Introductory Algebra

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Izabela Mazur

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BCCAMPUS
VICTORIA, B.C.

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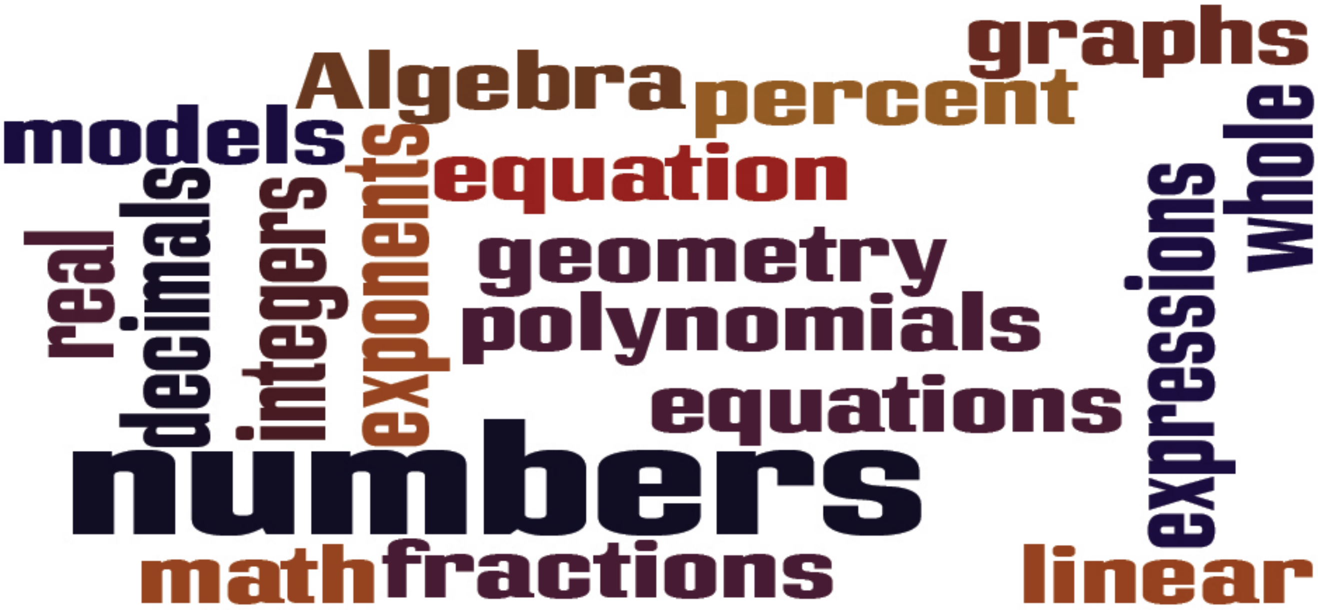
CHAPTER 1 Whole Numbers, Integers, and Introduction to Algebra

Algebra has a language of its own. The picture shows just some of the words you may see and use in your study of algebra.



You may not realize it, but you already use algebra every day. Perhaps you figure out how much to tip a server in a restaurant. Maybe you calculate the amount of change you should get when you pay for something. It could even be when you compare batting averages of your favorite players. You can describe the algebra you use in specific words, and follow an orderly process. In this chapter, you will explore the words used to describe algebra and start on your path to solving algebraic problems easily, both in class and in your everyday life.

1.1 Whole Numbers

Learning Objectives

By the end of this section, you will be able to:

- Use place value with whole numbers
- Identify multiples and and apply divisibility tests
- Find prime factorization and least common multiples

As we begin our study of intermediate algebra, we need to refresh some of our skills and vocabulary. This chapter and the next will focus on whole numbers, integers, fractions, decimals, and real numbers. We will also begin our use of algebraic notation and vocabulary.

Use Place Value with Whole Numbers

The most basic numbers used in algebra are the numbers we use to count objects in our world: 1, 2, 3, 4, and so on. These are called the counting number**s**. Counting numbers are also called *natural numbers*. If we add zero to the counting numbers, we get the set of whole number**s**.

Counting Numbers: 1, 2, 3, …

Whole Numbers: 0, 1, 2, 3, …

The notation "…" is called ellipsis and means "and so on," or that the pattern continues endlessly.

We can visualize counting numbers and whole numbers on a number line .See Figure 1.
The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left. While this number line shows only the whole numbers 0 through 6, the numbers keep going without end.

