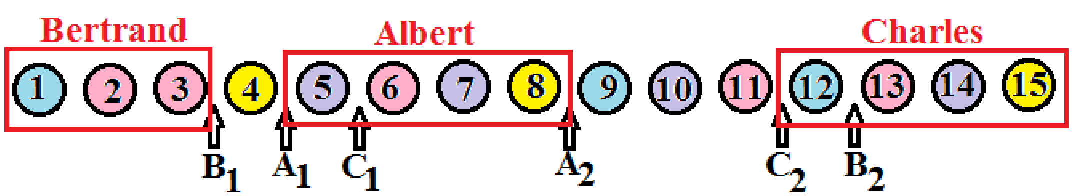
Figure 8.4.21: Allocate the Last Fair Share



Step 4: Typically some objects are left over at this point. Objects numbered 4, 9, 10 and 11 are left over in this game. The three players could draw straws to determine an order. Then each player in order would choose an object until all the object are allocated.

Note: Normally when we do the method of markers, we only draw the figure once.

Figure 8.4.22: Combined Figure for All Three Shares



Summary of the Method of Markers for n players:

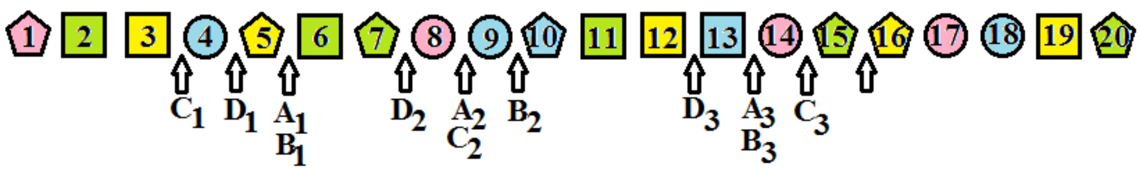
1. Arrange the objects in a line. Each of the n players places n-1 markers among the objects.
2. Find the 1st first marker, say A_1. Give player A all the objects from the beginning of the line to the 1st first marker. Player A removes his/her remaining markers and leaves the game for now.
3. Find the 1st second marker, say B_2. Give player B all the objects from the 1st second marker back to B's previous marker B_1. In other words, all the objects from B_1 to B_2. Player B removes his/her remaining markers and leaves the game for now.

4. Find the 1st third marker, say C_3. Give player C all the objects from the 1st third marker back to C's previous marker C_2. In other words, all the objects from C_2 to C_3. Player C removes his/her remaining markers and leaves the game for now.
5. Continue this pattern until one player remains. Give the last player all the objects from his/her last marker to the end of the line of objects.
6. Divide up the remaining objects. If many objects remain, do the method of markers again. If only a few objects remain, randomly choose an order then let each player choose an object in order until all the objects are gone.

Example 8.4.5: Method of Markers, #2

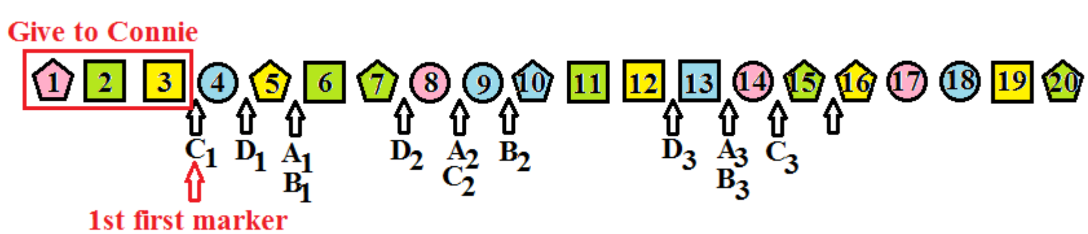
Four cousins, Amy, Becky, Connie and Debbie wish to use the method of markers to divide a collection of jewels. The jewels are lined up and the cousins place their markers as shown below in Figure 8.4.23. What is the final allocation of the jewels?

Figure 8.4.23: Jewels and Markers for Four Cousins



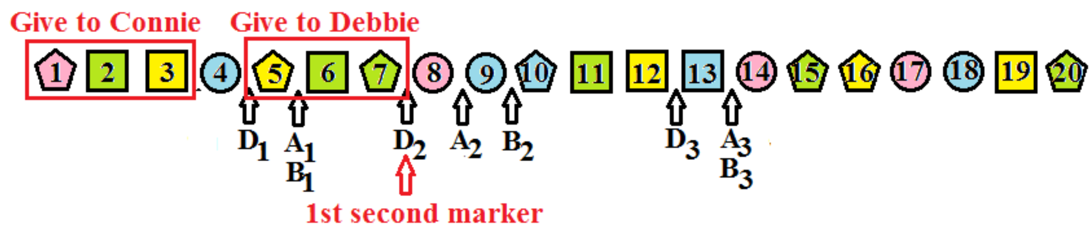
The 1st first marker is C_1 so give Connie all the jewels from the beginning of the line to her first marker. Connie removes her remaining markers and leaves the game for now.

Figure 8.4.24: Allocate the First Fair Share



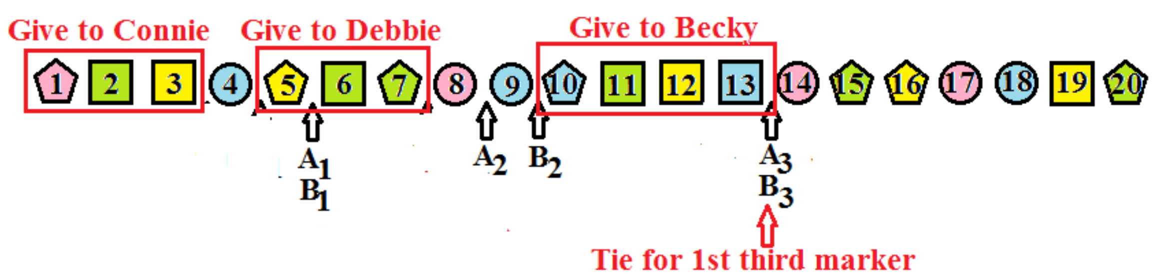
The 1st second marker is D_2 so give Debbie all the jewels between markers D_1 and D_2. Debbie removes her remaining markers and leaves the game for now.

Figure 8.4.25: Allocate the Second Fair Share



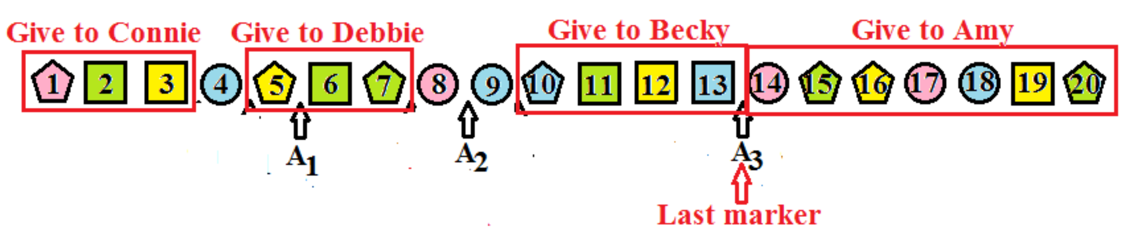
The 1st third marker is a tie between A_3 and B_3 so randomly choose one. One possibility is to have Amy and Becky toss a coin and the winner gets the next fair share. Assume Becky wins the coin toss. Give Becky all the jewels between markers B_2 and B_3. Becky removes her remaining markers and leaves the game for now.

Figure 8.4.26: Allocate the Third Fair Share



Amy is the last player in the game. Give Amy all the jewels from her last marker to the end of the line.

Figure 8.4.27: Allocate the Last Fair Share



Jewels numbered 4, 8, and 9 are left over. The players can draw straws to determine an order. Each player, in order, chooses a jewel until all the jewels have been allocated.

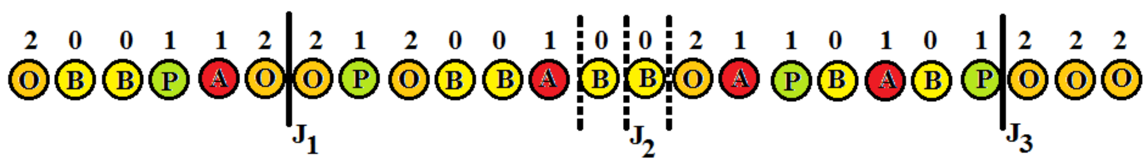
Example 8.4.6: Determining Where to Place the Markers

Four roommates want to split up a collection of fruit consisting of 8 oranges (O), 8 bananas (B), 4 pears (P), and 4 apples (A). The fruit are lined up as shown below in Figure 8.4.28.

Figure 8.4.28: Line of Fruit

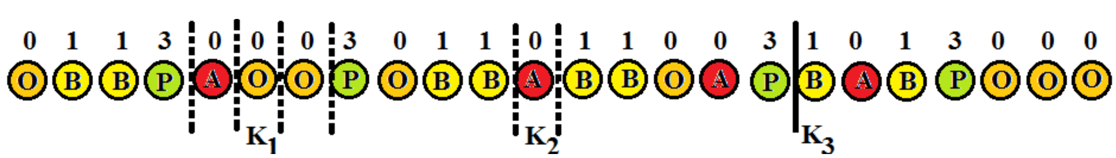
To determine where to place the markers, each player assigns a value to each type of fruit. Jack loves oranges, likes apples and pears equally, but dislikes bananas. He assigns a value of $1 to each apple and each pear, a value of $2 to each orange, and a value of $0 to each banana. In Jack's value system, the collection of fruit is worth $24. Jack's fair share is $6. He needs to place his markers so that the fruit is divided into groups worth $6. It can be helpful to work from both ends of the line. Jack has no choice about the placement of his first and third markers. Because he sees the bananas as worth $0 he has three possible places for his second marker. These possibilities are shown below in Figure 8.4.29 as dotted lines.

Figure 8.4.29: How Jack Values the Fruit



Kent dislikes apples and oranges, like bananas and really loves pears. He assigns a value of $0 to each apple or orange, a value of $1 to each banana, and a value of $3 to each pear. In Kent's value system, the collection of fruit is worth $20. Since there are four players, Kent's fair share is $5. He needs to place his markers so that the fruit is divided into groups worth $5. Jack has no choice about the placement of his third marker. He has a few possibilities for his first two markers. The possibilities are shown below in Figure 8.4.30 as dotted lines.

Figure 8.4.30: How Kent Sees the Fruit.



> The other two roommates would follow the same process to place their markers. Once all the markers are placed, the allocation by the method of markers begins.

Imagine if the order of the fruit in Example 8.4.6 was rearranged. It might not be possible for Kent to divide up the line of fruits into groups worth \$5. He might have to use a group worth \$6 next to a group worth \$4. This is a good time to remember that none of our fair division methods are perfect. They work well most of the time but sometimes we just have to make do. If Kent was allocated a group of fruit worth only \$4 he might get some of the missing value back when the left over fruits are allocated.

Chapter 8 Homework

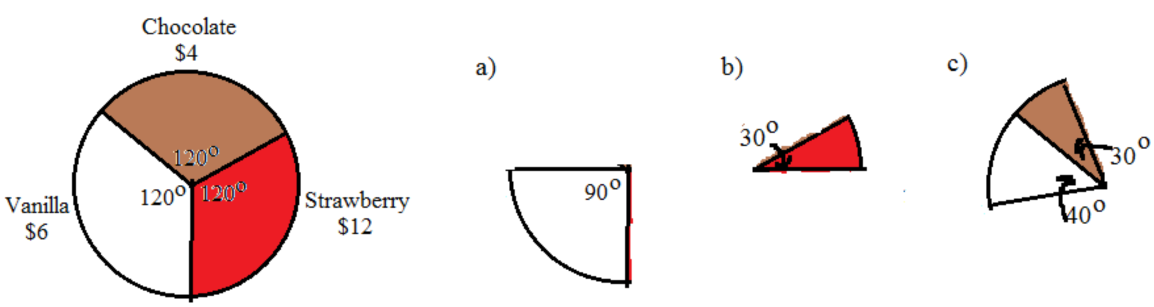
1. Three players are dividing a business. The assets are divided into three shares, S_1, S_2, and S_3. The following table shows how each player sees each share. For each player, list the shares that the player considers a fair share.

	S_1	S_2	S_3
Doug	40%	30%	30%
Eddie	33 1/3%	33 1/3%	33 1/3%
Fred	35%	30%	35%

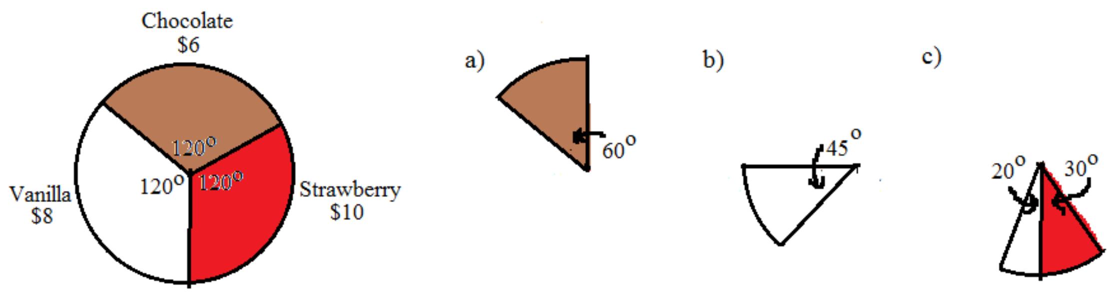
2. Four cousins are dividing a pizza. The pizza has been divided into four pieces, S_1, S_2, S_3, and S_4. The following table shows how each cousin sees each piece. For each cousin, list the pieces that the cousin considers a fair share.

	S_1	S_2	S_3	S_4
Anne	0%	0%	50%	50%
Bob	30%	30%	30%	10%
Cathy	20%	30%	20%	30%
Don	25%	25%	25%	25%

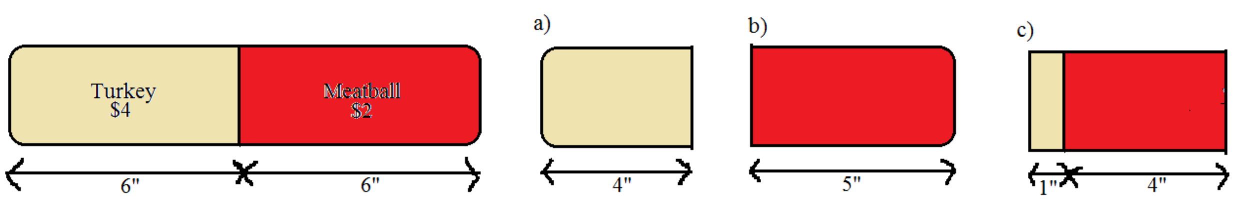
3. A three-flavored cake is one-third chocolate, one-third vanilla, and one-third strawberry. If the chocolate part is worth \$4, the vanilla part is worth \$6 and the strawberry part is worth \$12 to Francis, find the value of each of the following slices.



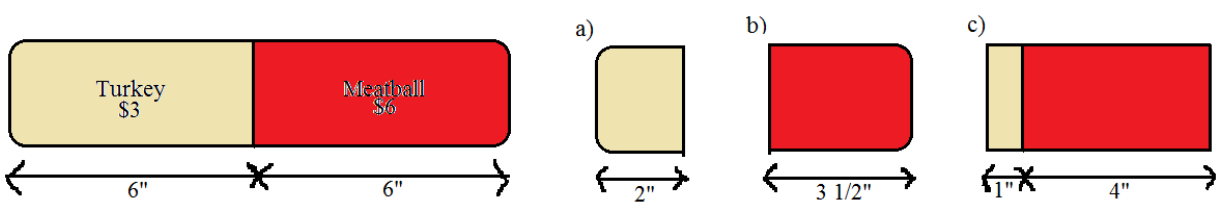
4. A three-flavored cake is one-third chocolate, one-third vanilla, and one-third strawberry. If the chocolate part is worth \$6, the vanilla part is worth \$8 and the strawberry part is worth \$10 to George, find the value of each of the following slices.



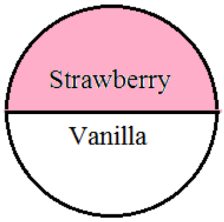
5. A 12-inch sandwich worth \$6 is half turkey and half meatball. To Jack, the turkey half is worth \$4 and the meatball half is worth \$2. Find the value of the following slices of the sandwich.



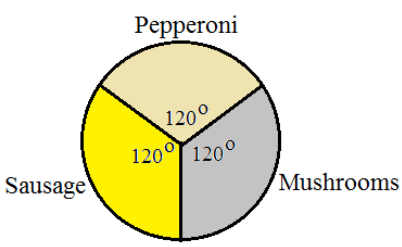
6. A 12-inch sandwich worth \$9 is half turkey and half meatball. To Jack, the turkey half is worth \$3 and the meatball half is worth \$6. Find the value of the following slices of the sandwich.



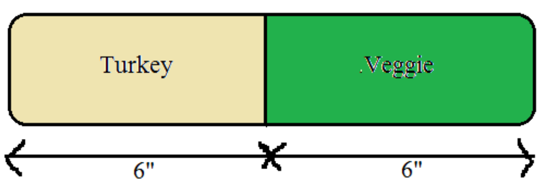
7. Alice wants to divide a half-strawberry half-vanilla cake worth $12 into two pieces of equal value. She likes strawberry three times as much as vanilla. How should she cut the cake so that each piece is a fair share to her?



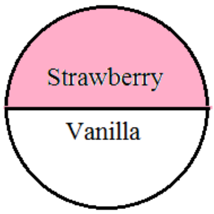
8. Sam has a pizza that is one-third pepperoni, one-third mushrooms, and one-third sausage. He likes both pepperoni and mushrooms twice as much as sausage. He wants to split the pizza into two pieces to share with his roommate. How should Sam cut the pizza so that each piece is a fair share to him?



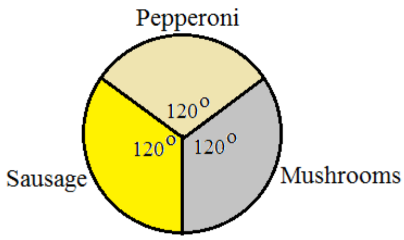
9. Luke wants to split a twelve-inch half turkey and half veggie sub sandwich worth $12 with a friend. Luke likes turkey twice as much as veggies. How should he cut the sandwich so that each piece is a fair share to him?



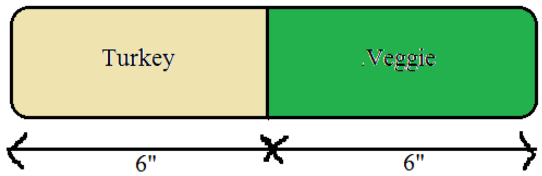
10. Alice and Betty want to divide a half-strawberry half-vanilla cake worth \$12 by the divider/chooser method. Alice likes strawberry three times as much as vanilla and Betty likes vanilla twice as much as strawberry. A coin is tossed and Alice is the divider.
    a. How should Alice cut the cake?
    b. Which piece should Betty choose and what is its value to her?



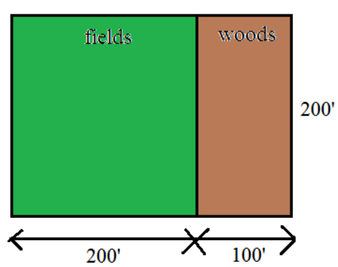
11. Sam and Ted have a pizza that is one-third pepperoni, one-third mushrooms, and one-third sausage. Sam likes pepperoni and sausage equally well but does not like mushrooms. Ted likes pepperoni twice as much as sausage and likes mushrooms twice as much as pepperoni. They want to split the pizza by the divider/chooser method. After drawing straws, Sam is the divider.
    a. How should Sam cut the pizza?
    b. Which piece should Ted choose and what is its value to him?



12. Luke and Mark want to use the divider/chooser method to split a twelve-inch half turkey and half veggie sub sandwich worth $12. Luke likes turkey three times as much as veggies and Mark like veggies twice as much as turkey. They draw cards and Mark is the divider.
    a. How should Mark cut the sandwich?
    b. Which piece should Luke choose and what is its value to him?



13. Two brothers want to divide up a piece of land their grandfather left them. The piece of land, valued at $300,000 is made up of two distinct parts as shown in the following figure. Joseph likes the woods twice as much as the fields. Kevin likes the fields but does not like the woods at all. The brothers decide to use the divider/chooser method to divide the land. They toss a coin and Joseph is the divider.
    a. How should Joseph divide the land if he makes one horizontal cut on the map: Which piece of land should Kevin choose: What is its value to him?
    b. How should Joseph divide the land if he makes one vertical cut on the map: Which piece of land should Kevin choose: What is its value to him?
    c. If Joseph can make more than one cut (i.e. cut out a rectangle or a triangle) how should he cut the piece of land: Hint: there is more than one answer.

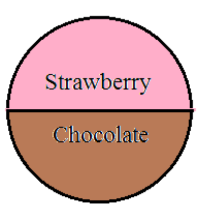


14. Amy, Becky, Charles and Doug want to use the lone divider method to split a piece of land they inherited from their father. They draw cards to determine that Doug is the divider. After Doug divides the land, Amy bids $\{S_2, S_3\}$, Becky bids $\{S_1, S_2\}$, and Charles bids $\{S_3\}$. Describe the fair division.

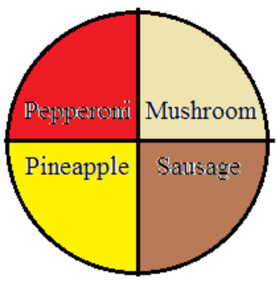
15. Edward, Frank, George, and Harold want to use the lone divider method to split a piece of land they inherited from their Grandfather. They draw cards to determine that George is the divider. After George divides the land, Edward bids $\{S_3, S_4\}$, Frank bids $\{S_2\}$, and Harold bids $\{S_3, S_4\}$. Describe the fair division.