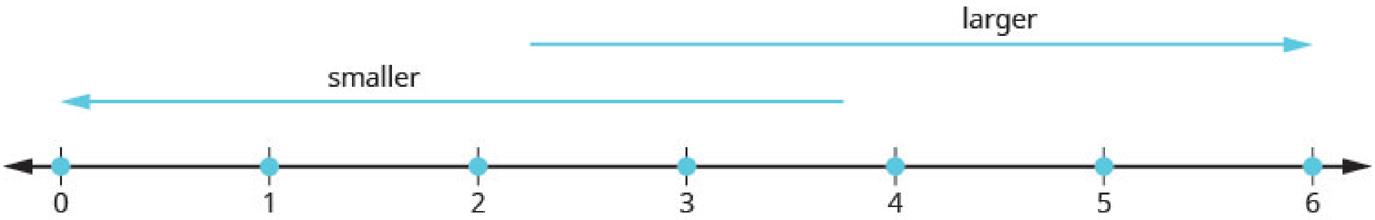
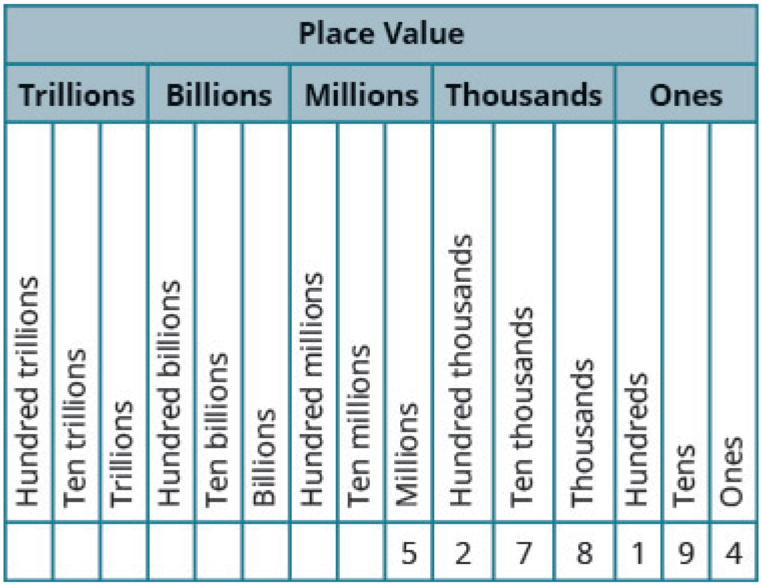


Figure 1

Our number system is called a place value system, because the value of a digit depends on its position in a number. Figure 2 shows the place values. The place values are separated into groups of three, which are called periods. The periods are *ones, thousands, millions, billions, trillions*, and so on. In a written number, commas separate the periods.

The number 5,278,194 is shown in the chart. The digit 5 is in the millions place. The digit 2 is in the hundred-thousands place. The digit 7 is in the ten-thousands place. The digit 8 is in the thousands place. The digit 1 is in the hundreds place. The digit 9 is in the tens place. The digit 4 is in the ones place.



Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

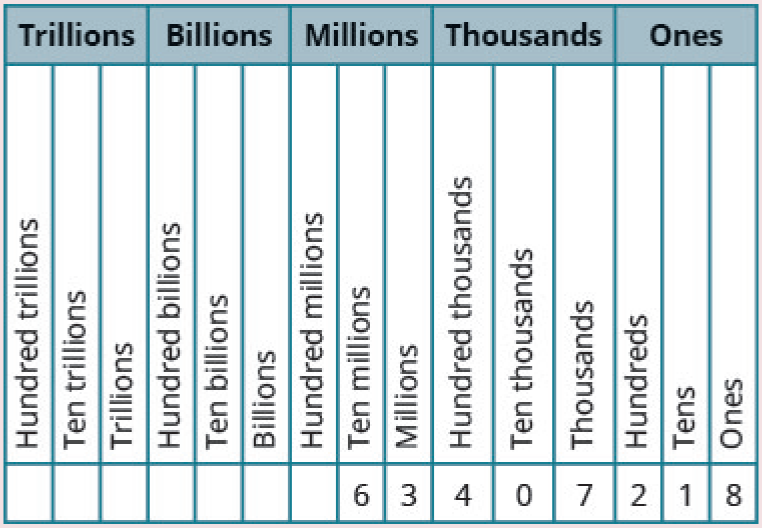
Figure 2

EXAMPLE 1

In the number 63,407,218, find the place value of each digit:

a. 7
b. 0
c. 1
d. 6
e. 3

Solution
Place the number in the place value chart:



Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

a) The 7 is in the thousands place.
b) The 0 is in the ten thousands place.
c) The 1 is in the tens place.
d) The 6 is in the ten-millions place.
e) The 3 is in the millions place.

TRY IT 1.1

For the number 27,493,615, find the place value of each digit:

a) 2 b) 1 c) 4 d) 7 e) 5

Show answer
a) ten millions b) tens c) hundred thousands d) millions e) ones

TRY IT 1.2

For the number 519,711,641,328, find the place value of each digit:

a) 9 b) 4 c) 2 d) 6 e) 7

Show answer
a) billions b) ten thousands c) tens d) hundred thousands e) hundred millions

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period, followed by the name of the period, without the *s* at the end. Start at the left, where the periods have the largest value. The ones period is not named. The commas separate the periods, so wherever there is a comma in the number, put a comma between the words (see Figure 3).

The number 74,218,369 is written as seventy-four million, two hundred eighteen thousand, three hundred sixty-nine.

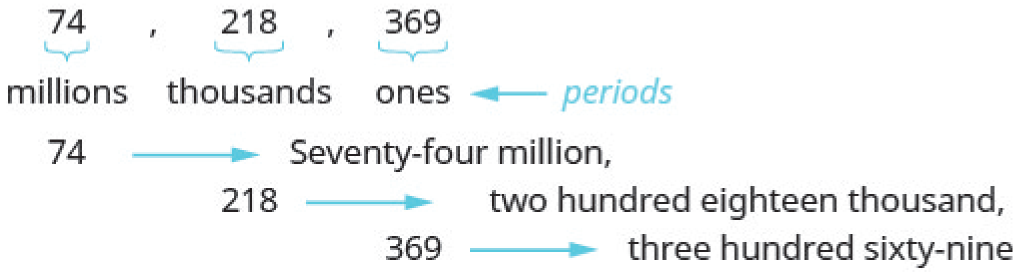


Figure 3

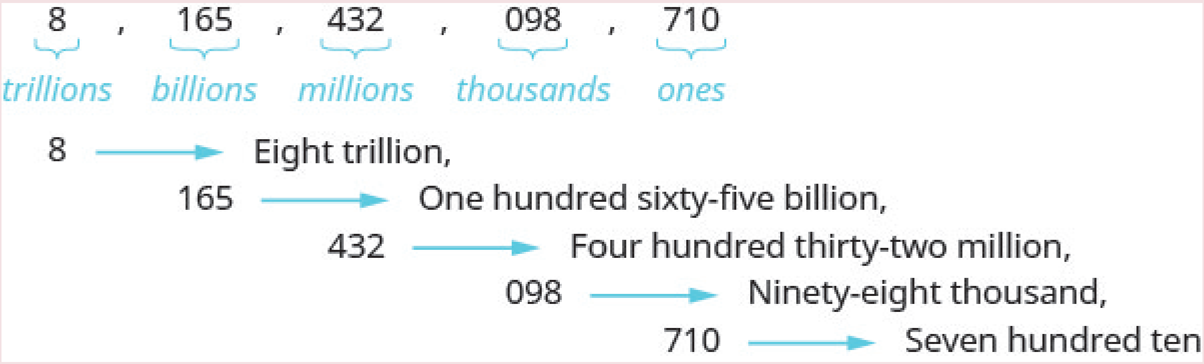
HOW TO: Name a Whole Number in Words.

1. Start at the left and name the number in each period, followed by the period name.
2. Put commas in the number to separate the periods.
3. Do not name the ones period.

EXAMPLE 2

Name the number 8,165,432,098,710 using words.

Solution
Name the number in each period, followed by the period name.



Put the commas in to separate the periods.

So, $8,165,432,098,710$ is named as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

TRY IT 2.1

Name the number $9,258,137,904,061$ using words.

Show answer
nine trillion, two hundred fifty-eight billion, one hundred thirty-seven million, nine hundred four thousand, sixty-one

TRY IT 2.2

Name the number $17,864,325,619,004$ using words.

Show answer
seventeen trillion, eight hundred sixty-four billion, three hundred twenty-five million, six hundred nineteen thousand four

We are now going to reverse the process by writing the digits from the name of the number. To write the number in digits, we first look for the clue words that indicate the periods. It is helpful to draw three blanks for the needed periods and then fill in the blanks with the numbers, separating the periods with commas.