16. Inez, Jackie, Kelly, and Louise want to use the lone divider method to split a piece of land they inherited from their father. They draw cards to determine that Jackie is the divider.
    a. If Inez bids $\{S_2\}$, Kelly bids $\{S_1, S_2\}$, and Louise bids $\{S_2, S_4\}$ describe the fair division.
    b. At the last minute Louise changes her bid to $\{S_2\}$. If Inez and Kelly do not change their bids, describe the fair division.

17. Frank, Greg, and Harriet want to divide a cake worth \$24 that is half chocolate and half strawberry. Frank likes all cake equally well. Greg likes chocolate twice as much as strawberry and Harriet like strawberry three times as much as chocolate. They decide to use the lone chooser method and draw cards to determine that Greg will cut the cake first and Harriet will be the chooser.
    a. How should Greg cut the cake?
    b. Which piece will Frank choose and what is its value to him?
    c. How will Greg subdivide his piece of the cake?
    d. How will Frank subdivide his piece of the cake?
    e. Which pieces of cake will Harriet choose?
    f. Describe the final division of the cake. Which pieces does each player receive and what are their values?



18. Paul, Rachel and Sally want to divide a four-topping pizza using the lone chooser method. The draw straws to determine that Rachel is the chooser and Paul will make the first cut. Paul likes pepperoni twice as much as mushrooms, likes sausage three times as much as mushrooms and does not like pineapple. Rachel likes mushrooms and pineapple equally well but does not like pepperoni or sausage. Sally likes all pizza equally well.
    a. How should Paul cut the pizza?
    b. Which piece should Sally choose and what is its value to her?
    c. How will Paul subdivide his piece of pizza?
    d. How will Sally subdivide her piece of pizza?
    e. Which pieces of pizza will Rachel choose?
    f. Describe the final division of the pizza. Which pieces does each player receive and what are their values?



19. Eight heirs inherit a large piece of property. They decide to use the last diminisher method to divide the property. They draw straws to choose an order. Assume the order of the players is P_1, P_2, P_3, P_4, P_5, P_6, P_7, and P_8. In round one, P_3, P_4 and P_7 are the only diminishers. In round two, no one diminishes the piece. In round three, P_3 and P_4 are the only diminishers. No one diminishes the piece in rounds four and five. Every player diminishes the piece in round 6.
    a. Who keeps the piece at the end of round one?
    b. Who cuts the piece at the beginning of round three?
    c. In round six, who cuts the piece and who keeps the piece?
    d. Which players are left after round six and how do they finish the division?

20. Seven heirs inherit a large piece of property. They decide to use the last diminisher method to divide the property. They draw straws to choose an order. Assume the order of the players is P_1, P_2, P_3, P_4, P_5, P_6, and P_7. In round one, P_3, P_4 and P_7 are the only diminishers. In round two, every player diminishes the piece. In round three, P_3 and P_4 are the only diminishers. No one diminishes the piece in rounds four or five. Every player diminishes the piece in round 6.
    a. Who keeps the piece at the end of round two?
    b. In round three, who cuts the piece and who keeps the piece?
    c. Who cuts the piece at the beginning of round five?
    d. Which players are left after round five and how do they finish the division?

21. Three heirs are dividing an estate consisting of a house, a lakeside cabin, and a small business using the method of sealed bids. The bids are listed in the following table.

	Mary	Nancy	Olivia
House	$350,000	$380,000	$362,000
Cabin	$280,000	$257,000	$270,000
Business	$537,000	$500,000	$520,000

Describe the final settlement including who gets each item and how much money they pay or receive.

22. Five heirs are dividing an estate using the method of sealed bids. The bids are listed in the following table.

	A	B	C	D	E
Item 1	$352	$295	$395	$368	$324
Item 2	$98	$102	$98	$95	$105
Item 3	$460	$449	$510	$501	$476
Item 4	$852	$825	$832	$817	$843
Item 5	$513	$501	$505	$505	$491
Item 6	$725	$738	$750	$744	$761

Describe the final settlement including who gets each item and how much money they pay or receive.

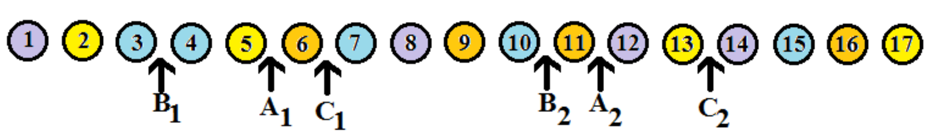
23. Albert, Brett and Carl own a hot dog stand together. Unfortunately, circumstances are forcing them to dissolve the partnership. They decide to use the method of sealed bids with the understanding that one of them will get the hot dog stand and the other two will get cash. Albert bids \$81,000, Brett bids \$78,000, and Carl bids \$87,000. Who gets the hot dog stand and how much does he pay each of the other two partners?

24. The method of sealed bids can be used to divide up negative items like a list of chores that must be done. The main difference in the method is that the item or chore is given to the lowest bidder rather than the highest. You also need to be careful in the “owes to estate”/”estate owes” step. Three roommates need to divide up four chores in order to get their security deposit back. They use the method of sealed bids to divide the chores. The bids are summarized in the following table.

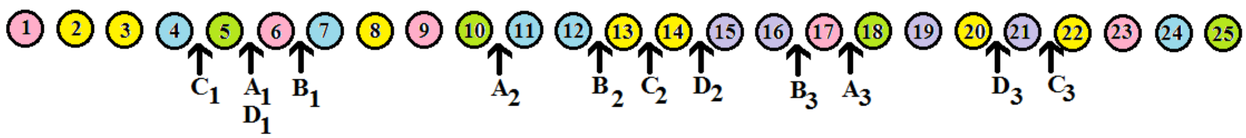
	Harry	Ingrid	Jeff
Clean Bathrooms	\$65	\$70	\$55
Patch and Paint Wall	\$100	\$85	\$95
Scrub Baseboards	\$60	\$50	\$45
Wash Windows	\$75	\$80	\$90

Describe the final outcome of the division. State which chores each roommate does and how much money each roommate gets or pays.

25. Albert, Bernard, and Charles want to divide up a collection of 17 small objects using the method of markers. The objects are laid out in a straight line and the players place their markers as shown in the following figure. Describe the final division, including which objects each player gets and how they deal with any leftover objects.



26. Alex, Bobby, Carrie, and Doug want to divide a collection of 25 small objects using the method of markers. The objects are laid out in a straight line and the players place their markers as shown in the following figure. Describe the final division, including which objects each player gets and how they deal with any leftover objects.



27. Jack, Kelly, and Larry want to divide a collection of 25 small objects using the method of markers. The objects are laid out in a straight line as shown in the following figure. Jack values each red object at $2, each blue object at $1, each green object at $0.50, and each yellow object at $0. Kelly values each red object at $0, each blue object at $1.50, each green object at $1, and each yellow object at $2. Larry values each red object at $1.50, each blue object at $1.50, each green object at $2, and each yellow object at $0.50. Determine where each player should place his markers. Draw the figure placing each player's markers in the correct places. Do not determine the division of objects.
Note: The numbers do not come out evenly so you might have to round off a bit.

Chapter 9: Apportionment

Apportionment involves dividing something up, just like fair division. In fair division we are dividing objects among people while in apportionment we are dividing people among places. Also like fair division, the apportionment processes that are widely used do not always give the best answer, and apportionment is still an open field of mathematics.

Apportionment is used every day in American politics. It is used to determine the size of voting districts and to determine the number of representatives from each state in the U.S. House of Representatives. Another example of how apportionment can be used is to assign a group of new fire fighters to the fire stations in town in an equitable way. Overall, apportionment is used to divide up resources (human or otherwise) in as fair a way as possible.

Section 9.1 Basic Concepts of Apportionment and Hamilton's Method

Apportionment can be thought of as dividing a group of people (or other resources) and assigning them to different places.

Example 9.1.1: Why We Need Apportionment

> Tom is moving to a new apartment. On moving day, four of his friends come to help and stay until the job is done since Tom promised they will split a case of beer afterwards. It sounds like a fairly simple job to split the case of beer between the five friends until Tom realizes that 24 is not evenly divisible by five. He could start by giving each of them (including himself) four beers. The question is how to divide the four remaining beers among the five friends assuming they only get whole beers. Apportionment methods can help Tom come up with an equitable solution

Basic Concepts of Apportionment:

The apportionment methods we will look at in this chapter were all created as a way to divide the seats in the U.S. House of Representatives among the states based on the size of the population for each state. The terminology we use in apportionment reflects this history. An important concept is that the number of seats a state has is proportional to the population of the state. In other words, states with large populations get lots of seats and states with small populations only get a few seats.

The **seats** are the people or items that are to be shared equally. The **states** are the parties that will receive a proportional share of the seats.

The first step in any apportionment problem is to calculate the standard divisor. This is the ratio of the total population to the number of seats. It tells us how many people are represented by each seat.

The **standard divisor** is $\text{SD} = \dfrac{\text{total population}}{\text{\# seats}}$.

The next step is to find the standard quota for each state. This is the exact number of seats that should be allocated to each state if decimal values were possible.

The **standard quota** is $\text{SQ} = \dfrac{\text{state population}}{\text{standard divisor}}$

Example 9.1.2: Finding the Standard Quota

Hamiltonia, a small country consisting of six states is governed by a senate with 25 members. The number of senators for each state is proportional to the population of the state. The following table shows the population of each state as of the last census.

Table 9.1.1: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

Find the standard divisor and the standard quotas for each of the states of Hamiltonia.

Standard Divisor: $\text{SD} = \dfrac{\text{total population}}{\text{\# seats}} = \dfrac{237,000}{25} = 9480$

This means that each seat in the senate corresponds to a population of 9480 people.

Standard Quotas:

$$\text{Alpha: } SQ = \frac{\text{state population}}{\text{standard divisor}} = \frac{24,000}{9480} = 2.532$$

$$\text{Beta: } SQ = \frac{\text{state population}}{\text{standard divisor}} = \frac{56,000}{9480} = 5.907$$

If fractional seats were possible, Alpha would get 2.532 seats and Beta would get 5.907 seats.

Use similar calculations for the other states.

Table 9.1.2: Standard Quotas for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001

Notice that the sum of the standard quotas is 25.001, the total number of seats. This is a good way to check your arithmetic.

Note: Do not worry about the 0.001. That is due to rounding and is negligible.

The standard quota for each state is usually a decimal number but in real life the number of seats allocated to each state must be a whole number. Rounding off the standard quota by the usual method of rounding does not always work. Sometimes the total number of seats allocated is too high and other times it is too low. In Example 9.1.2 the total number of seats allocated would be 26 if we used the usual rounding rule.

When we round off the standard quota for a state the result should be the whole number just below the standard quota or the whole number just above the standard quota. These values are called the lower and upper quotas, respectively. In the extremely rare case that the standard quota is a whole number, use the standard quota for the lower quota and the next higher integer for the upper quota.

The **lower quota** is the standard quota rounded down. The **upper quota** is the standard quota rounded up.

Example 9.1.3: Upper and Lower Quotas for Hamiltonia

Find the lower and upper quotas for each of the states in Hamiltonia.

Table 9.1.3: Upper and Lower Quotas for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001
Lower Quota	2	5	2	1	6	4	20
Upper Quota	3	6	3	2	7	5	26

Note: The total of the lower quotas is 20 (below the number of seats to be allocated) and the total of the upper quotas is 26 (above the number of seats to be allocated).

Hamilton's Method

The U.S. Constitution requires that the seats for the House of Representatives be apportioned among the states every ten years based on the sizes of the populations. Since 1792, five different apportionment methods have been proposed and four of these methods have been used to apportion the seats in the House of Representatives. The number of seats in the House has also changed many times. In many situations the five methods give the same results. However, in some situations, the results depend on the method used. As we will see in the next section, each of the methods has at least one weakness. Because it was important for a state to have as many representatives as possible, senators tended to pick the method that would give their state the most representatives. In 1941, the number of seats in the House was fixed at 435 and an official method was chosen. This took the politics out of apportionment and made it a purely mathematical process.

Alexander Hamilton proposed the first apportionment method to be approved by Congress. Unfortunately for Hamilton, President Washington vetoed its selection. This veto was the first presidential veto utilized in the new U.S. government. A different method proposed by Thomas Jefferson was used instead for the next 50 years. Later, Hamilton's method was used off and on between 1852 and 1901.

Summary of Hamilton's Method:

1. Use the standard divisor to find the standard quota for each state.
2. Temporarily allocate to each state its lower quota of seats. At this point, there should be some seats that were not allocated.
3. Starting with the state that has the largest fractional part and working toward the state with the smallest fractional part, allocate one additional seat to each state until all the seats have been allocated.

Example 9.1.4: Hamilton's Method for Hamiltonia

Use Hamilton's method to finish the allocation of seats in Hamiltonia.

Let's use red numbers below in Table 9.1.4 to rank the fractional parts of the standard quotas from each state in order from largest to smallest. For example, Zeta's standard quota, 4.958, has the largest fractional part, 0.958. Also find the sum of the lower quotas to determine how many seats still need to be allocated.

Table 9.1.4: Fractional Parts for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532(6)	5.907(3)	2.954(2)	1.793(5)	6.857(4)	4.958(1)	25.001
Lower Quota	2	5	2	1	6	4	20

Twenty of the 25 seats have been allocated so there are five remaining seats. Allocate the seats, in order, to Zeta, Gamma, Beta, Epsilon and Delta.

Table 9.1.5: Final Allocation for Hamiltonia Using Hamilton's Method

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001
Lower Quota	2	5	2	1	6	4	20
Final Allocation	2	6	3	2	7	5	25

Overall, Alpha gets two senators, Beta gets six senators, Gamma gets three senators, Delta gets two senators, Epsilon gets seven, and Zeta gets five senators.

According to Ask.com, "a paradox is a statement that apparently contradicts itself and yet might be true." (Ask.com, 2014) Hamilton's method and the other apportionment methods discussed in section 9.2 are all subject to at least one paradox. None of the apportionment methods is perfect. The Alabama paradox was first noticed in 1881 when the seats in the U.S. House of Representatives were reapportioned after the 1880 census. At that time the U.S. Census Bureau created a table which showed the number of seats each state would have for various possible sizes of the House of Representatives. They did this for possible sizes of the House from 275 total seats to 350 total seats. This table showed a strange occurrence as the size of the House of Representatives increased from 299 to 300. With 299 total seats, Alabama would receive 8 seats. However, if the house size was increased to 300 total seats, Alabama would only receive 7 seats. Increasing the overall number of seats caused Alabama to lose a seat.

The **Alabama paradox** happens when an increase in the total number of seats results in a decrease in the number of seats for a given state.

Example 9.1.5: The Alabama Paradox

A mother has an incentive program to get her five children to read more. She has 30 pieces of candy to divide among her children at the end of the week based on the number of minutes each of them spends reading. The minutes are listed below in Table 9.1.6.

Table 9.1.6: Reading Times

Child	Abby	Bobby	Charli	Dave	Ed	Total
Population	188	142	138	64	218	750

Use Hamilton's method to apportion the candy among the children.

The standard divisor is $SD = \frac{750}{30} = 25.$ After dividing each child's time by the standard divisor, and finding the lower quotas for each child, there are three pieces of candy left over. They will go to Ed, Bobby, and Dave, in that order, since they have the largest fractional parts of their quotas.

Table 9.1.7: Apportionment with 30 Pieces of Candy

Child	Abby	Bobby	Charli	Dave	Ed	Total
Population	188	142	138	64	218	750
Standard Quota	7.520	5.680	5.520	2.560	8.720	30.000
Lower Quota	7	5	5	2	8	27
Final Allocation	7	6	5	3	9	30

At the last minute, the mother finds another piece of candy and does the apportionment again. This time the standard divisor will be 24.19. Bobby, Abby, and Charli, in that order, will get the three left over pieces this time.

Table 9.1.8: Apportionment with 31 Pieces of Candy

Child	Abby	Bobby	Charli	Dave	Ed	Total
Population	188	142	138	64	218	750
Standard Quota	7.772	5.870	5.705	2.646	9.012	31.005
Lower Quota	7	5	5	2	9	28
Final Allocation	8	6	6	2	9	31

Notice that adding another piece of candy (a seat) caused Dave to lose a piece while Abby and Charli gain a piece. This is an example of the Alabama paradox.

Section 9.2: Apportionment: Jefferson's, Adams's, and Webster's Methods

Jefferson's method was the first method used to apportion the seats in the U.S. House of Representatives in 1792. It was used through 1832. That year, New York had a standard quota of 38.59 but was granted 40 seats by Jefferson's method. At that time, John Quincy Adams and Daniel Webster each proposed new apportionment methods but the proposals were defeated and Jefferson's method was still used. Webster's method was later chosen to be used in 1842 but Adams's method was never used. Webster's method and Hamilton's method often give the same result. For many of the years between 1852 and 1901, Congress used a number of seats for the House that would result in the same apportionment by either Webster's or Hamilton's methods. After Hamilton's method was finally scrapped in 1901, Webster's method was used in 1901, 1911, and 1931. There were irregularities in the process in 1872 and just after the 1920 census. In 1941, the House size was fixed at 435 seats and the Huntington-Hill method became the permanent method of apportionment.

Jefferson's, Adams's, and Webster's methods are all based on the idea of finding a divisor that will apportion all the seats under the appropriate rounding rule. There should be no seats left over after the number of seats are rounded off. For this to happen we have to adjust the standard divisor either up or down. The difference between the three methods is the rule for rounding off the quotas. Jefferson's method rounds the quotas down to their lower quotas, Adams' method rounds the quotas up to their upper quotas, and Webster's method rounds the quotas either up or down following the usual rounding rule.

Jefferson's Method:

Jefferson's method divides all populations by a modified divisor and then rounds the results down to the lower quota. Sometimes the total number of seats will be too large and other times it will be too small. We keep guessing modified divisors until the method assigns the correct total number of seats. Dividing by a larger modified divisor will make each quota smaller so the sum of the lower quotas will be smaller. It is easy to remember which way to go. If the sum is too large, make the divisor larger. If the sum is too small, make the divisor smaller. All the quotas are rounded down so using the standard divisor will give a sum that is too small. Our guess for the first modified divisor should be a number smaller than the standard divisor.

Summary of Jefferson's Method:

1. Find the standard divisor, $\text{SD} = \dfrac{\text{total population}}{\text{\# seats}}$.
2. Pick a modified divisor, d, that is slightly less than the standard divisor.
3. Divide each state's population by the modified divisor to get its modified quota.
4. Round each modified quota down to its lower quota.
5. Find the sum of the lower quotas.
6. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too large, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps three through six until the correct number of seats are apportioned.

Example 9.2.1: Jefferson's Method

Use Jefferson's method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.2.1: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

From Example 9.1.2 we know the standard divisor is 9480 and the sum of the lower quotas is 20. In Jefferson's method the standard divisor will always give us a sum that is too small so we begin by making the standard divisor smaller. There is no formula for this, just guess something. Let's try the modified divisor, d = 9000.

Table 9.2.2: Quotas for d = 9000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9000	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24

The sum of 24 is too small so we need to try again by making the modified divisor smaller. Let's try d = 8000.

Table 9.2.3: Quotas for d = 8000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9000	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24
d = 8000	3.00	7.00	3.50	2.13	8.13	5.88	
Lower Quota	3	7	3	2	8	5	28

This time the sum of 28 is too big. Try again making the modified divisor larger. We know the divisor must be between 8000 and 9000 so let's try 8500.

Table 9.2.4: Quotas for d = 8500

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9000	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24
d = 8000	3.00	7.00	3.50	2.13	8.13	5.88	
Lower Quota	3	7	3	2	8	5	28
d = 8500	2.82	6.59	3.29	2.00	7.65	5.53	
Lower Quota	2	6	3	2	7	5	25

This time the sum is 25 so we are done. Alpha gets two senators, Beta gets six senators, Gamma gets three senators, Delta gets two senators, Epsilon gets seven senators, and Zeta gets five senators.

Note: This is the same result as we got using Hamilton's method in Example 9.1.4. The two methods do not always give the same result.

Adams's Method:

Adams's method divides all populations by a modified divisor and then rounds the results up to the upper quota. Just like Jefferson's method we keep guessing modified divisors until the method assigns the correct number of seats. All the quotas are rounded up so the standard divisor will give a sum that is too large. Our guess for the first modified divisor should be a number larger than the standard divisor.

Summary of Adams's Method:

1. Find the standard divisor, $\text{SD} = \dfrac{\text{total population}}{\text{\# seats}}$.
2. Pick a modified divisor, d, that is slightly more than the standard divisor.
3. Divide each state's population by the modified divisor to get the modified quota.
4. Round each modified quota up to the upper quota.
5. Find the sum of the upper quotas.
6. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too big, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps three through six until the correct number of seats are apportioned.

Example 9.2.2: Adams's Method

Use Adams's method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.2.5: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

From Example 9.1.2 we know the standard divisor is 9480 and the sum of the upper quotas is 26. In Adams's method the standard divisor will always give us a sum that is too big so we begin by making the standard divisor larger. There is no formula for this, just guess something. Let's try the modified divisor, d = 10,000.

Table 9.2.6: Quotas for d = 10,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 10,000	2.40	5.60	2.80	1.70	6.50	4.70	
Upper Quota	3	6	3	2	7	5	26

The total number of seats, 26, is too big so we need to try again by making the modified divisor larger. Try d = 11,000.

Table 9.2.7: Quotas for d = 11,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 10,000	2.40	5.60	2.80	1.70	6.50	4.70	
Upper Quota	3	6	3	2	7	5	26
d = 11,000	2.18	5.09	2.55	1.55	5.91	4.27	
Upper Quota	3	6	3	2	6	5	25

This time the total number of seats is 25, the correct number of seats to be apportioned. Give Alpha three seats, Beta six seats, Gamma three seats, Delta two seats, Epsilon six seats, and Zeta five seats.

Note: This is not the same result as we got using Hamilton's method in Example 9.1.4.

Webster's Method:

Webster's method divides all populations by a modified divisor and then rounds the results up or down following the usual rounding rules. Just like Jefferson's method we keep guessing modified divisors until the method assigns the correct number of seats. Because some quotas are rounded up and others down we do not know if the standard divisor will give a sum that is too large or too small. Our guess for the first modified divisor should be the standard divisor.