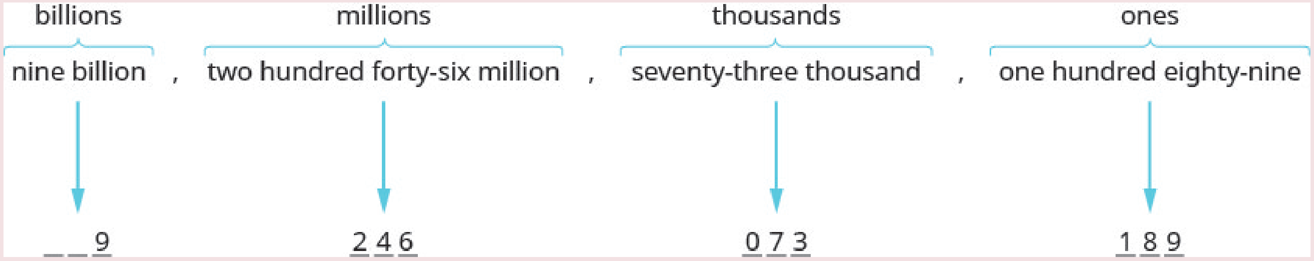
HOW TO: Write a Whole Number Using Digits.

1. Identify the words that indicate periods. (Remember, the ones period is never named.)
2. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
3. Name the number in each period and place the digits in the correct place value position.

EXAMPLE 3

Write *nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine* as a whole number using digits.

Solution
Identify the words that indicate periods.
Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
Then write the digits in each period.



The number is 9,246,073,189.

TRY IT 3.1

Write the number two billion, four hundred sixty-six million, seven hundred fourteen thousand, fifty-one as a whole number using digits.

Show answer

$2,466,714,051$

TRY IT 3.2

Write the number eleven billion, nine hundred twenty-one million, eight hundred thirty thousand, one hundred six as a whole number using digits.

Show answer

$11,921,830,106$

In 2016, Statistics Canada estimated the population of Toronto as 13,448,494. We could say the population of Toronto was approximately 13.4 million. In many cases, you don’t need the exact value; an approximate number is good enough.

The process of approximating a number is called rounding. Numbers are rounded to a specific place value, depending on how much accuracy is needed. Saying that the population of Toronto is approximately 13.4 million means that we rounded to the hundred thousands place.

EXAMPLE 4

Round 23,658 to the nearest hundred.

Solution

Step 1. Locate the given place value with an arrow. All digits to the left do not change.	Locate the hundreds place in 23,658.	hundredths place ↓ 23,658
Step 2. Underline the digit to the right of the given place value.	Underline the 5, which is to the right of the hundreds place.	hundredths place ↓ $23{,}6\underline{5}8$
Step 3. Is this digit greater than or equal to 5? Yes–add 1 to the digit in the given place value. No–do not change the digit in the given place value.	Add 1 to the 6 in the hundreds place, since 5 is greater than or equal to 5.	23,658 add 1 ↗
Step 4. Replace all digits to the right of the given place value with zeros.	Replace all digits to the right of the hundreds place with zeros.	$23{,}7\underbrace{00}$ add 1 ↗ ↖ replace with 0s So, 23,700 is rounded to the nearest hundred.

TRY IT 4.1

Round to the nearest hundred: $17,852$.

Show answer
$17,900$

TRY IT 4.2

Round to the nearest hundred: $468,751$.

Show answer
$468,800$

HOW TO: Round Whole Numbers.

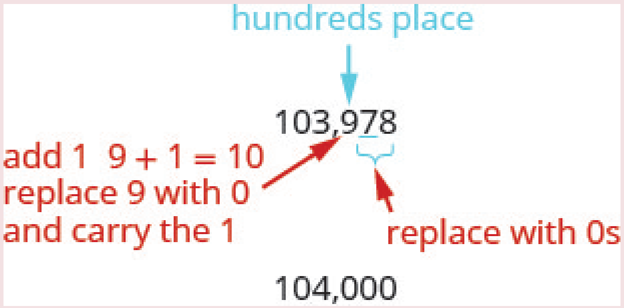
1. Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
2. Underline the digit to the right of the given place value.
3. Is this digit greater than or equal to 5?
    - Yes–add 1 to the digit in the given place value.
    - No–do not change the digit in the given place value.
4. Replace all digits to the right of the given place value with zeros.