Summary of Webster's Method:

1. Find the standard divisor, $SD = \frac{\text{total population}}{\text{\# seats}}$. Use the standard divisor as the first modified divisor.
2. Divide each state's population by the modified divisor to get the modified quota.
3. Round each modified quota to the nearest integer using conventional rounding rules.
4. Find the sum of the rounded quotas.
5. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too big, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps two through five until the correct number of seats are apportioned.

Example 9.2.3: Webster's Method

Use Webster's method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.2.8: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

From Example 9.1.2 we know the standard divisor is 9480. Because some quotas will be rounded up and other quotas will be rounded down we do not know immediately whether the total number of seats is too big or too small. Unlike Jefferson's and Adam's method, we do not know which way to adjust the modified divisor. This forces us to use the standard divisor as the first modified divisor.

Note that we must use more decimal places in this example than in the last few examples. Using two decimal places gives more information about which way to round correctly. Think about Alpha's standard quota. Both 2.48 and 2.53 would round off to 2.5. However, 2.48 should be rounded down to 2 while 2.53 should be rounded up to 3 according to Webster's method. This situation has not happened in any of the previous examples.

Table 9.2.9: Quotas for d = 9480

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26

Since the total of 26 seats is too big we need to make the modified divisor larger. Try d = 11,000.

Table 9.2.10: Quotas for d = 11,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26
d = 11,000	2.18	5.09	2.55	1.55	5.91	4.27	
Rounded Quota	2	5	3	2	6	4	22

The total number of seats is smaller like we hoped but 22 is way too small. That means that d = 11,000 is much too big. We need to pick a new modified divisor between 9480 and 11,000. Try a divisor closer to 9480 such as d = 10,000.

Table 9.2.11: Quotas for d = 10,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26
d = 11,000	2.18	5.09	2.55	1.55	5.91	4.27	
Rounded Quota	2	5	3	2	6	4	22
d = 10,000	2.40	5.60	2.80	1.70	6.50	4.70	
Rounded Quota	2	6	3	2	7	5	25

Note: This is the same apportionment we found using Hamilton's and Jefferson's methods, but not Adam's method.

9.3: Huntington-Hill Method

The Huntington-Hill method is the method currently used to apportion the seats for the U.S. House of Representatives. As with the other apportionment methods, the method of rounding off the quotas is what distinguishes this method from the others. The Huntington-Hill method starts out similarly to Webster's method since some quotas are rounded up and some quotas are rounded down. The difference is that the cut-off for rounding is not 0.5 anymore. Now the cut-off depends on the geometric mean between the lower and upper quotas.

The **geometric mean** G of two positive numbers A and B is $G = \sqrt{AB}$

Example 9.3.1: Geometric Mean

Find the geometric mean between 5 and 6.

$$G = \sqrt{5 \cdot 6} = \sqrt{30} \approx 5.477$$

Note that the geometric mean between A and B must be a number between A and B. In this example the geometric mean of 5.477 is between 5 and 6.

Summary of the Huntington-Hill Method:

1. Find the standard divisor, $\text{SD} = \frac{\text{total population}}{\text{\# seats}}$. Use the standard divisor as the first modified divisor.
2. Divide each state's population by the modified divisor to get the modified quota.
3. Round each modified quota to the nearest integer using the geometric mean as the cut off. If the quota is less than the geometric mean between the upper and lower quotas, round the quota down to the lower quota. If the quota is more than the geometric mean between the upper and lower quotas, round the quota up to the upper quota.
4. Find the sum of the rounded quotas.
5. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too big, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps two through five until the correct number of seats are apportioned.

Example 9.3.2: Huntington-Hill Method

Use the Huntington-Hill method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.3.1: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

The first step is to use the standard divisor as the first modified divisor. We also include a row for the geometric mean between the upper and lower quotas for each state.

Table 9.3.2: Quotas for d = 9480

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Lower Quota	2	5	2	1	6	4	
Upper Quota	3	6	3	2	7	5	
Calculation for G	$\sqrt{2 \bullet 3}$	$\sqrt{5 \bullet 6}$	$\sqrt{2 \bullet 3}$	$\sqrt{1 \bullet 2}$	$\sqrt{6 \bullet 7}$	$\sqrt{4 \bullet 5}$	
Geometric Mean	2.449	5.477	2.449	1.414	6.481	4.472	
Rounded Quota	3	6	3	2	7	5	26

The total number of seats, 26, is too big so we need to try again by making the modified divisor larger. Try d = 10,500.

Table 9.3.3: Quotas for d = 10,500

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Geometric Mean	2.449	5.477	2.449	1.414	6.481	4.472	
Rounded Quota	3	6	3	2	7	5	26
d = 10,500	2.29	5.33	2.67	1.62	6.19	4.48	
Geometric Mean	2.449	5.477	2.449	1.414	6.481	4.472	
Rounded Quota	2	5	3	2	6	5	23

The total number of seats, 23 is too small. We need to try again with a modified divisor between 9480 and 10,500. Since 23 is further from 25 than 26 is, try a divisor closer to 9480. Try d = 9800.

Table 9.3.4: Quotas for d = 9800

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Geometric Mean	2.449	5.477	2.450	1.414	6.481	4.472	
Rounded Quota	3	6	3	2	7	5	26
d = 10,500	2.29	5.33	2.67	1.62	6.19	4.48	
Geometric Mean	2.449	5.480	2.450	1.410	6.480	4.470	
Rounded Quota	2	5	3	2	6	5	23
d = 9800	2.4490	5.71	2.86	1.73	6.63	4.80	
Geometric Mean	2.4495	5.480	2.450	1.410	6.480	4.470	
Rounded Quota	2	6	3	2	7	5	25

Note: It was necessary to use more decimal places for Alpha's quota than the other quotas in order to see which way to round off.

This is the same apportionment we got with most of the other methods.

Example 9.3.3: Comparison of all Apportionment Methods

In the city of Adamstown, 42 new firefighters have just completed their training. They are to be assigned to the five firehouses in town in a manner proportional to the population in each fire district. The populations are listed in the following table.

Table 9.3.5: Populations for the Fire Districts of Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465

Apportion the new firefighters to the fire houses using Hamilton's, Jefferson's, Adams's, Webster's, and Huntington-Hill's methods.

The standard divisor is $SD = \dfrac{69,465}{42} \approx 1654$.

Hamilton's Method:

Table 9.3.6: Hamilton's Method for Adamstown

Start by dividing each population by the standard divisor and rounding each standard quota down.

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
Standard Quota	15.121	5.296	7.007	5.456	9.117	41.998
Lower Quota	15	5	7	5	9	41
Final Allocation	15	5	7	6	9	42

Using the lower quotas, there is one firefighter left over. Assign this firefighter to District D since D has the largest fractional part.

Jefferson's Method:

Jefferson's method always rounds down making the sum of the lower quotas too small. Make the standard divisor smaller to get the first modified divisor. The results are summarized below in Table 9.3.7.

Guess #1: d = 1600. The sum of 41 is still too small so make the modified divisor smaller.

Guess #2: d = 1550. The sum is 42 so we are done.

Table 9.3.7: Jefferson's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1600	15.631	5.475	7.244	5.641	9.425	
quota	15	5	7	5	9	41
d = 1550	16.135	5.652	7.477	5.823	9.729	
Final Allocation	16	5	7	5	9	42

Adams's Method:

Adams's method always rounds up making the sum of the upper quotas too large. Make the standard divisor larger to get the first modified divisor. The results are summarized below in Table 9.3.8.

Guess #1: d = 1700. The total is too still too large so make the modified divisor larger.

Guess #2: d = 1900. Now the total is too small so make the modified divisor smaller.

Guess #3: d = 1750. The total is too large again so make the modified divisor larger.

Guess #4: d = 1775. The sum is 42 so we are done.

Table 9.3.8: Adams's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1700	14.712	5.153	6.818	5.309	8.871	
quota	15	6	7	6	9	43
d = 1800	13.894	4.867	6.439	5.014	8.378	
quota	14	5	7	6	9	41
d = 1750	14.291	5.006	6.623	5.157	8.617	
quota	15	6	7	6	9	43
d = 1775	14.090	4.935	6.530	5.085	8.496	
Final Allocation	15	5	7	6	9	42

Webster's Method:

Webster's method rounds the usual way so we cannot tell if the sum is too large or too small right away. Try the standard divisor as the first modified divisor. The results are summarized below in Table 9.3.9.

Guess #1: d = 1654. The sum of 41 is too small so make the modified divisor smaller.

Guess #2: d = 1600. The sum of 43 is too large so make the modified divisor larger.

Guess #3: d = 1625. The sum is 42 so we are done.

Table 9.3.9: Webster's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1654	15.121	5.296	7.007	5.456	9.117	
quota	15	5	7	5	9	41
d = 1600	15.631	5.475	7.244	5.641	9.425	
quota	16	5	7	6	9	43
d = 1625	15.391	5.391	7.132	5.554	9.280	
Final Allocation	15	5	7	6	9	42

Huntington-Hill's Method:

Huntington-Hill's method rounds off according to the geometric mean. Use the standard divisor as the first modified divisor. The results are summarized below in Table 9.3.10.

Guess #1: d = 1654. The sum of 41 is too small so make the modified divisor smaller. Look at District D. It was really close to being rounded up rather than rounded down so we do not need to change the modified divisor by very much.

Guess #2: d = 1625. The sum is 42 so we are done.

Table 9.3.10: Huntington-Hill's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1654	15.121	5.296	7.007	5.456	9.117	
Geometric mean	15.492	5.477	7.483	5.477	9.487	
quota	15	5	7	5	9	41
d = 1625	15.391	5.391	7.132	5.554	9.280	
Geometric mean	15.492	5.477	7.483	5.477	9.487	
Final Allocation	15	5	7	6	9	42

Hamilton's, Adams's, Webster's, and Huntington-Hill's methods all gave the same apportionment: 15 firefighters to District A, five to District B, seven to District C, six to District D, and nine to District E.

Jefferson's method gave a different apportionment: 16 firefighters to District A, five to District B, seven to District C, five to District D, and nine to District E.

9.4: Apportionment Paradoxes

Each of the apportionment methods has at least one weakness. Some potentially violate the quota rule and some are subject to one of the three paradoxes.

The **quota rule** says that each state should be given either its upper quota of seats or its lower quota of seats.

Example 9.4.1: Quota Rule Violation

A small college has three departments. Department A has 98 faculty, Department B has 689 faculty, and Department C has 212 faculty. The college has a faculty senate with 100 representatives. Use Jefferson's method with a modified divisor of d = 9.83 to apportion the 100 representatives among the departments.

Table 9.4.1: Quota Rule Violation

State	A	B	C	Total
Population	98	689	212	999
Standard Quota	9.810	68.969	21.221	100.000
d = 9.83	9.969	70.092	21.567	
quota	9.000	70.000	21.000	100

District B has a standard quota of 68.969 so it should get either its lower quota, 68, or its upper quota, 69, seats. Using this method, District B received 70 seats, one more than its upper quota. This is a Quota Rule violation.

The **population paradox** occurs when a state's population increases but its allocated number of seats decreases.

Example 9.4.2: Population Paradox

A mom decides to split 11 candy bars among three children based on the number of minutes they spend on chores this week. Abby spends 54 minutes, Bobby spends 243 minutes and Charley spends 703 minutes. Near the end of the week, Mom reminds the children of the deal and they each do some extra work. Abby does an extra two minutes, Bobby an extra 12 minutes and Charley an extra 86 minutes. Use Hamilton's method to apportion the candy bars both before and after the extra work.

Table 9.4.2: Candy Bars Before the Extra Work

State	Abby	Bobby	Charley	Total
Population	54	243	703	1,000
Standard Quota	0.594	2.673	7.734	11.000
Lower Quota	0	2	7	9
Apportionment	0	3	8	11

With the extra work:
Abby now has 54 + 2 = 56 minutes
Bobby has 243 + 12 = 255
Charley has 703 + 86 = 789 minutes

Table 9.4.3: Candy Bars After the Extra Work

State	Abby	Bobby	Charley	Total
Population	56	255	789	1,100
Standard Quota	0.560	2.550	7.890	11.000
Lower Quota	0	2	7	9
Apportionment	1	2	8	11

Abby's time only increased by 3.7% while Bobby's time increased by 4.9%. However, Abby gained a candy bar while Bobby lost one. This is an example of the Population Paradox.

The **new states paradox** occurs when a new state is added along with additional seats and existing states lose seats.

Example 9.4.3: New-States Paradox

A small city is made up of three districts and governed by a committee with 100 members. District A has a population of 5310, District B has a population of 1330, and District C has a population of 3308. The city annexes a small area, District D with a population of 500. At the same time the number of committee members is increased by five. Use Hamilton's method to find the apportionment before and after the annexation.

$$SD = \frac{9948}{100} = 99.48$$

Table 9.4.4: Apportionment Before the Annexation

State	A	B	C	Total
Population	5,310	1,330	3,308	9,948
Standard Quota	53.378	13.370	33.253	100.000
Lower Quota	53	13	33	99
Apportionment	54	13	33	100

$$SD = \frac{10,448}{105} = 99.505$$

Table 9.4.5: Apportionment After the Annexation

State	A	B	C	D	Total
Population	5,310	1,330	3,308	500	10,448
Standard Quota	53.364	13.366	33.245	5.025	105.000
Lower Quota	53	13	33	5	104
Apportionment	53	14	33	5	105

District D has a population of 500 so it should get five seats. When District D is added with its five seats, District A loses a seat and District B gains a seat. This is an example of the New-States Paradox.

In 1980, Michael Balinski (State University of New York at Stony Brook) and H. Peyton Young (Johns Hopkins University) proved that all apportionment methods either violate the quota rule or suffer from one of the paradoxes. This means that it is impossible to find the "perfect" apportionment method. The methods and their potential flaws are listed in the following table.

Table 9.4.6: Methods, Quota Rule Violations, and Paradoxes

		Paradoxes		
Method	**Quota Rule**	**Alabama**	**Population**	**New-States**
Hamilton	No violations	Yes	Yes	Yes
Jefferson	Upper-quota violations	No	No	No
Adams	Lower-quota violations	No	No	No
Webster	Lower- and upper-quota violations	No	No	No
Huntington-Hill	Lower- and upper-quota violations	No	No	No

Chapter 9 Homework

1. A solar system consisting of five planets has a governing council of 135 members who are apportioned proportional to the populations of the planets. The population for each planet is listed in the following table.

Planet	Ajax	Borax	Calax	Delphi	Eljix	Total
Population	183,000	576,000	274,000	749,000	243,000	2,025,000

   For each planet, find the standard quota, the upper quota and the lower quota. Give your answers in a table.

2. A city has seven fire districts and 585 firefighters. The number of firefighters assigned to each district is proportional to the population of the district. The population for each district is given in the following table.

District	1	2	3	4	5	6	7	Total
Population	23,400	41,800	36,200	28,800	34,900	48,500	16,300	229,900

   For each district, find the standard quota, the upper quota and the lower quota. Give your answers in a table.

3. The country named Erau has five states and a total of 200 seats available in its House of Representatives. The number of seats that each state receives is proportional to the population of that state. The populations of the states are given in the table below.

State	1	2	3	4	5	Total
Population	3,500,000	1,200,000	530,000	999,000	771,000	7,000,000

   For each state, find the standard quota, the upper quota and the lower quota. Give your answers in a table.

4. Use Hamilton’s method to apportion the 585 firefighters in problem #2.

5. Use Hamilton’s method to apportion the 135 council members in problem #1.

6. Use Hamilton's method to apportion the 200 seats in the House of Representatives in problem #3.

7. A small country is made up of three separate islands: Eno, with a population of 100,300, Owt, with a population of 9,405, and Eerht with a population of 90,295. The country has a senate with 200 members whose seats are apportioned proportional to the population of each island.
   a. Use Hamilton's method to apportion the 200 seats.
   b. The senate decides to add another seat so that they have an odd number of senators. Use Hamilton's method to apportion the 201 seats.
   c. Compare your results from parts (a) and (b). This is an example of which paradox?

8. Use Jefferson's method to apportion the 585 firefighters in problem #2.

9. Use Jefferson's method to apportion the 135 council members in problem #1.

10. Use Jefferson's method to apportion the 200 seats in the House of Representatives in problem #3.

11. Use Adams's method to apportion the 135 council members in problem #1.

12. Use Adams's method to apportion the 585 firefighters in problem #2.

13. Use Adam's method to apportion the 200 seats in the House of Representatives in problem #3.

14. Use Webster's method to apportion the 585 firefighters in problem #2.

15. Use Webster's method to apportion the 135 council members in problem #1.

16. Use Webster's method to apportion the 200 seats in the House of Representatives in problem #3.

17. Use Huntington-Hill's method to apportion the 135 council members in problem #1.

18. Use Huntington-Hill's method to apportion the 585 firefighters in problem #2.

19. Use Huntington-Hill's method to apportion the 200 seats in the House of Representatives in problem #3.

20. Last year a city had three school districts: North, with a population of 5,200 children, South, with a population of 10,600 children, and West, with a population of 15,100.
    a. Use Hamilton's method to apportion 50 speech therapists among the districts using the populations for last year.
    b. This year, the city took over another school district. The new East district has a population of 9,500 children. If the number of speech therapists is increased by 15 to accommodate the new district, use Hamilton's method to apportion the 65 speech therapists.
    c. Compare your results to parts (a) and (b). This is an example of which paradox?

21. After the census in 1950, planet Ajax had a standard quota of 11.87 but was awarded 13 seats using Jefferson's method. Why were people on the other planets upset about this? What rule was violated?

22. Five co-workers work together on a large design project. As a reward for their effort, their boss wishes to split 50 gift cards of equal value among the employees based on the amount of time spend on the project. The time, in hours, worked by each employee is listed in the following table.

Employee	Jack	Kim	Lisa	Mark	Nancy	Total
Time	150	173	78	295	204	900

a. Use Hamilton's method to apportion the 50 gift cards among the employees.
b. At the last minute, the boss made a minor change to the design. It took Kim 8 more hours and Mark one more hour to incorporate the change. Use Hamilton's method to apportion the 50 gift cards using the new total times for Kim (181) and Mark (296).
c. Compare your answers to parts (a) and (b). Which one of the paradoxes occurred? Explain your answer.

Chapter 10: Geometric Symmetry and the Golden Ratio

Patterns and geometry occur in nature and humans have been noticing these patterns since the dawn of humanity. In this chapter, topics in geometry will be examined. These topics include transformation and symmetry of geometric shapes, similar figures, gnomons, Fibonacci numbers, and the Golden Ratio.

Section 10.1: Transformations Using Rigid Motions

In this section we will learn about isometry or rigid motions. An isometry is a transformation that preserves the distances between the vertices of a shape. A rigid motion does not affect the overall shape of an object but moves an object from a starting location to an ending location. The resultant figure is congruent to the original figure.

A **rigid motion** is when an object is moved from one location to another and the size and shape of the object have not changed.

Two figures are **congruent** if and only if there exists a rigid motion that sets up a correspondence of one figure as the image of the other. Side lengths remain the same and interior angles remain the same.

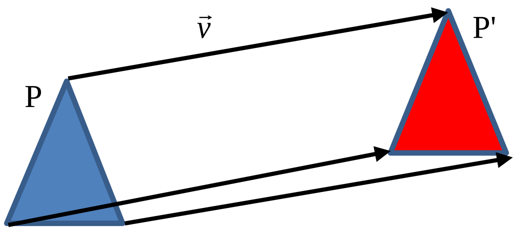
An **identity motion** is a rigid motion that moves an object from its starting location to exactly the same location. It is as if the object has not moved at all.

There are four kinds of rigid motions: translations, rotations, reflections, and glide-reflections. When describing a rigid motion, we will use points like P and Q, located on the geometric shape, and identify their new location on the moved geometric shape by P' and Q'.

We will start with the rigid motion called a translation. When translating an object, we move the object in a specific direction for a specific length, along a vector $\vec{v}$.

Figure 10.1.1: Translation

The translation of the blue triangle with point P was moved along the vector $\vec{v}$ to the location of the red triangle with point P'. Also note that the other vertices of the blue triangle also moved along the vector $\vec{v}$ to corresponding vertices on the red triangle.



A **translation** of an object moves the object along a directed line segment called a vector $\vec{v}$ for a specific distance and in a specific direction. The motion is completely determined by two points P and P' where P is on the original object and P' is on the translated object.

In regular language, a translation of an object is a *slide* from one position to another. You are given a geometric figure and an arrow which represents the vector. The vector gives you the direction and distance which you slide the figure.

Example 10.1.1 Translation of a Triangle

You are given a blue triangle and a vector $\vec{v}$. Move the triangle along vector $\vec{v}$.

Figure 10.1.2: Blue Triangle and Vector $\vec{v}$

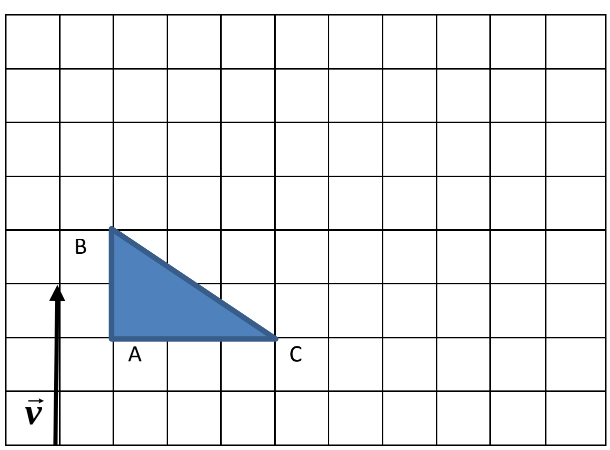
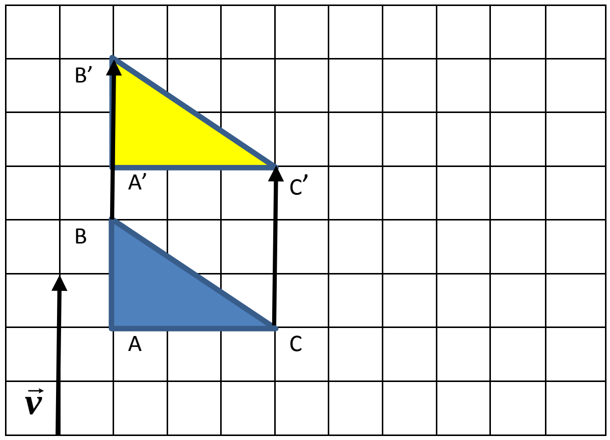


Figure 10.1.3: Result of the Translation



Properties of a Translation

1. A translation is completely determined by two points P and P'
2. Has no fixed points
3. Has identity motion $-\vec{v}$

Note: the vector $-\vec{v}$ *has the same length as vector* $\vec{v}$ *, but points in the opposite direction.*

Example 10.1.2 Translation of an Object

Given the L-shape figure below, translate the figure along the vector $\vec{v}$. The vector $\vec{v}$ moves horizontally three units to the right and vertically two units up. Move each vertex three units to the right and two units up. The red figure is the position of the L-shape figure after the slide.

Figure 10.1.4: L-Shape and Vector $\vec{v}$

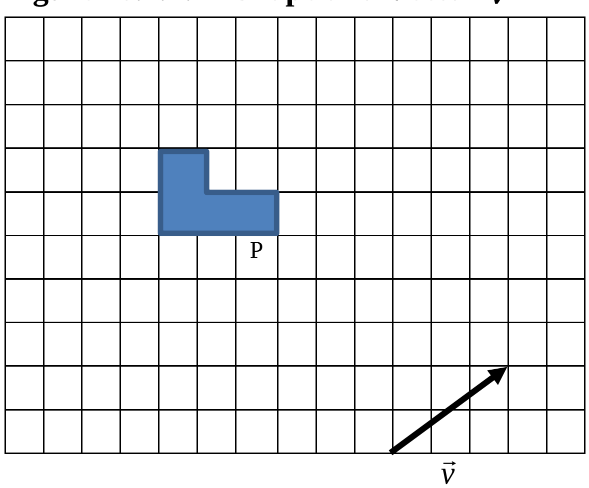
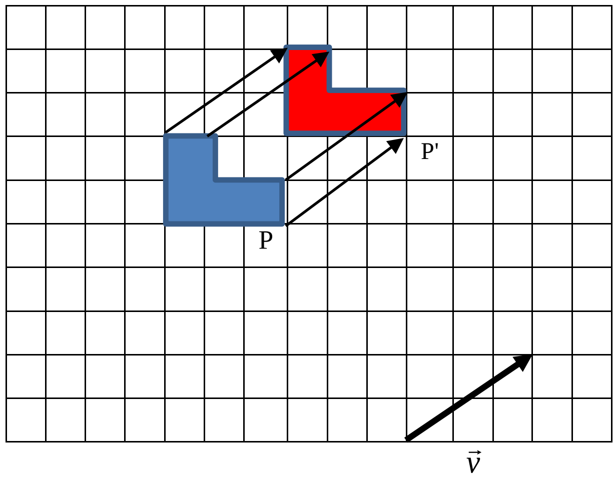


Figure 10.1.5: Result of the L-Shape Translated by Vector $\vec{v}$



The next type of transformation (rigid motion) that we will discuss is called a rotation. A rotation moves an object about a fixed point R called the rotocenter and through a specific angle. The blue triangle below has been rotated 90° about point R.

> A **rotation** of an object moves the object around a point called the rotocenter R a certain angle θ either clockwise or counterclockwise.

Note: the rotocenter R can be outside the object, inside the object or on the object.

Figure 10.1.6: A Triangle Rotated 90° around the Rotocenter R outside the Triangle

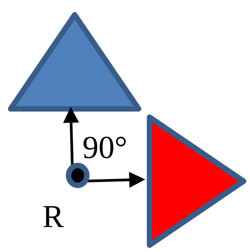
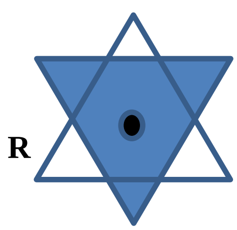


Figure 10.1.7: A Triangle Rotated 180° around the Rotocenter R inside the Triangle



Properties of a Rotation

1. A Rotation is completely determined by two pairs of points; P and P' and Q and Q'
2. Has one fixed point, the rotocenter R
3. Has identity motion the 360° rotation

Example 10.1.3: Rotation of an L-Shape

Given the diagram below, rotate the L-shaped figure 90° clockwise about the rotocenter R. The point Q rotates 90°. Move each vertex 90° clockwise.

Figure 10.1.8: L-Shape and Rotocenter R

The L-shaped figure will be rotated 90° clockwise and vertex Q will move to vertex Q'. Each vertex of the object will be rotated 90°.

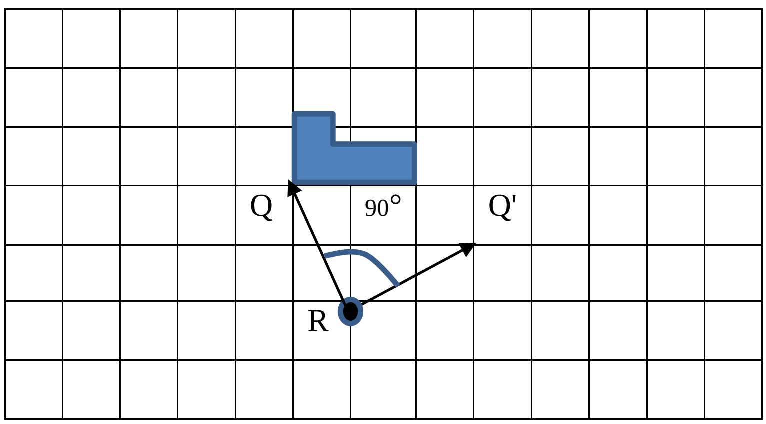
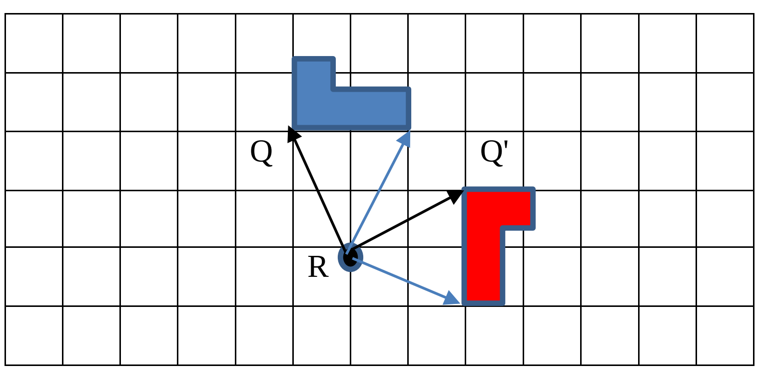


Figure 10.1.9: Result of the 90° Clockwise Rotation



Example 10.1.4: 45° Clockwise Rotation of a Rectangle

Figure 10.1.10: Rectangle and Rotocenter R

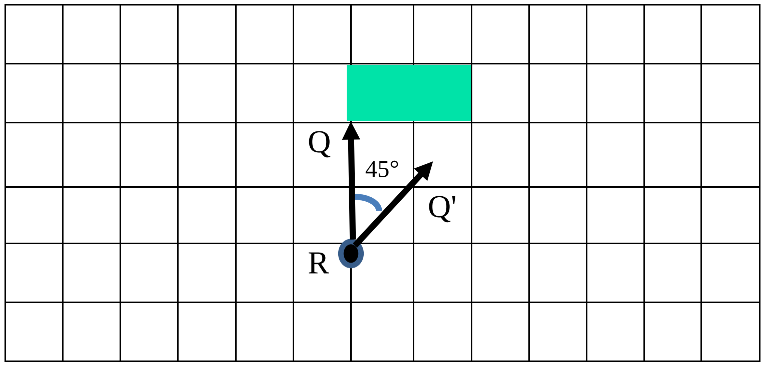
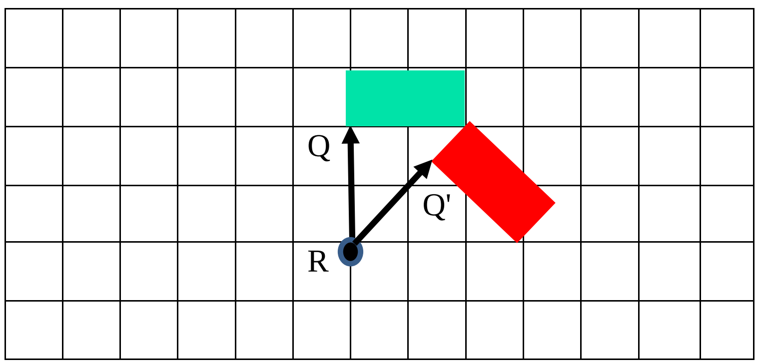


Figure 10.1.11: Result of 45° Clockwise Rotation



Example 10.1.5: 180° Clockwise Rotation of an L-Shape

Figure 10.1.12: L-Shape and Rotocenter R

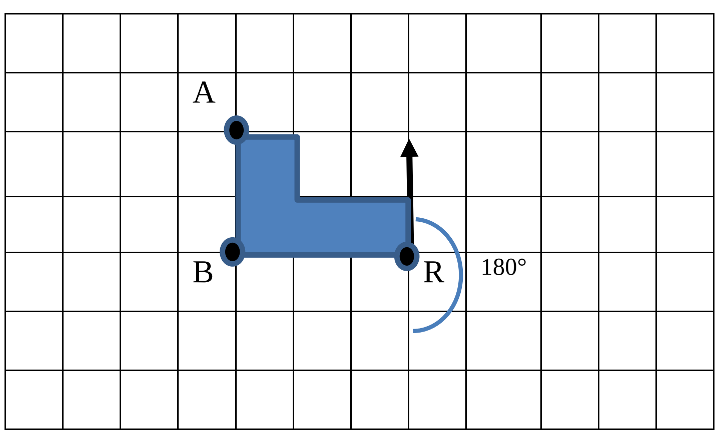
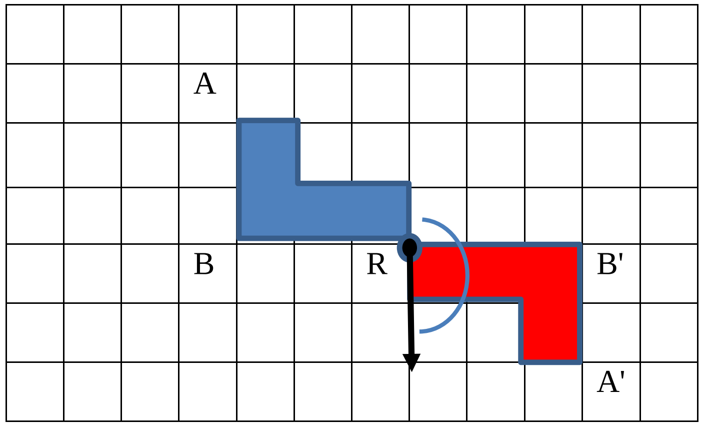


Figure 10.1.13: Result of the 180° Clockwise Rotation



The next type of transformation (rigid motion) is called a reflection. A reflection is a mirror image of an object, or can be thought of as "flipping" an object over.

> **Reflection:** If each point on a line l corresponds to itself, and each other point P in the plane corresponds to a unique point P' in the plane, such that l is the perpendicular bisector of PP', then the correspondence is called the reflection in line l.

In regular language, a reflection is a mirror image across a line l. The line l is the midpoint of the line between the two points, P in the original figure and P' in the reflection. P goes to P'.

Figure 10.1.14: Reflection of an Object about a Line *l*

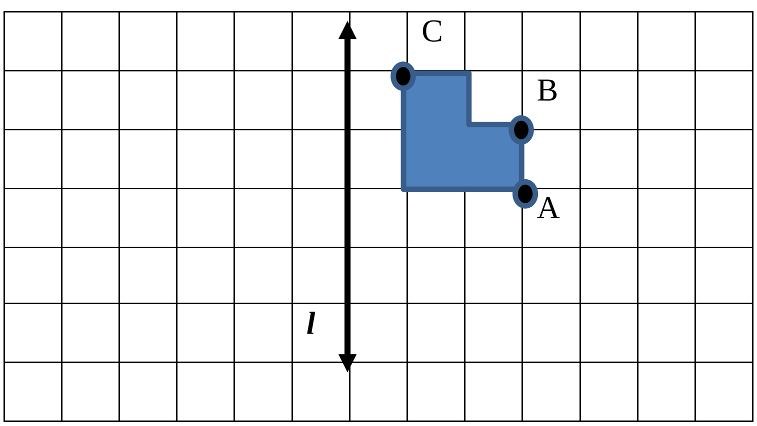
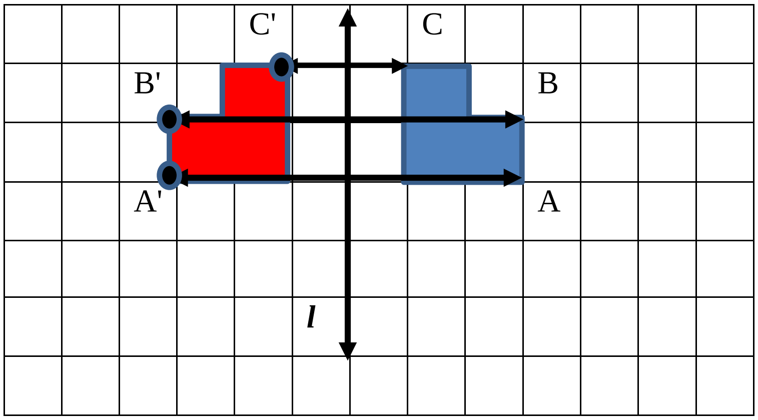


Figure 10.1.15: Result of the Reflection over Line *l*

The reflection places each vertex along a line perpendicular to *l* and equidistant from *l*.

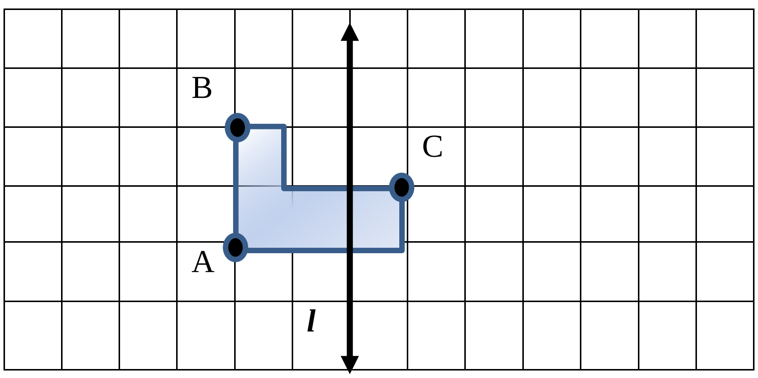


Properties of a Reflection

1. A reflection is completely determined by a single pair of points; P and P’
2. Has infinitely many fixed points: the line of reflection *l*
3. Has identity motion the reverse reflection

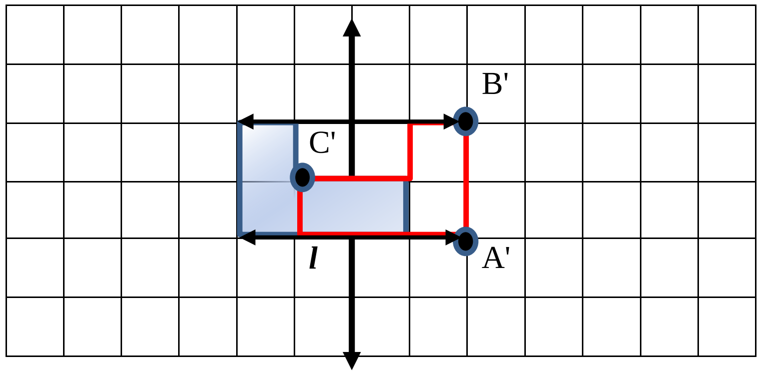
Example 10.1.6 Reflect an L-Shape across a Line *l*

Figure 10.1.16: L-shape and Line *l*



Reflect the L-shape across line *l*. The red L-shape shown below is the result after the reflection. The original position of each vertex is on a line with the reflected position of each vertex. This line that connects the original and reflected positions of the vertex is perpendicular to line *l* and the original and reflected positions of each vertex are equidistant to line *l*.

Figure 10.1.17: Result of Reflection over Line *l*



Example 10.1.7: Reflect another L-Shape across Line *l*

First identify the vertices of the figure. From each vertex, draw a line segment perpendicular to line *l* and make sure its midpoint lies on line *l*. Now draw the new positions of the vertices, making the transformed figure a mirror image of the original figure.

Figure 10.1.18: L-Shape and Line *l*

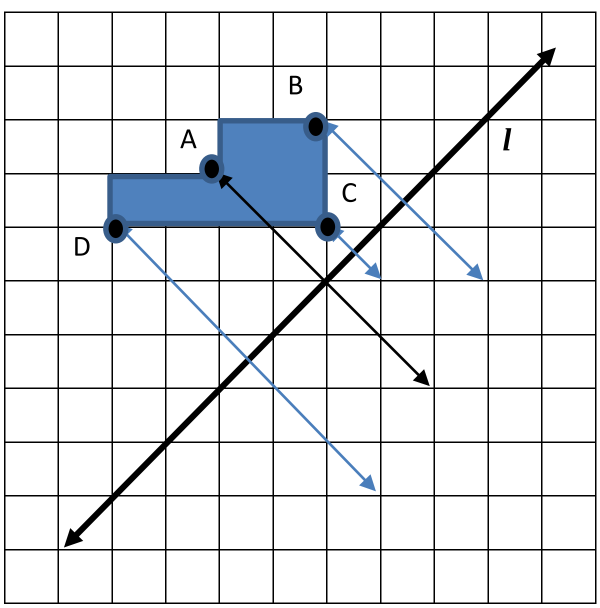
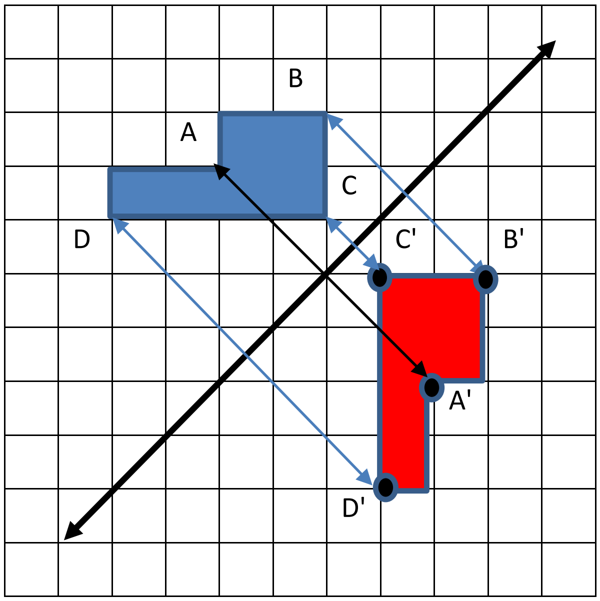


Figure 10.1.19: Result of Reflection over Line *l*



The final transformation (rigid motion) that we will study is a glide-reflection, which is simply a combination of two of the other rigid motions.

A **glide-reflection** is a combination of a reflection and a translation.

Example 10.1.8 Glide-Reflection of a Smiley Face by Vector $\vec{v}$ and Line *l*

Figure 10.1.20: Smiley Face, Vector $\vec{v}$, and Line *l*

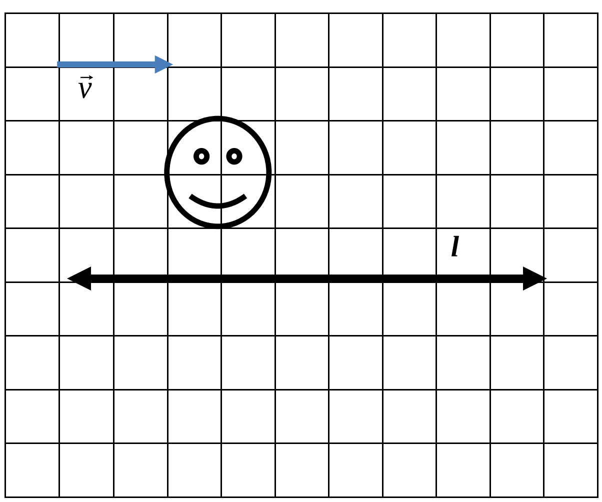


Figure 10.1.21: Smiley Face Glide-Reflection Step One

First slide the smiley face two units to the right along the vector $\vec{v}$.

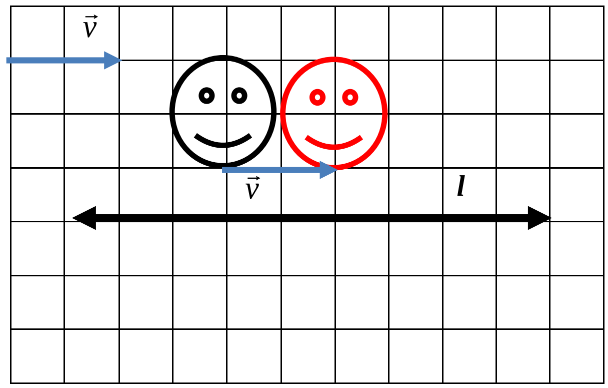
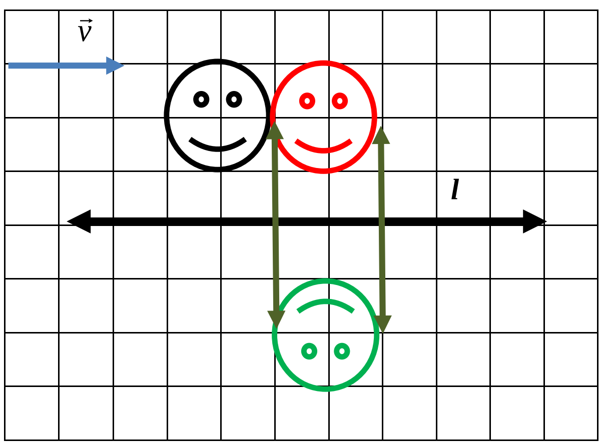


Figure 10.1.22: Smiley Face Glide-Reflection Step Two

Then reflect the smiley face across line *l*. The final result is the green upside-down smiley face.



Properties of a Glide-Reflection

1. A reflection is completely determined by a single pair of points; P and P.
2. Has infinitely fixed points: the line of reflection *l*.
3. Has identity motion the reverse glide-reflection**.**

Example 10.1.9: Glide-Reflection of a Blue Triangle

Figure 10.1.23: Blue Triangle, Vector $\vec{v}$, and Line *l*

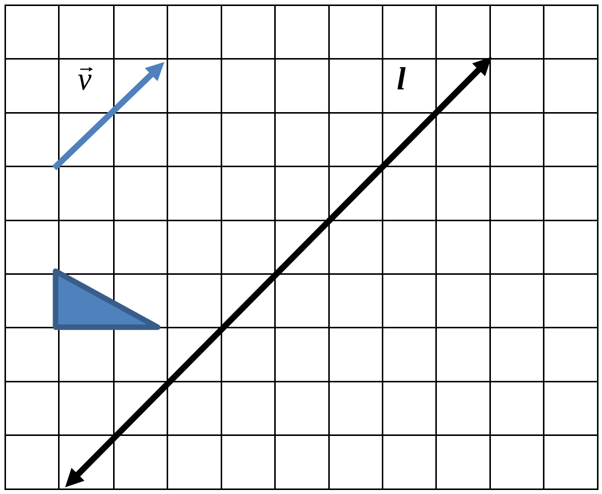


Figure 10.1.24: Triangle Glide-Reflection Step One

First, slide the triangle along vector $\vec{v}$.

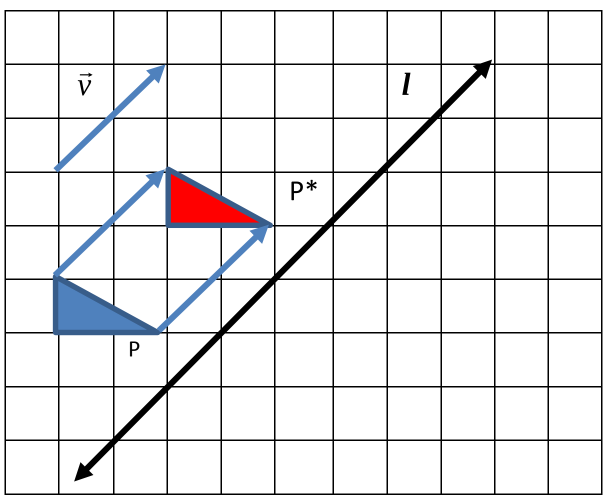
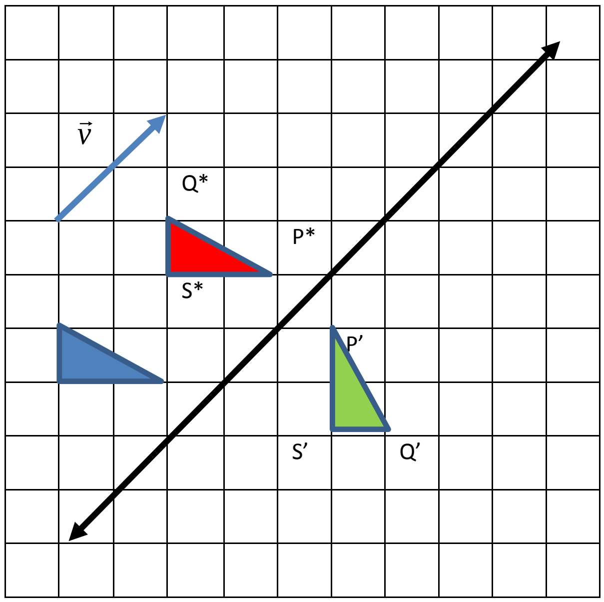


Figure 10.1.25: Triangle Glide-Reflection Step Two

Then, reflect the triangle across line *l*. The final result is the green triangle below line *l*.



Example 10.1.10: Glide-Reflection of an L-Shape

Figure 10.1.26: L-Shape, Vector $\vec{v}$, and Line *l*

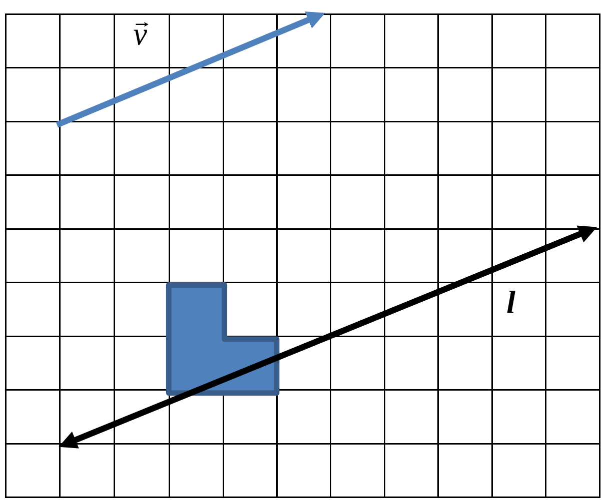


Figure 10.1.27: L-Shape Glide-Reflection Step One

First slide the L-shape along vector $\vec{v}$.

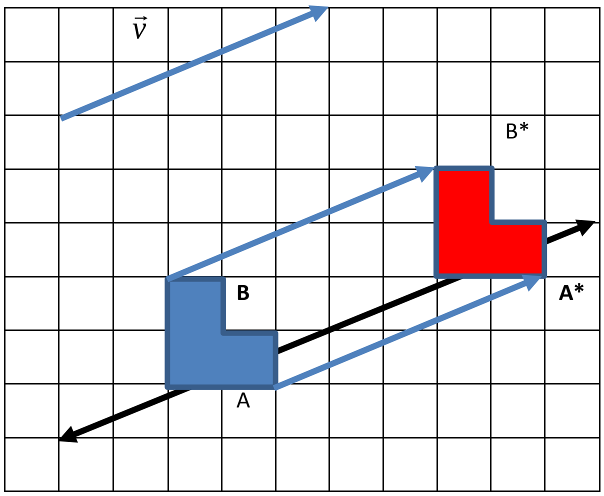
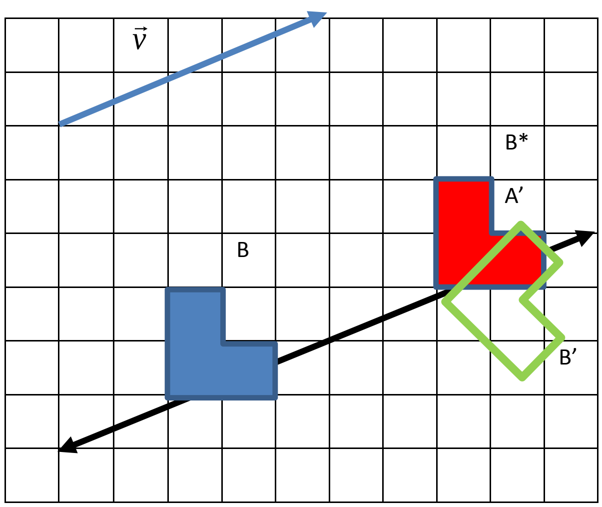


Figure 10.1.28: L-Shape Glide-Reflection Step Two

Then reflect the L-shape across line *l*. The result is the green open shape below the line *l*.



Section 10.2: Connecting Transformations and Symmetry

Humans have long associated symmetry with beauty and art. In this section, we define symmetry and connect it to rigid motions.

A **symmetry** of an object is a rigid motion that moves an object back onto itself.

There are two categories of symmetry in two dimensions, reflection symmetries and rotation symmetries.

A **reflection symmetry** occurs when an object has a line of symmetry going through the center of the object, and you can fold the object on this line and the two halves will "match." An object may have no reflection symmetry or may have one or more reflection symmetries.

A **rotation symmetry** occurs when an object has a rotocenter in the center of the object, and the object can be rotated about the rotocenter some degree less than or equal to 360° and is a "match" to the original object. Every object has one or more rotation symmetries.

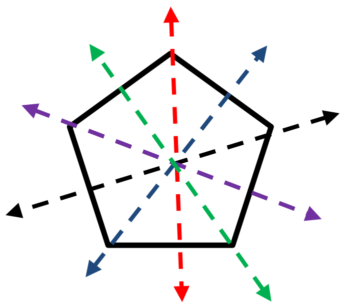
D-Type Symmetry: Objects that have both reflection symmetries and rotation symmetries are Type D_n where n is either the number of reflection symmetries or the number of rotation symmetries. If an object has both reflection and rotation symmetries, then it is always the same number, n, of each kind of symmetry.

Z-Type Symmetry: Objects that have no reflection symmetries and only rotation symmetries are Type Z_n where n is the number of the rotation symmetries.

Example 10.2.1: Symmetries of a Pentagon

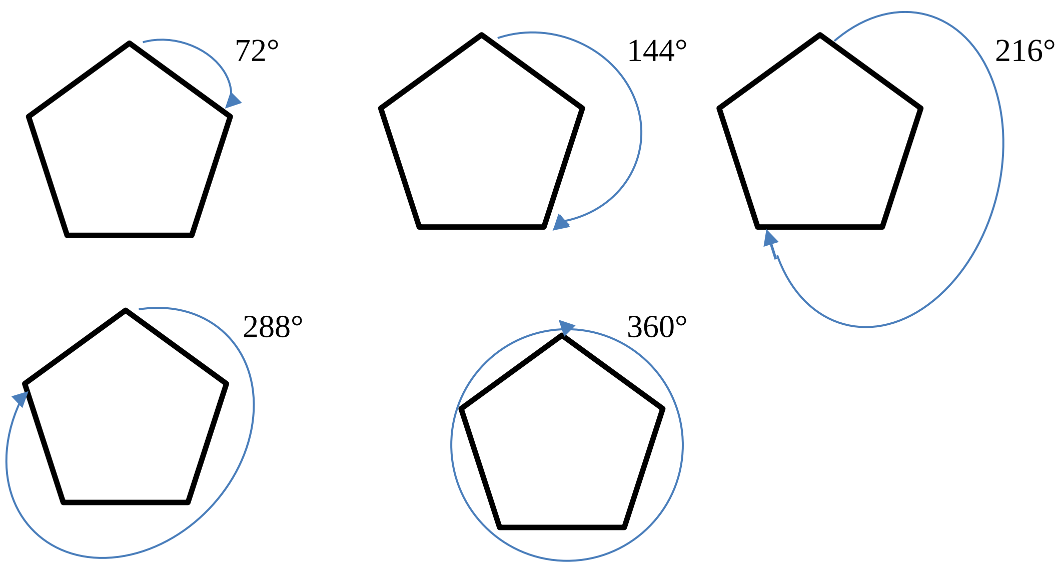
Identify the reflection and the rotation symmetries of the pentagon. The five dashed lines shown on the figure below are lines of reflection. The pentagon can be folded along these lines back onto itself and the two halves will "match" which means that the pentagon has a reflection symmetry along each line.

Figure 10.2.1: Reflection Symmetries of a Pentagon



Also, there are five vertices of the pentagon and there are five rotation symmetries. The angle of rotation for each rotation symmetry can be calculated by dividing 360° by the number of vertices of the object: $\frac{360^\circ}{5} = 72^\circ$. So, if you rotate the upper vertex of the pentagon to any other vertex, the resulting object will be a match to the original object, and thus a symmetry.

Figure 10.2.2: Rotation Symmetries of a Pentagon

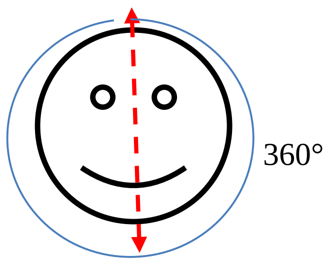


When an object has the same number of reflection symmetries as rotation symmetries, we say it has symmetry type D_n Therefore, the pentagon has symmetry type D_5 because it has five reflection symmetries and five rotation symmetries.

Example 10.2.2: Symmetries of a Smiley Face

Identify the rotation and the reflection symmetries of the smiley face. There is one line of reflection that will produce a reflection symmetry as shown below, and the only rotation symmetry is 360°, also shown below.

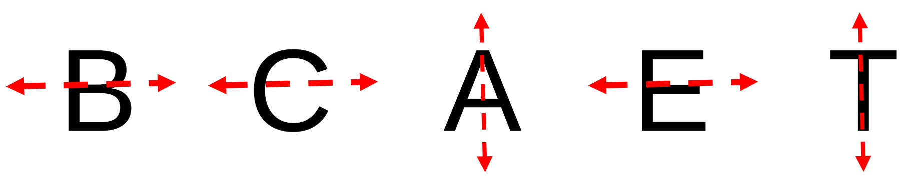
Figure 10.2.3: The Smiley Face has Symmetry Type D_1



Example 10.2.3: Symmetry Type D_1

Figure 10.2.4: Some Letters with Symmetry Type D_1

The following letters are all examples of symmetry type D_1 since they each have only one axis of reflection that will produce a symmetry as shown below, and they each have only one rotation symmetry, 360°.



Example 10.2.4: Symmetries of a Pinwheel

Identify the rotation and reflection symmetries of a pinwheel.

Figure 10.2.5: There are No Reflection Symmetries of the Pinwheel

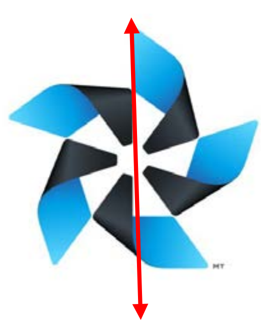


Figure 10.2.6: There are Five Rotation Symmetries of the Pinwheel

We find the angle by dividing 360° by five pinwheel points; $\frac{360^\circ}{5} = 72^\circ$. The rotation symmetries of the pinwheel are 72°, 144°, 216°, 288°, and 360°.

72°

144°

216°

288°

360°

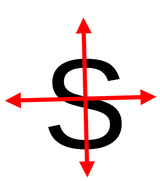
When an object has no reflection symmetries and only rotation symmetries, we say it has symmetry type Z_n**.** The pinwheel has symmetry type Z_5.

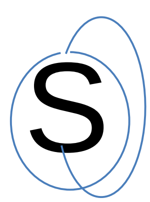
Example 10.2.5: Symmetries of the Letter S

Identify the rotation and the reflection symmetries of the letter S.

Figure 10.2.7: The Letter S

There are no reflection symmetries and two rotation symmetries; 180° and 360°, therefore the letter S has symmetry type Z_2.





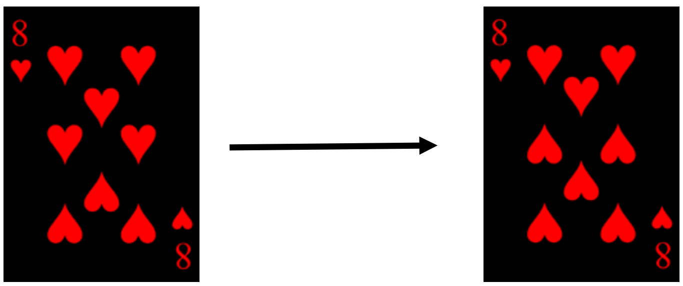
180° and 360°

Example 10.2.6: Symmetries of the Card the Eight of Hearts

Identify the rotation and reflection symmetries of card the eight of hearts.

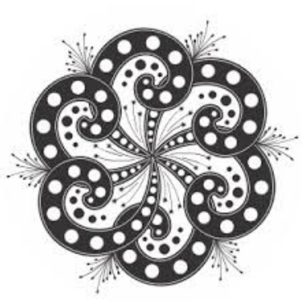
The card shown below has no reflection symmetries since any reflection would change the orientation of the card. At first, it may appear that the card has symmetry type Z_2. However, when rotated 180°, the top five hearts will turn upside-down and it will not be the same. Therefore, this card has only the 360° rotation symmetry, and so it has symmetry type Z_1.

Figure 10.2.8: The Eight of Hearts and its 180° Rotation



Example 10.2.7: Other Examples of Symmetry Type Z_n

Figure 10.2.9: A Design and the Letter K

a. 

b. K

a. The design has symmetry type Z_6, no reflection symmetries and six rotation symmetries. To find the degrees for the rotation symmetries, divide 360° by the number of points of the design: $\frac{360°}{6} = 60°$. Thus, the six rotation symmetries are 60°, 120°, 180°, 240°, 300°, and 360°.

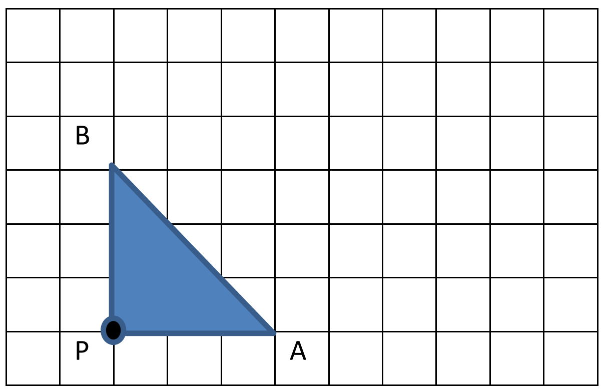
b. The letter K has symmetry type Z_1, no reflection symmetries and one rotation symmetry (360°).

Section 10.3: Transformations that Change Size and Similar Figures

This section covers transformations that either enlarge or shrink an object from a point, P. The point P is called the **center of the size transformation**. The multiplier used to enlarge or shrink the object is called the **scale factor**, *k*. The calculations used to make the transformation depend on the distances from point P to the vertices of the object. These distances are multiplied by *k*. So, to enlarge or shrink an object, find the distance from the point P to a vertex A of the object. Multiply this distance by *k* to get *k*PA, where PA represents the distance from P to A. Then, measure this new distance, *k*PA, from point P in the direction of vertex A. This distance gives the new location of vertex A after the object has been size-transformed. Repeat for all vertices of the object.

Example 10.3.1: Enlarge a Triangle by a Factor of Two

Figure 10.3.1: Triangle to be Size-Transformed by a Factor of Two



Step 1: Measure the distances from point P to each vertex of the triangle. One vertex of the triangle is on point P, so that vertex will remain at point P.

The distance from point P to vertex A is three units.

The distance from point P to vertex B is also three units.

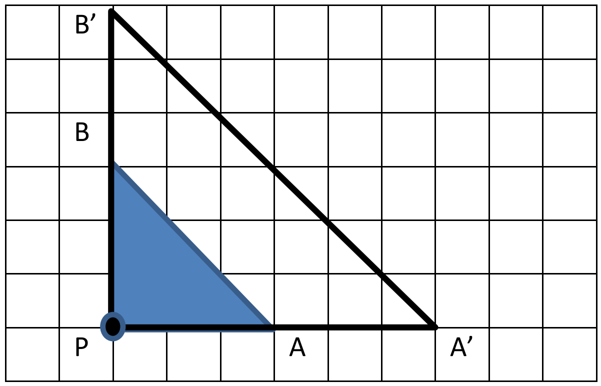
Step 2: Multiply these distance by the scale factor two.

2PA = 2(3) = 6

2PB = 2(3) = 6

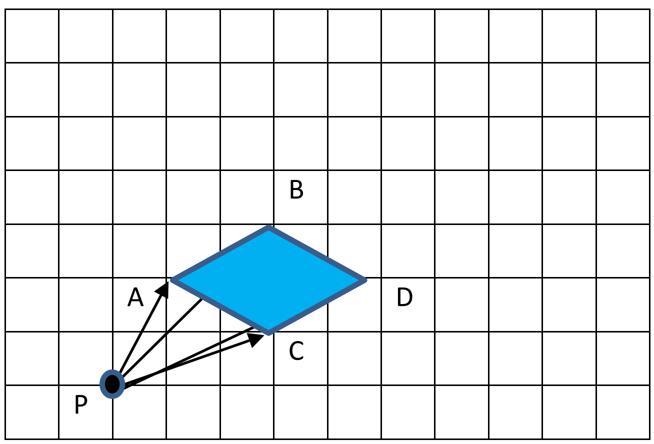
Step 3: Measure six units from point P in the direction of vertex A and measure six units from point P in the direction of vertex B. The new locations of A and B are each six units from P in their corresponding directions as shown below.

Figure 10.3.2: Triangle Enlarged by a Factor of Two



Example 10.3.2: Enlarge a Diamond by a Factor of Two

Figure 10.3.3: Diamond to be Size-Transformed by a Factor of Two



Step 1: Measure the distances from point P to each vertex of the diamond.

The distance from point P to vertex A is PA

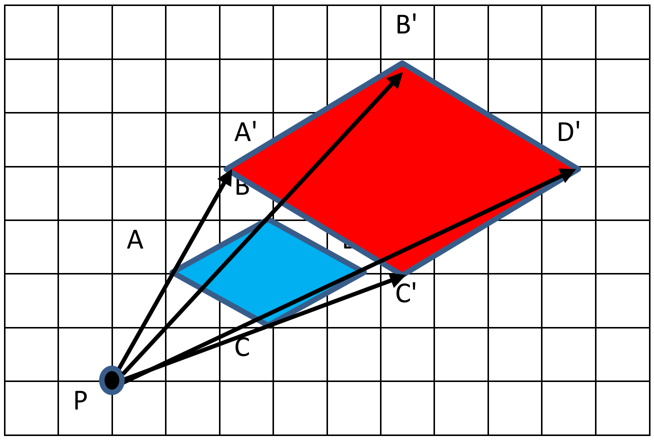
Likewise, the distances from point P to the other three vertices are PB, PC, and PD, respectively.

Step 2: Multiply these distance by the scale factor two.

The distances of the new points from P are: 2PA, 2PB, 2PC, and 2PD.

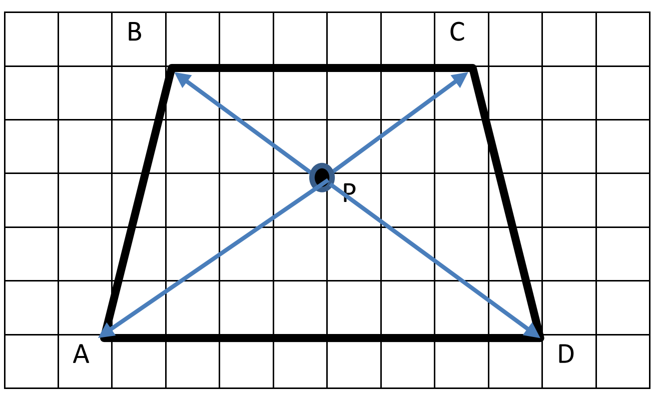
Step 3: Measure these distances from point P in the direction of each vertex A, B, C, and D as shown below.

Figure 10.3.4: Diamond Enlarged by a Factor of Two



Example 10.3.3: Shrink a Trapezoid by a Factor of $\frac{1}{2}$

Figure 10.3.5: Trapezoid to be Size-Transformed by a Factor of $\frac{1}{2}$



Step 1: Measure the distances from point P to each vertex of the trapezoid.

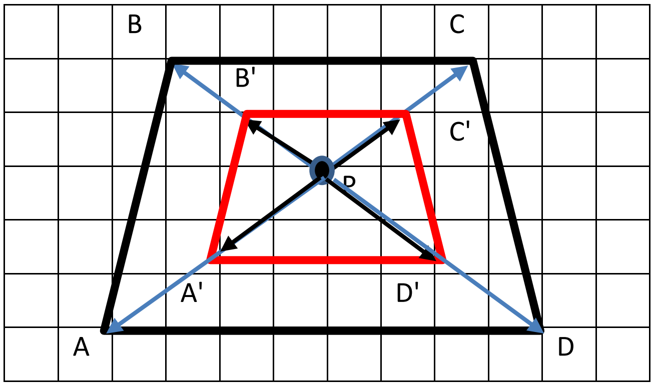
The distances from point P to the vertices are PA, PB, PC, and PD, respectively.

Step 2: Multiply these distances by the scale factor $\frac{1}{2}$.

The distances of the new points from P are: $\frac{1}{2}$PA, $\frac{1}{2}$PB, $\frac{1}{2}$PC, and $\frac{1}{2}$PD.

Step 3: Measure these distances from point P in the direction of each vertex A, B, C, and D as shown below.

Figure 10.3.6: Trapezoid Shrunk by a Factor of $\frac{1}{2}$



Shapes that have been transformed by a enlarging or shrinking are similar figures to the original shape.

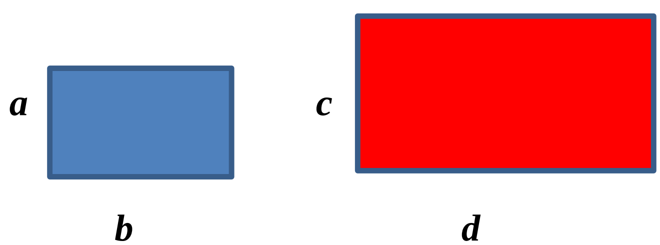
Similarity Using Transformations

Two figures are similar if and only if there exists a combination of an isometry (rigid motion) and a size transformation that generates one figure as the image of the other.

Similar figures are figures that have the same shape but not necessarily the same size. Side lengths and interior angles of similar figures are proportional to each other.

Figure 10.3.7: Similar Figures

The two rectangles below are similar if the sides are proportional to each other. In other words, they are similar if $\frac{a}{b}=\frac{c}{d}$.



The two rectangles are related by the scale factor *k*. Therefore, the sides of the rectangles are related to each other by: $c = ka$ and $d = kb$.

Let P = perimeter of the smaller rectangle and P' = perimeter of the larger rectangle.

$$P' = c + c + d + d$$

$P' = 2c + 2d$, but remember that $c = ka$ and $d = kb$, so

$P' = 2ka + 2kb$, now factor out the k to get

$P' = k(2a + 2b)$, and also, $P = 2a + 2b$, so

$P' = kP$

If P represents the perimeter of the smaller rectangle and P' represents the perimeter of the larger rectangle, then the two **perimeters are related by** $P' = kP$.

Let A = area of the smaller rectangle and A' = the area of the larger rectangle.

$A' = c \bullet d$, but remember that $c = ka$ and $d = kb$, so

$A' = ka \bullet kb$, rearrange to get

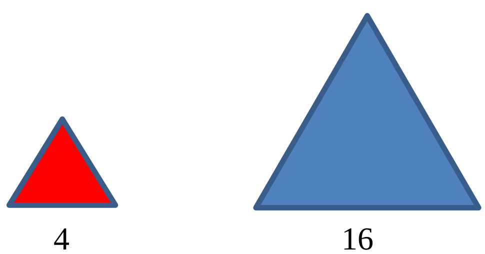
$A' = k^2 a \bullet b$, and also, $A = a \bullet b$, so

$A' = k^2 A$

If A represents the area of the smaller rectangle and A' represents the area of the larger rectangle, then the two **areas are related by** $A' = k^2 A$.

Example 10.3.4: Areas and Perimeters of Similar Triangles

Figure 10.3.8: The Triangles in this Figure are Similar Triangles



a. Determine the perimeter P' of the larger triangle if the perimeter P of the smaller triangle is 12 *mm*.

b. Determine the area A' of the larger triangle if the area A of the smaller triangle is 6.9 mm^2

First, find the scale factor using the fact that similar triangles have sides that are proportional to each other: $k = \frac{16}{4} = 4$.

a. $P' = kP$

$P' = 4 \cdot 12 = 48$

$P' = 48\ mm$

b. $A' = k^2 A$

$A' = 4^2 \cdot 6.9 = 16 \cdot 6.9 = 110.4$

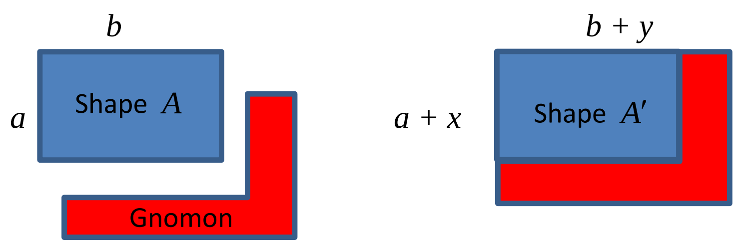
$A' = 110.4\ mm^2$

Gnomons

A **gnomon** is a shape which, when added to a shape A, yields another shape A' that is similar to the original shape A. Alternately, a **gnomon** is also the shape which, when subtracted from a shape A', yields another shape A which is similar to the original shape A'.

Figure 10.3.9: What is a Gnomon?

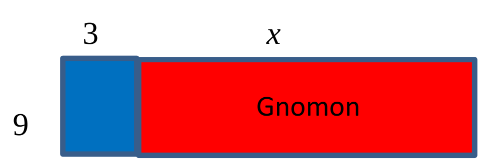
The blue rectangle shown below is the original shape A. When the red L-shape is attached to the blue rectangle, a new rectangle is formed, called shape A'. If the red L-shape is a gnomon to shape A, then the blue rectangle (shape A) is similar to the blue and red rectangle (shape A'). Also, since the rectangles are similar, the side lengths are proportional: $\frac{a}{b} = \frac{a+x}{b+y}$



Example 10.3.5: Rectangular Gnomon

Find the value of x so that the red larger rectangle on the right is a gnomon to the blue smaller rectangle on the left.

Figure 10.3.10: Rectangular Gnomon



To calculate this, set up a proportion so that the sides of the small blue rectangle on the left are proportional to the sides of the blue and red rectangles (left and right) combined. To set up the proportion, make ratios of the width over the length of the small blue rectangle on the left and the width over the length of the combined rectangles.

$$\frac{\text{width}}{\text{length}}$$

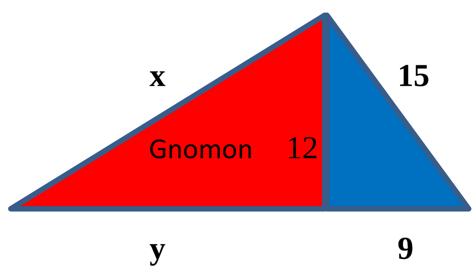
$$\frac{3}{9} = \frac{9}{3+x}$$
$$3(3+x) = 81$$
$$9+3x = 81$$
$$3x = 72$$
$$x = 24$$

If $x = 24$, then the red larger rectangle is a gnomon to the blue smaller rectangle.

Example 10.3.6: Triangular Gnomon

Find the values of *x* and y so that the larger red triangle on the left is a gnomon to the smaller blue triangle on the right.

Figure 10.3.11: Triangular Gnomon



To calculate this, set up proportions, one with x and one with y, so that the sides of the smaller blue triangle on the right are proportional to the sides of the blue and red triangles combined. To set up the proportion for x, make ratios of the longer leg over the shorter leg of the smaller blue triangle on the right and the longer leg over the shorter leg of the combined triangles.

$$\frac{9}{12} = \frac{15}{x}$$
$$9x = 180$$
$$x = 20$$

Now to set up the proportion for y, make ratios of the shorter leg over the hypotenuse of the smaller blue triangle on the right and the shorter leg over the hypotenuse of the combined triangles.

$$\frac{9}{15} = \frac{15}{y+9}$$
$$9(\text{y}+9) = 225$$
$$9y + 81 = 225$$
$$9y = 144$$
$$y = 16$$

If $x = 20$ and $y = 16$, then the red larger triangle on the left is a gnomon to the blue smaller triangle on the right.

Section 10.4: Fibonacci Numbers and the Golden Ratio

A famous and important sequence is the Fibonacci sequence, named after the Italian mathematician known as Leonardo Pisano, whose nickname was Fibonacci, and who lived from 1170 to 1230. This sequence is:

$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55, \ldots\ldots..\}$$

This sequence is defined **recursively.** This means each term is defined by the previous terms.

Given $f_1 = 1, f_2 = 1,$ *then*

$$f_3 = f_2 + f_1 = 1+1 = 2,$$
$$f_4 = f_3 + f_2 = 2+1 = 3,$$
$$f_5 = f_4 + f_3 = 3+2 = 5$$

and so on.

The **Fibonacci sequence** is defined by $f_n = f_{n-1} + f_{n-2}$, for all $n \ge 3$, when $f_1 = 1$ and $f_2 = 1$.

In other words, to get the next term in the sequence, add the two previous terms.

$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 55+34 = 89,\ 89+55 = 144, \cdots\}$$

The notation that we will use to represent the Fibonacci sequence is as follows:

$f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, $f_7 = 13$, $f_8 = 21$, $f_9 = 34$, $f_{10} = 55$, $f_{11} = 89$, $f_{12} = 144$, $\ldots$

Example 10.4.1: Finding Fibonacci Numbers Recursively

Find the 13th, 14th, and 15th Fibonacci numbers using the above recursive definition for the Fibonacci sequence.

First, notice that there are already 12 Fibonacci numbers listed above, so to find the next three Fibonacci numbers, we simply add the two previous terms to get the next term as the definition states.

$$f_{13} = f_{12} + f_{11} = 144 + 89 = 233$$

$$f_{14} = f_{13} + f_{12} = 233 + 144 = 377$$

$$f_{15} = f_{14} + f_{13} = 377 + 233 = 610$$

Therefore, the 13th, 14th, and 15th Fibonacci numbers are 233, 377, and 610 respectively.

Calculating terms of the Fibonacci sequence can be tedious when using the recursive formula, especially when finding terms with a large n. Luckily, a mathematician named Leonhard Euler discovered a formula for calculating any Fibonacci number. This formula was lost for about 100 years and was rediscovered by another mathematician named Jacques Binet. The original formula, known as Binet's formula, is below.

Binet's Formula: The nth Fibonacci number is given by the following formula:

$$f_n = \frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}{\sqrt{5}}$$

Binet's formula is an example of an **explicitly** defined sequence. This means that terms of the sequence are not dependent on previous terms.

A somewhat more user-friendly, simplified version of Binet's formula is sometimes used instead of the one above.

Binet's Simplified Formula: The nth Fibonacci number is given by the following formula:

$$f_n = \left[\!\!\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} \right]\!\!\right]$$

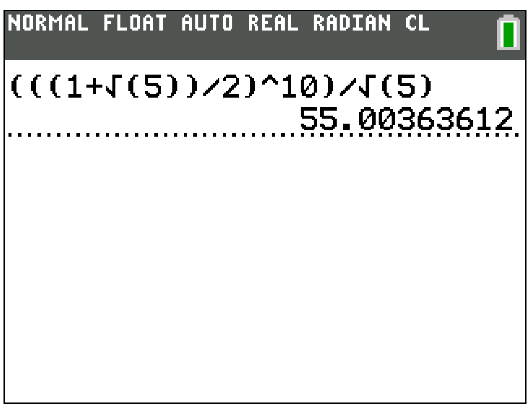
Note: The symbol $[\![\;\;]\!]$ *means "round to the nearest integer."*

Example 10.4.2: Finding f_{10} Explicitly

Find the value of f_{10} using Binet's simplified formula.

$$f_{10} = \left[\!\!\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{10}}{\sqrt{5}} \right]\!\!\right] = \left[\!\!\left[\frac{1.618033989^{10}}{\sqrt{5}} \right]\!\!\right] = [\![55.003636]\!] = 55$$

Figure 10.4.1: Calculator Work for f_{10}

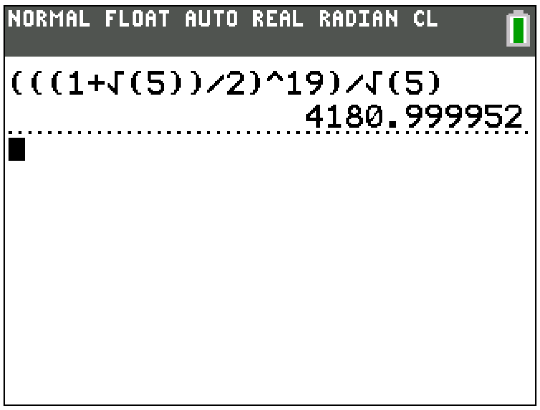


Example 10.4.3: Finding f_{19} Explicitly

Find the value of f_{19} using Binet's simplified formula.

$$f_{19} = \left[\!\!\left[\frac{\left[\left(\frac{1+\sqrt{5}}{2} \right)^{19} \right]}{\sqrt{5}} \right]\!\!\right] = \left[\!\!\left[\frac{1.618033989^{19}}{\sqrt{5}} \right]\!\!\right] = [\![4180.999952]\!] = 4181$$

Figure 10.4.2: Calculator Work for f_{19}

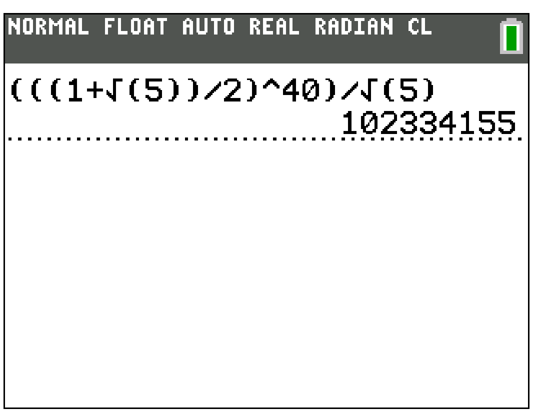


Example 10.4.4: Finding f_{40} Explicitly

Find the value of f_{40} using Binet's simplified formula.

$$f_{40} = \left[\!\!\left[\frac{\left[\left(\frac{1+\sqrt{5}}{2} \right)^{40} \right]}{\sqrt{5}} \right]\!\!\right] = \left[\!\!\left[\frac{1.618033989^{40}}{\sqrt{5}} \right]\!\!\right] = 102,334,155$$

Figure 10.4.3: Calculator Work for f_{40}



All around us we can find the Fibonacci numbers in nature. The number of branches on some trees or the number of petals of some daisies are often Fibonacci numbers

Figure 10.4.4: Fibonacci Numbers and Daisies

a. Daisy with 13 petals

b. Daisy with 21 petals

a. 

b. 

(Daisies, n.d.)

Fibonacci numbers also appear in spiral growth patterns such as the number of spirals on a cactus or in sunflowers seed beds.

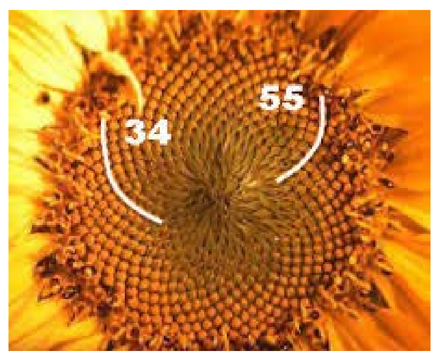
Figure 10.4.5: Fibonacci Numbers and Spiral Growth

a. Cactus with 13 clockwise spirals

b. Sunflower with 34 clockwise spirals and 55 counterclockwise spirals

a. 

(Cactus, n.d.)

b. 

(Sunflower, n.d.)

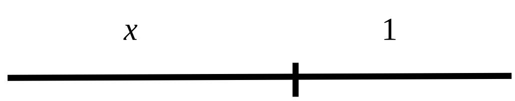
Another interesting fact arises when looking at the ratios of consecutive Fibonacci numbers.

$\frac{1}{1}=1$, $\frac{2}{1}=2$, $\frac{3}{2}=1.5$, $\frac{5}{3}=1.\overline{66}$, $\frac{8}{5}=1.6$, $\frac{13}{8}=1.625$, $\frac{21}{13}=1.615385$,
$\frac{34}{21}=1.619048$, $\frac{55}{34}=1.617647$, $\frac{89}{55}=1.618182$, $\frac{144}{89}=1.617978$, $\frac{233}{144}=1.618056$,
$\frac{377}{233}=1.618026$, $\frac{610}{377}=1.618037$, $\frac{987}{610}=1.618033$, …

It appears that these ratios are approaching a number. The number that these ratios are getting closer to is a special number called the Golden Ratio which is denoted by ϕ (the Greek letter phi). You have seen this number in Binet's formula.

The Golden Ratio: $\phi=\frac{1+\sqrt{5}}{2}$
The Golden Ratio has the decimal approximation of $\phi=1.6180339887$.

The Golden Ratio is a special number for a variety of reasons. It is also called the divine proportion and it appears in art and architecture. It is claimed by some to be the most pleasing ratio to the eye. To find this ratio, the Greeks cut a length into two parts, and let the smaller piece equal one unit. The most pleasing cut is when the ratio of the whole length $x+1$ to the long piece x is the same as the ratio of the long piece x to the short piece 1.



$\frac{x+1}{x}=\frac{x}{1}$ cross-multiply to get

$x+1=x^2$ rearrange to get

$x^2-x-1=0$ solve this quadratic equation using the quadratic formula.

$a=1$, $b=-1$, and $c=-1$

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{5}}{2}$$

$$x=\frac{1+\sqrt{5}}{2} \text{ and } x=\frac{1-\sqrt{5}}{2}$$

The Golden Ratio is a solution to the quadratic equation $x^2 = x+1$ meaning it has the property $\phi^2 = \phi+1$. This means that if you want to square the Golden Ratio, just add one to it. To check this, just plug in ϕ.

$$\phi^2 = \phi+1$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+\sqrt{5}}{2}+1$$

$$(1.618)^2 = 1.618+1$$

$$2.618 = 2.618$$

It worked!

Another interesting relationship between the Golden Ratio and the Fibonacci sequence occurs when taking powers of ϕ.

$$\phi^3 = \phi\cdot\phi^2 = \phi(\phi+1) = \phi^2+\phi = (\phi+1)+\phi = 2\phi+1$$
$$\phi^3 = 2\phi+1$$
$$\phi^4 = \phi\cdot\phi^3 = \phi(2\phi+1) = 2\phi^2+\phi = 2(\phi+1)+\phi = 3\phi+2$$
$$\phi^4 = 3\phi+2$$
$$\phi^5 = \phi\cdot\phi^4 = \phi(3\phi+2) = 3\phi^2+2\phi = 3(\phi+1)+2\phi = 5\phi+3$$
$$\phi^5 = 5\phi+3$$
$$\phi^6 = \phi\cdot\phi^5 = \phi(5\phi+3) = 5\phi^2+3\phi = 5(\phi+1)+3\phi = 8\phi+5$$
$$\phi^6 = 8\phi+5$$

And so on.

Notice that the coefficients of ϕ and the numbers added to the ϕ term are Fibonacci numbers. This can be generalized to a formula known as the Golden Power Rule.

Golden Power Rule: $\phi^n = f_n\phi + f_{n-1}$
where f_n is the nth Fibonacci number and ϕ is the Golden Ratio.

Example 10.4.5: Powers of the Golden Ratio

Find the following using the golden power rule: a. ϕ^8 and b. ϕ^{15}

a. $\phi^n = f_n\phi + f_{n-1}$

$\phi^8 = f_8\phi + f_{8-1}$

$\phi^8 = f_8\phi + f_7$

$\phi^8 = 21(1.618) + 13 = 46.978$

b. $\phi^n = f_n\phi + f_{n-1}$

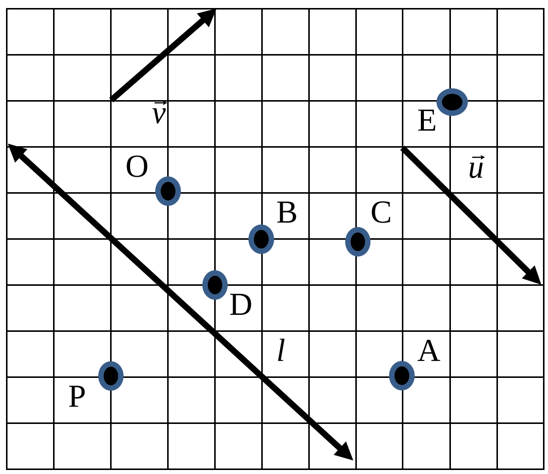
$\phi^{15} = f_{15}\phi + f_{15-1}$

$\phi^{15} = f_{15}\phi + f_{14}$

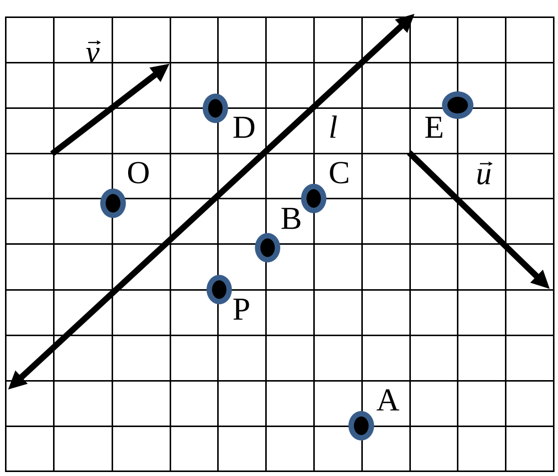
$\phi^{15} = 610(1.618) + 377 = 1363.98$

Chapter 10 Homework

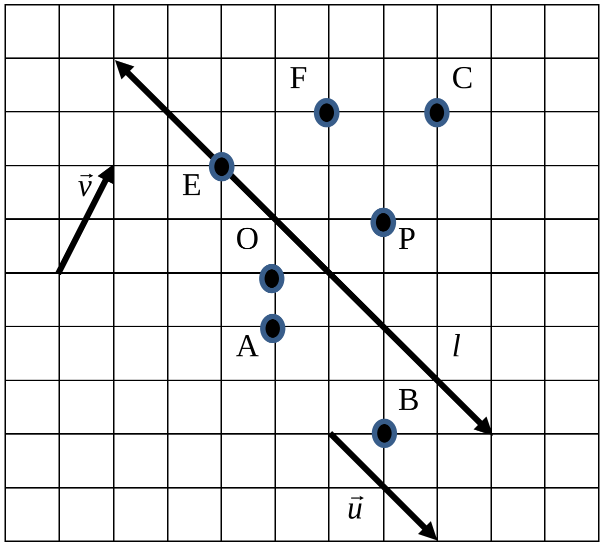
1. Identify the image of point P under the following transformations.
    a. a translation along vector $\vec{v}$
    b. a reflection across line l
    c. a counterclockwise rotation of $90°$ about point O
    d. A glide-reflection across l and along $\vec{u}$



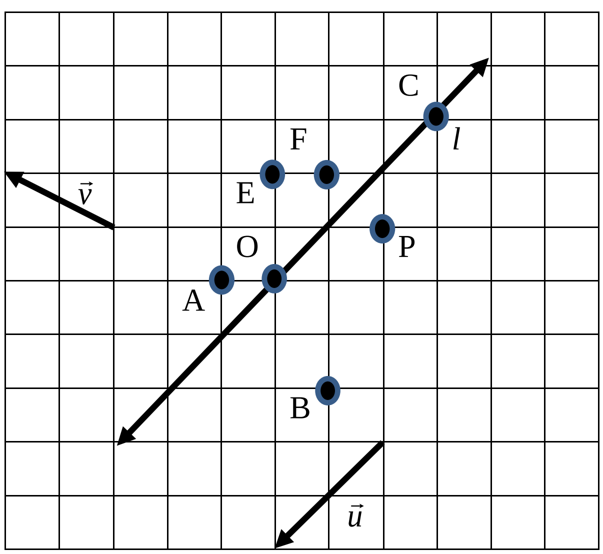
2. Identify the image of point P under the following transformations.
    a. a translation along vector $\vec{u}$
    b. a reflection across line l
    c. a counterclockwise rotation of $90°$ about point O
    d. A glide-reflection across l and along $\vec{v}$



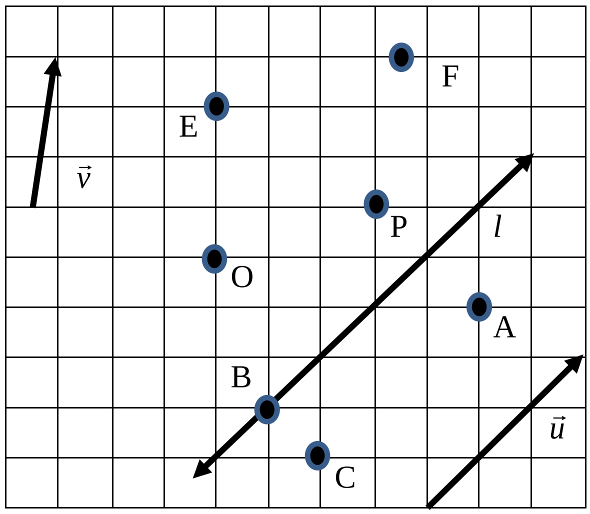
3. Identify the image of point P under the following transformations.
   a. a translation along vector $\vec{v}$
   b. a reflection across line l
   c. a counterclockwise rotation of $90°$ about point O
   d. A glide-reflection across l and along $\vec{u}$



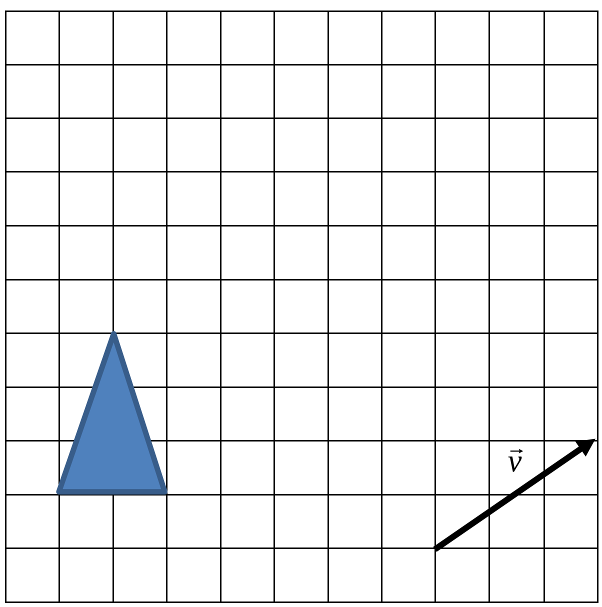
4. Identify the image of point P under the following transformations.
   a. a translation along vector $\vec{v}$
   b. a reflection across line l
   c. a clockwise rotation of $90°$ about point O
   d. A glide-reflection across l and along $\vec{u}$



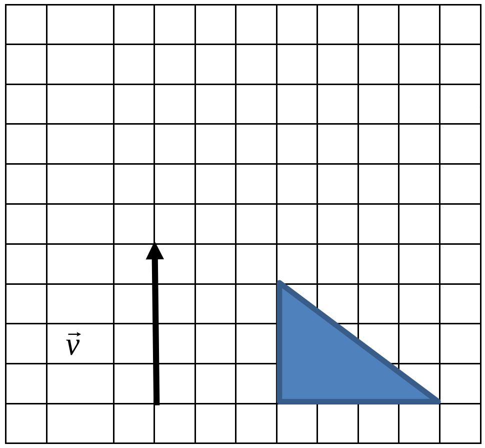
5. Identify the image of point P under the following transformations.
   a. a translation along vector $\vec{v}$
   b. a reflection across line l
   c. a clockwise rotation of $90°$ about point O
   d. A glide-reflection across l and along $-\vec{u}$



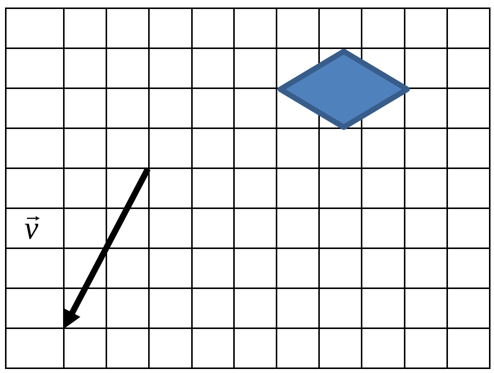
6. Translate the figure along vector $\vec{v}$.



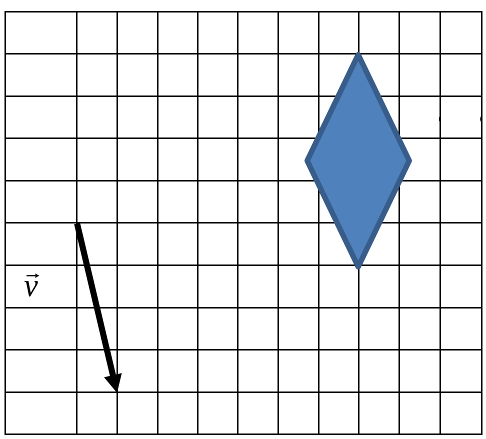
7. Translate the figure along vector $\vec{v}$.



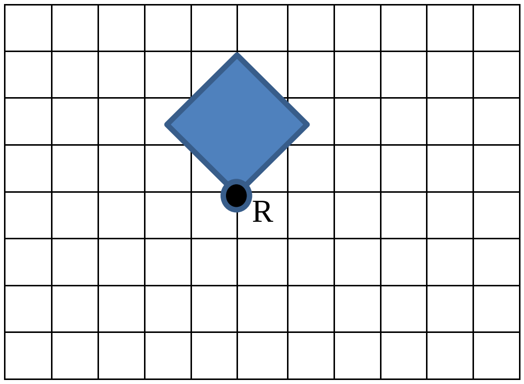
8. Translate the figure along vector $\vec{v}$.



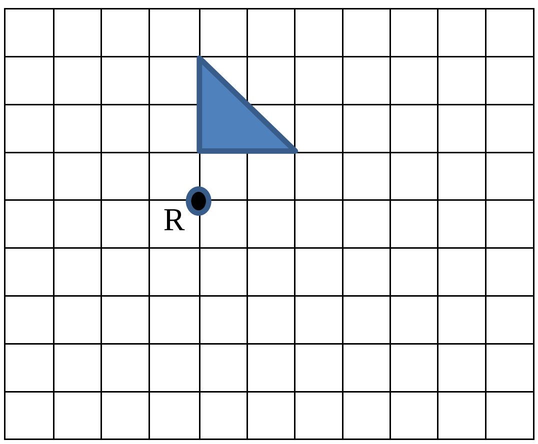
9. Translate the figure along vector $\vec{v}$.



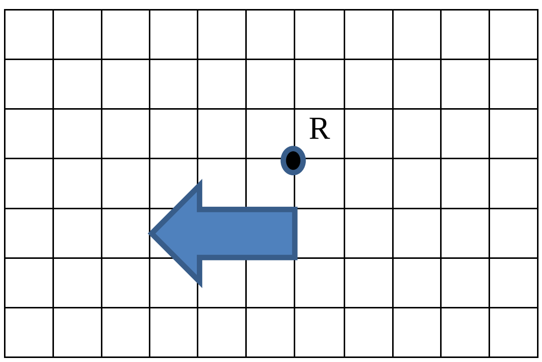
10. Rotate the figure 90° clockwise about the rotocenter R.



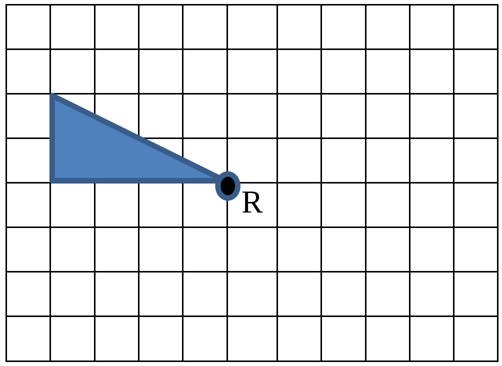
11. Rotate the figure 90° clockwise about the rotocenter R.



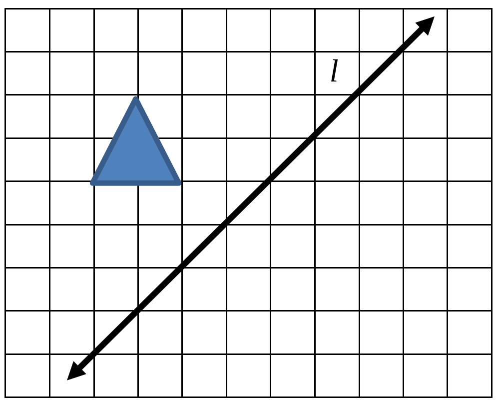
12. Rotate the figure 180° clockwise about the rotocenter R.



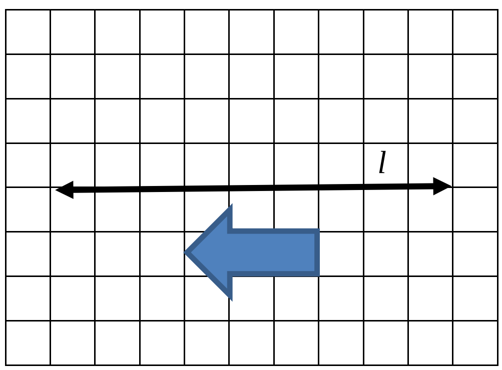
13. Rotate the figure 180° clockwise about the rotocenter R.



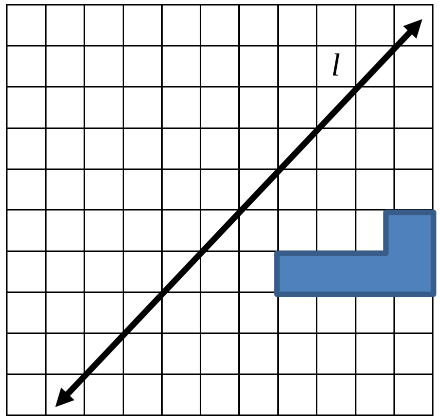
14. Reflect the figure over the line *l*.



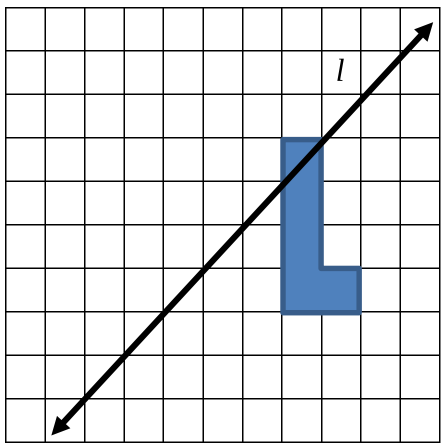
15. Reflect the figure over the line *l*.



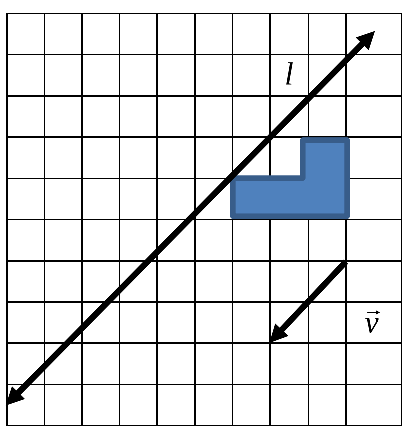
16. Reflect the figure over the line *l*.



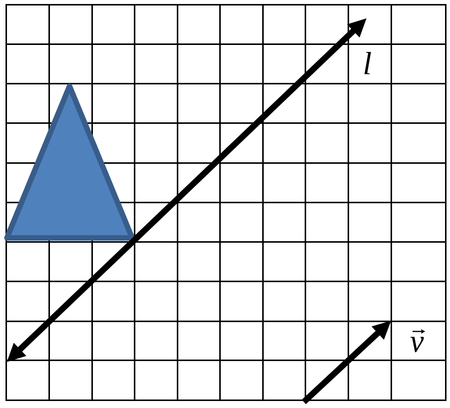
17. Reflect the figure over the line *l*.



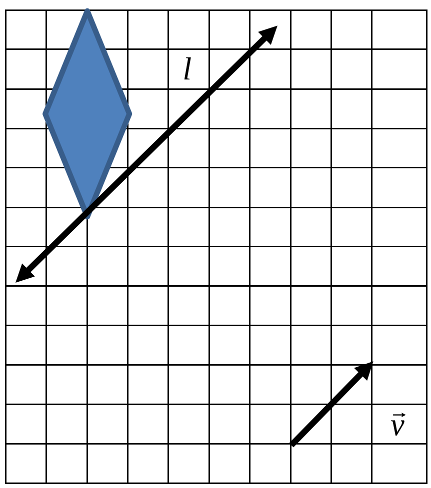
18. Glide-reflect the figure over the line *l* and along the vector $\vec{v}$.



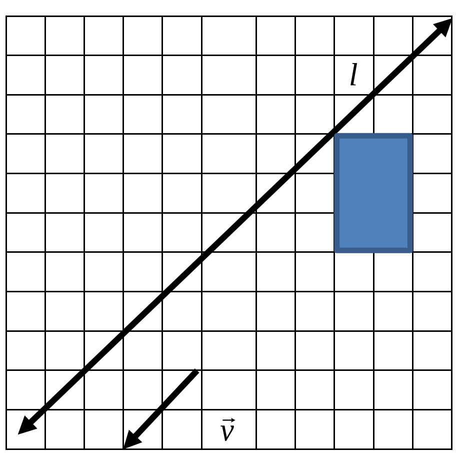
19. Glide-reflect the figure over the line l and along the vector $\vec{v}$.



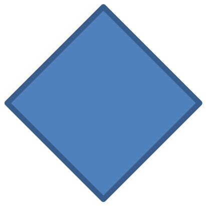
20. Glide-reflect the figure over the line l and along the vector $\vec{v}$.



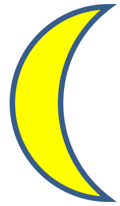
21. Glide-reflect the figure over the line l and along the vector $\vec{v}$.



22. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

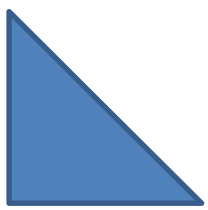


a.

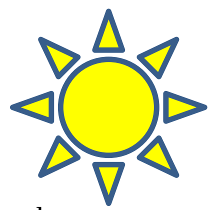


b.

23. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

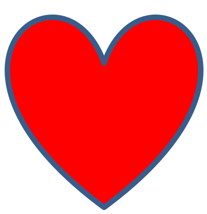


a.

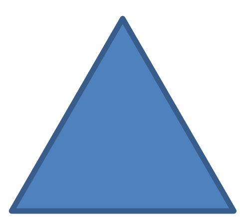


b.

24. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.



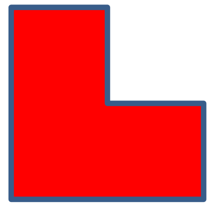
a.



b.

25. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

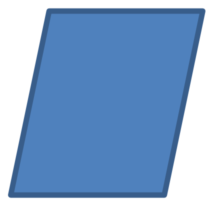
a.



b.

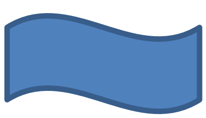
26. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

a.

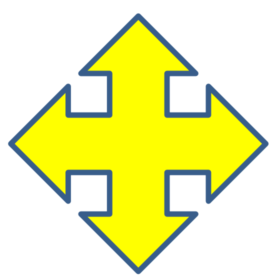


b.

27. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

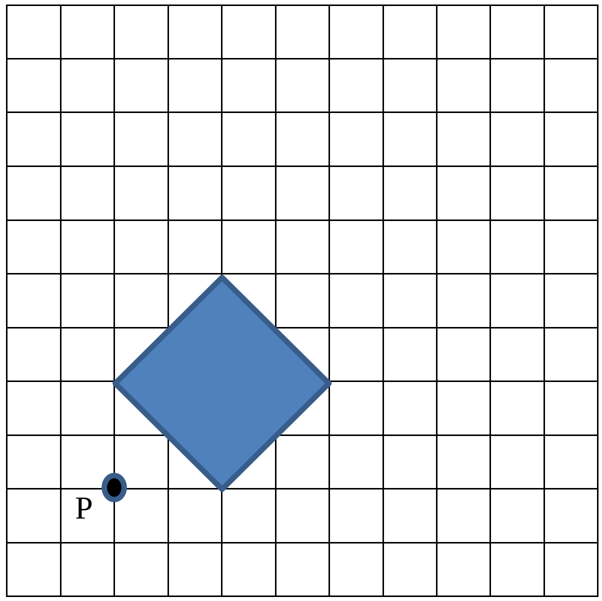


a.

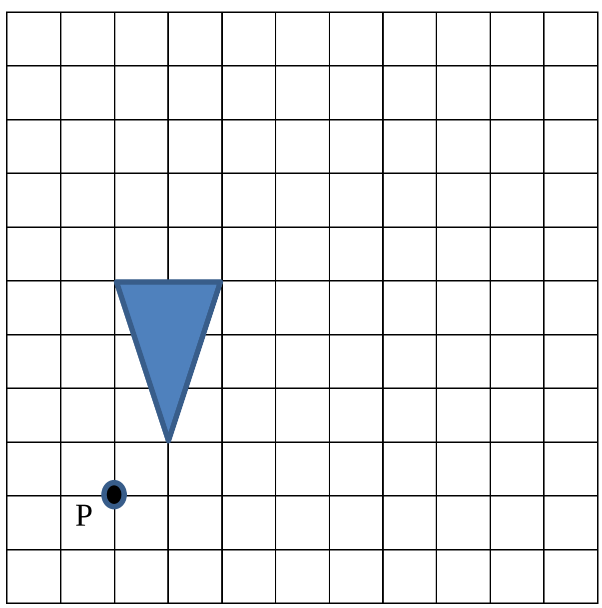


b.

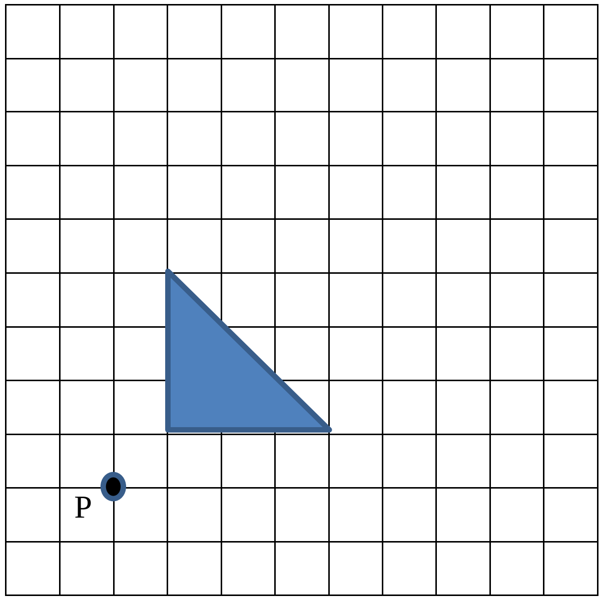
28. Enlarge the figure with respect to the point P by a factor of 2.



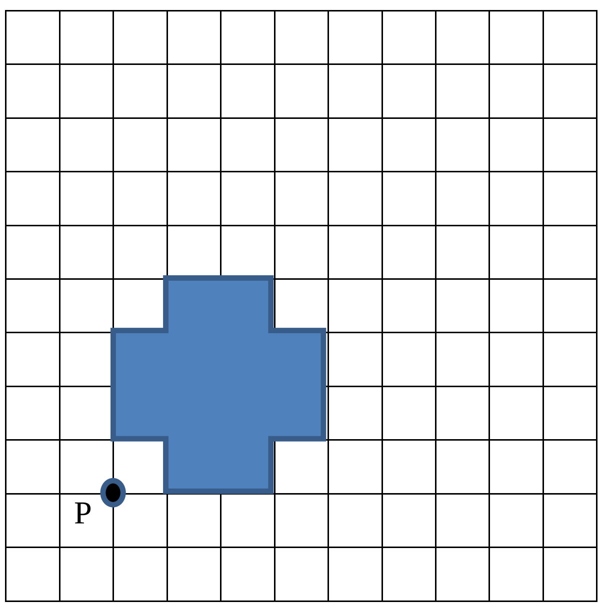
29. Enlarge the figure with respect to the point P by a factor of 2.



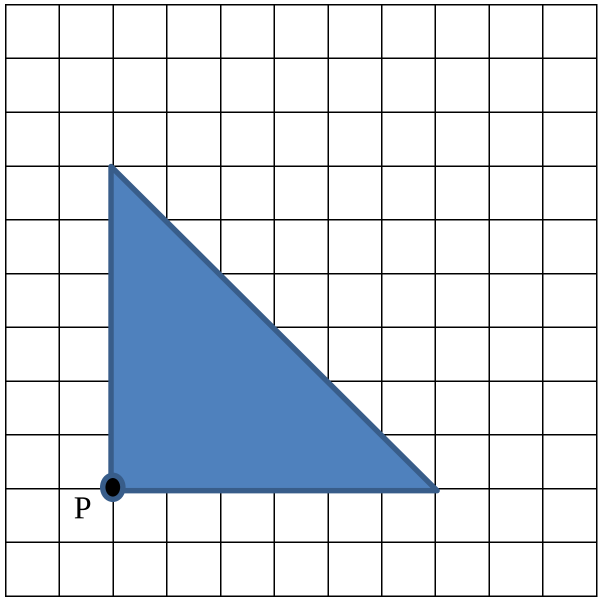
30. Enlarge the figure with respect to the point P by a factor of 2.



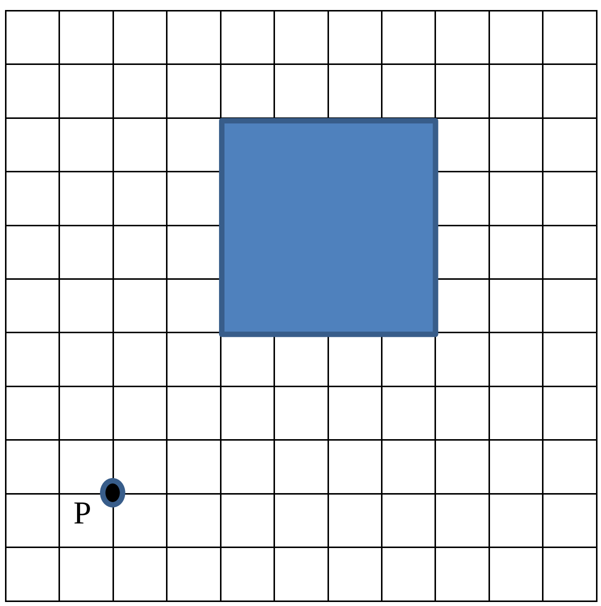
31. Enlarge the figure with respect to the point P by a factor of 2.



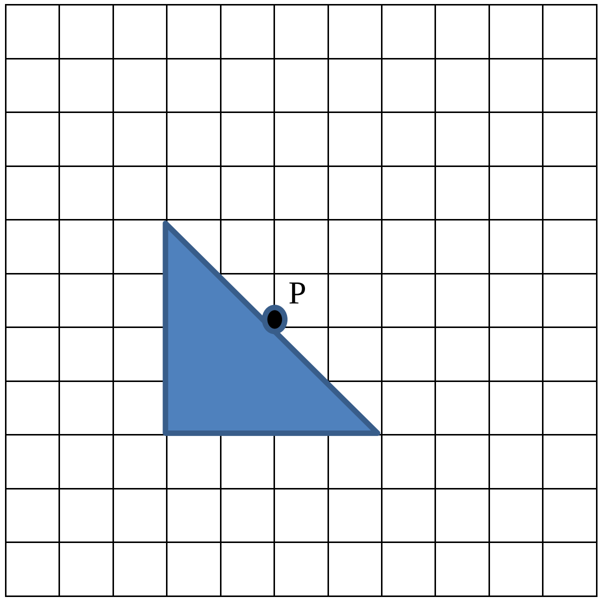
32. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



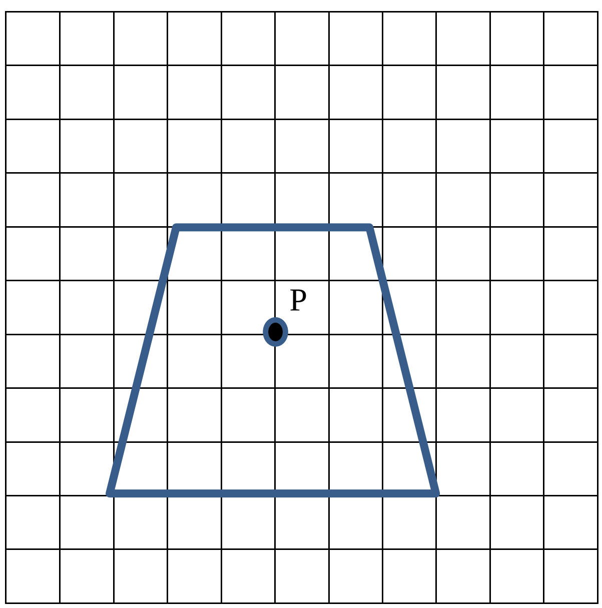
33. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



34. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



35. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



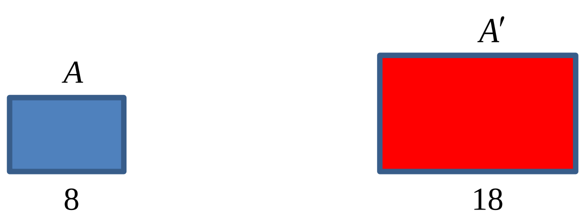
36. Triangles A and A' are similar and are related by the scale factor of 3.
    a. If the perimeter of triangle A is 12 *ft*, find the perimeter of A'.
    b. If the area of triangle A is 8 ft^2, find the area of A'.

37. Triangles A and A' are similar and are related by the scale factor of 4.
    a. If the perimeter of triangle A is 48 *ft*, find the perimeter of A'.
    b. If the area of triangle A is 140 ft^2, find the area of A'.

38. Triangles A and A' are similar and are related by the scale factor of 5.
    a. If the perimeter of triangle A is 42 *ft*, find the perimeter of A'.
    b. If the area of triangle A is 68 ft^2, find the area of A'.

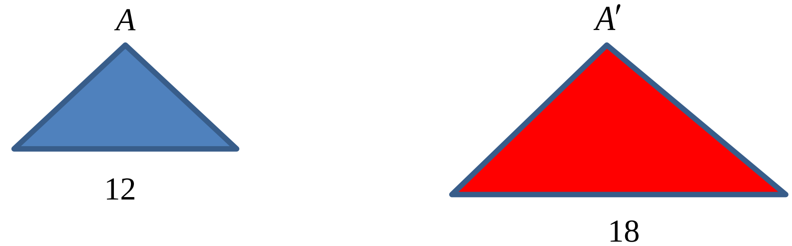
39. Triangles A and A' are similar and are related by the scale factor of 2.
    a. If the perimeter of triangle A is 42 *ft*, find the perimeter of A'.
    b. If the area of triangle A is 80 ft^2, find the area of A'.

40. The shapes A and A' are similar.



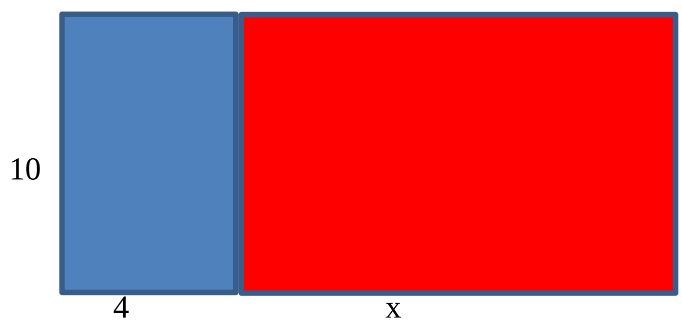
    a. If the perimeter of rectangle A is 22 *ft*, find the perimeter of A'.
    b. If the area of rectangle A is 24 ft^2, find the area of A'.

41. The shapes A and A' are similar.

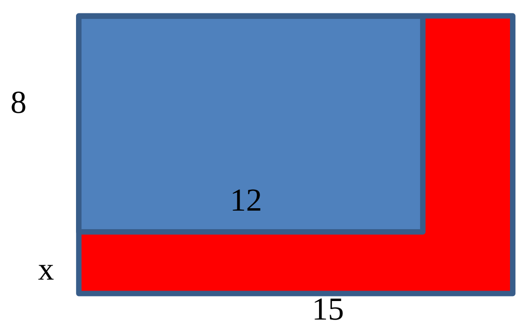


a. If the perimeter of triangle A is 32 *ft*, find the perimeter of A'.
b. If the area of triangle A is 48 ft^2, find the area of A'.

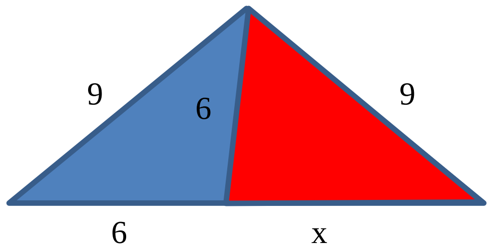
42. Find the value of x so that the red larger rectangle on the right is a gnomon to the blue smaller rectangle on the left.



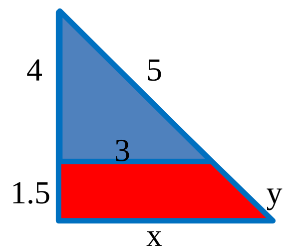
43. Find the value of x so that the red L-shape is a gnomon to the blue rectangle.



44. Find the value of x so that the red triangle on the right is a gnomon to the blue triangle on the left.



45. Find the values of x and y so that the red trapezoid is a gnomon to the blue triangle.



46. Compute the values of the following.
    a. f_{12}
    b. $f_4 + f_3 + f_2$
    c. f_{4+3+2}

47. Compute the values of the following.
    a. f_{13}
    b. $f_5 + f_3 + f_1$
    c. f_{5+3+1}

48. Compute the values of the following using Binet's simplified formula.
    a. f_{54}
    b. f_{65}
    c. f_{8+7+6}

49. Compute the values of the following using Binet's simplified formula.
    a. f_{35}
    b. f_{45}
    c. f_{9+8+7}

50. Given $f_{33} = 3{,}524{,}578$ and $f_{34} = 5{,}702{,}887$, find f_{32} and f_{35}.

51. Given $f_{40} = 102{,}334{,}155$ and $f_{41} = 165{,}580{,}141$, find f_{39} and f_{42}.

52. Solve the quadratic equation $x^2 - x - 3 = 0$.

53. Solve the quadratic equation $x^2 - 3x - 3 = 0$.

54. Solve the quadratic equation $x^2 - 5x - 3 = 0$.

55. Solve the quadratic equation $x^2 - 4x - 7 = 0$.

56. Solve the quadratic equation $x^2 - x - 7 = 0$.

57. Solve the quadratic equation $x^2 - x - 5 = 0$.

Works Cited

Arizona Gas Prices. (n.d.). *Lowest regular gas prices in the last 36 hours*. Retrieved July 16, 2013 from http://www.arizonagasprices.com/GasPriceSearch.aspx

Ask.com. (2014, April 18). Retrieved from http://www.ask.com/

Bloomberg Businessweek. (n.d.). *World's fastest-shrinking countries*. Retrieved March 15, 2014 from http://images.businessweek.com/ss/10/08/0813_fastest_shrinking_countries/2.htm

Case study I: The 1936 literary digest poll. (n.d.). Retrieved August 10, 2014, from http://www.math.upenn.edu/~deturck/m170/wk4/lecture/case1.html

Centers for Disease Control and Prevention. (2012, March 30). Prevalence of autism spectrum disorders—autism and developmental disabilities monitoring network, 14 sites, United States, 2008. *Morbidity and Mortality Weekly Report, Vol. 61 No. 3*. Retrieved from http://i2.cdn.turner.com/cnn/2012/images/03/29/ss6103.ebook.pdf

College Board. (2012, September 24). *State profile report, Arizona*. Retrieved from http://media.collegeboard.com/digitalServices/pdf/research/AZ_12_03_03_01.pdf

College Board. (2012, September 24). *State profile report, Michigan*. Retrieved from http://media.collegeboard.com/digitalServices/pdf/research/MI_12_03_03_01.pdf

Eurostat. (n.d.). *Harmonised unemployment rate by sex.* Retrieved May 21, 2013 from http://epp.eurostat.ec.europa.eu/tgm/table.do?tab=table&language=en&pcode=teilm020&tableSelection=1&plugin=1

Expedia. (2013, July 17). Retrieved from http://www.expedia.com/

Federal Reserve Bank of St. Louis. (2013). *Average hourly earnings of all employees: Professional and business services* (CEU6000000003). Retrieved from http://research.stlouisfed.org/fred2/series/CEU6000000003/downloaddata?cid=32321

Four-Step Budget Template (n.d.). Retrieved June 10, 2014, from https://drive.google.com/previewtemplate?id=0Aqko7Xi-nxN1dElRZ3RiUzJRY05fcngxaXRua3NEb0E&mode=public

Fuscaldo, D. (n.d.). *Car depreciation: 5 models that lose value.* Retrieved November 15, 2014,

Gentile, E., & Imberman, S. (2009, March 4). *Dressed for success: Do school uniforms improve student behavior, attendance, and achievement?*. Retrieved from http://www.uh.edu/econpapers/RePEc/hou/wpaper/2009-03.pdf

Kenya Open Data. (2013, July 15). *Health facility pie chart*.

Kiersz, A. (2015, March 27). *Here are the fastest growing and fastest shrinking counties in the US.* Retrieved from http://www.businessinsider.com/us-census-county-population-change-map-2015-3

McKenna, M. (2014, September 15). *Ebola mathematics stark warning of disease's spread.* Retrieved from http://www.wired.co.uk/news/archive/2014-09/15/ebola-epidemiology/viewgallery/338422

Motor Trend. (n.d.). *New sedans over 40 mpg.* Retrieved July 17, 2013 from http://www.motortrend.com/3_5_9_1_1/new_sedans_over_40_mpg.html

National Weather Service. (2013, July 16). *Graphical forecasts—conus area*. Retrieved from http://graphical.weather.gov/sectors/conus.php?element=T

Smith, A. (2011, July 11). *Platform differences in smartphone adoption*. Retrieved from http://www.pewinternet.org/Reports/2011/Smartphones/Section-3/Platform-differences-in-smartphone-adoption.aspx

Sustainability Victoria, (2009). *2001-02 to 2007-08 local government survey (Victoria)*. Retrieved from website: http://data.gov.au/dataset/2001-02-to-2007-08-local-government-survey-victoria/

U.S. Census. (2009). *Traffic fatalities by state and highest driver blood alcohol concentration (BAC) in the crash: 2009*.

U.S. Census. (2014) *The 15 Fastest-growing large cities with populations of 50,000 or more from July 1, 2012 to July 1, 2013*.

U. S. Department of Agriculture. (2010). *U.S. consumption of nutrients.* Retrieved from http://www.ers.usda.gov/data-products/fertilizer-use-and-price.aspx

U.S. Department of Energy, Office of Energy Efficiency and Renewable Energy. (2011). *Fuel economy guide* (DOE/EE-0333). Retrieved from website: http://www.fueleconomy.gov/feg/feg2011.pdf

U.S. Department of Labor, Bureau of Labor Statistics. (n.d.). *Employment projections.* Retrieved July 16, 2013 from http://www.bls.gov/emp/ep_chart_001.htm/ep_table_001.htm

U.S. Department of Labor, Bureau of Labor Statistics. (n.d.). *Unemployment rate—seasonally adjusted.* Retrieved June 22, 2013 from http://www.google.com/publicdata/explore?ds=z1ebjpgk2654c1_&met_y=unemployment_rate&hl=en&dl=en&idim=country:US&fdim_y=seasonality:S

U.S. Energy Information Administration. (n.d.). *International energy statistics*. Retrieved July 17, 2013 from http://www.eia.gov/cfapps/ipdbproject/IEDIndex3.cfm?tid=3&pid=26&aid=2

U.S. Inflation Calculator. (n.d.). *Historical inflation rates: 1914-2014*. Retrieved February 9, 2014 from http://www.usinflationcalculator.com/inflation/historical-inflation-rates/

The Weather Channel. (2013, May 18). *U.S.: Current temperatures.* Retrieved from http://www.weather.com/maps/maptype/currentweatherusnational/uscurrenttemperatures_large.html?from=wxcenter_maps

Weather Underground. (n.d.). *Weather history for Flagstaff, AZ.* Retrieved May 20, 2013 from http://www.wunderground.com/history/airport/KFLG/2013/5/21/MonthlyHistory.html

References

Weather Underground. (n.d.). *Weather history for Phoenix, AZ.* Retrieved July 17, 2013 from http://www.wunderground.com/history/airport/KPHX/2013/7/17/MonthlyHistory.html

Wikipedia. (n.d.). *List of U.S. states by population growth rate.* Retrieved January 15, 2015 from http://en.wikipedia.org/wiki/List_of_U.S._states_by_population_growth_rate

Wikipedia. (n.d.). *Shrinking cities in the United States*. Retrieved February 7, 2014 from http://en.wikipedia.org/wiki/Shrinking_cities_in_the_United_States

Wikipedia. (n.d.). *Topper site.* Retrieved December 17, 2014 from http://en.wikipedia.org/wiki/Topper_Site

World Bank. (2010). *Energy use (kg of oil equivalent per capita).* Retrieved from http://data.worldbank.org/indicator/EG.USE.PCAP.KG.OE?order=wbapi_data_value_2010 wbapi_data_value&sort=asc

World Health Rankings. (2011). *Life expectancy—Fertility rate*. Retrieved from http://www.worldlifeexpectancy.com/fertility-rate-by-country

Images:

Bookshelf Vector Illustration. (2011, May 14). Retrieved November 15, 2014 from http://www.clipartlogo.com/image/bookshelf-vector-illustration_344846.html

Cactus. (n.d.). Retrieved November 3, 2014 from https://www.google.com/search?as_st=y&tbm=isch&hl=en&as_q=fibonacci+sequence+in+nature&as_epq=&as_oq=&as_eq=&cr=&as_sitesearch=&safe=images&tbs=sur:fc#imgdii=_&imgrc=580HRZUyCnt-wM%253A%3BDcUYrSXT_5nOIM%3Bhttp%253A%252F%252Fupload.wikimedia.org%252Fwikipedia%252Fcommons%252F0%252F08%252FNautilusCutawayLogarithmicSpiral.jpg%3Bhttp%253A%252F%252Fen.wikipedia.org%252Fwiki%252FSpiral%3B2240%3B1693

Colwell, Taylor. (2013, July 21). *Arizona student suspended for asking that classes be taught in English.* Retrieved November 15, 2014 from http://media.townhall.com/townhall/reu/ha/2013/190/b00cc532-24d8-4028-9beb-877e2c63baf7.jpg

Daisies. (n.d.). Retrieved November 3, 2014 from https://www.google.com/search?as_st=y&tbm=isch&hl=en&as_q=daisy+flower&as_epq=&as_oq=&as_eq=&cr=&as_sitesearch=&safe=images

Logistic Growth Image 1. (n.d.). Retrieved June 17, 2014

Logistic Growth Image 2. (n.d.). Retrieved June 17, 2014 from https://encrypted-tbn2.gstatic.com/images?q=tbn:ANd9GcSx_fm5BCGLJ6h22_2nvg06UdkwJSlw6tVdepXssEfHHCJt-k0w

Ogle, Conor. (2009, May 10). *Spin*. Retrieved November 15, 2014 from https://www.flickr.com/photos/cmogle/3526750763/in/photostream/

Pine, Lucille. (2007, January 20). *Deck of cards*. Retrieved November 15, 2014 from https://www.flickr.com/photos/lulupine/363961229/

Sunflower. (n.d.). Retrieved November 3, 2014 from https://www.google.com/search?as_st=y&tbm=isch&hl=en&as_q=sun+flower&as_epq=&as_oq=&as_eq=&cr=&as_sitesearch=&safe=images&tbs=sur:fc

Thumbtack. (n.d.). Retrieved November 15, 2014 from PixaBar