EXAMPLE 5

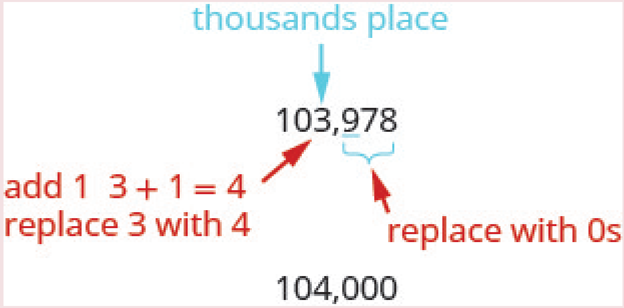
Round $103,978$ to the nearest:

a. hundred
b. thousand
c. ten thousand

Solution
a)

Locate the hundreds place in 103,978.	
Underline the digit to the right of the hundreds place.	
Since 7 is greater than or equal to 5, add 1 to the 9. Replace all digits to the right of the hundreds place with zeros.	
	So, 104,000 is 103,978 rounded to the nearest hundred.

b)

Locate the thousands place and underline the digit to the right of the thousands place.	
Since 9 is greater than or equal to 5, add 1 to the 3. Replace all digits to the right of the hundreds place with zeros.	
	So, 104,000 is 103,978 rounded to the nearest thousand.

c)

Locate the ten thousands place and underline the digit to the right of the ten thousands place.	
Since 3 is less than 5, we leave the 0 as is, and then replace the digits to the right with zeros.	100,000
	So, 100,000 is 103,978 rounded to the nearest ten thousand.

TRY IT 5.1

Round 206,981 to the nearest: a) hundred b) thousand c) ten thousand.

Show answer
a) 207,000 b) 207,000 c) 210,000

TRY IT 5.2

Round 784,951 to the nearest: a) hundred b) thousand c) ten thousand.

Show answer
a) 785,000 b) 785,000 c) 780,000

Identify Multiples and Apply Divisibility Tests

The numbers 2, 4, 6, 8, 10, and 12 are called multiples of 2. A multiple of 2 can be written as the product of a counting number and 2

$$\begin{array}{cccccc} 2, & 4, & 6, & 8, & 10, & 12, \ldots \\ 2\cdot 1, & 2\cdot 2, & 2\cdot 3, & 2\cdot 4, & 2\cdot 5, & 2\cdot 6 \end{array}$$

Similarly, a multiple of 3 would be the product of a counting number and 3

$$\begin{array}{cccccc} 3, & 6, & 9, & 12, & 15, & 18, \ldots \\ 3\cdot 1, & 3\cdot 2, & 3\cdot 3, & 3\cdot 4, & 3\cdot 5, & 3\cdot 6 \end{array}$$

We could find the multiples of any number by continuing this process.

The Table 1 below shows the multiples of 2 through 9 for the first 12 counting numbers.

Table 1

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108
Multiples of 10	10	20	30	40	50	60	70	80	90	100	110	120

Multiple of a Number

A number is a **multiple** of *n* if it is the product of a counting number and *n*.

Another way to say that 15 is a multiple of 3 is to say that 15 is divisible by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Divisible by a Number

If a number *m* is a multiple of *n*, then *m* is **divisible** by *n*.

Look at the multiples of 5 in Table 1. They all end in 5 or 0. Numbers with last digit of 5 or 0 are divisible by 5. Looking for other patterns in Table 1 that shows multiples of the numbers 2 through 9, we can discover the following divisibility tests:

Divisibility Tests

A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

EXAMPLE 6

Is 5,625 divisible by 2? By 3? By 5? By 6? By 10?

Solution

Is 5,625 divisible by 2?	
Does it end in 0,2,4,6, or 8?	No. 5,625 is not divisible by 2.
Is 5,625 divisible by 3?	
What is the sum of the digits?	$5 + 6 + 2 + 5 = 18$
Is the sum divisible by 3?	Yes. 5,625 is divisble by 3.
Is 5,625 divisible by 5 or 10?	
What is the last digit? It is 5.	5,625 is divisble by 5 but not by 10.
Is 5,625 divisible by 6?	
Is it divisible by both 2 or 3?	No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

TRY IT 6.1

Determine whether 4,962 is divisible by 2, by 3, by 5, by 6, and by 10

Show answer
by 2, 3, and 6

TRY IT 6.2

Determine whether 3,765 is divisible by 2, by 3, by 5, by 6, and by 10

Show answer
by 3 and 5

Find Prime Factorization and Least Common Multiples

In mathematics, there are often several ways to talk about the same ideas. So far, we’ve seen that if *m* is a multiple of *n*, we can say that *m* is divisible by *n*. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \cdot 9 = 72$, we say that 8 and 9 are factors of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72

$$\underbrace{8 \cdot 9}_{\textit{factors}} = \underbrace{72}_{\textit{product}}$$

Other ways to factor 72 are $1 \cdot 72, 2 \cdot 36, 3 \cdot 24, 4 \cdot 18,\ \text{and}\ 6 \cdot 12$. Seventy-two has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 36, and 72

Factors

If $a \cdot b = m$, then *a* and *b* are factors of *m*.

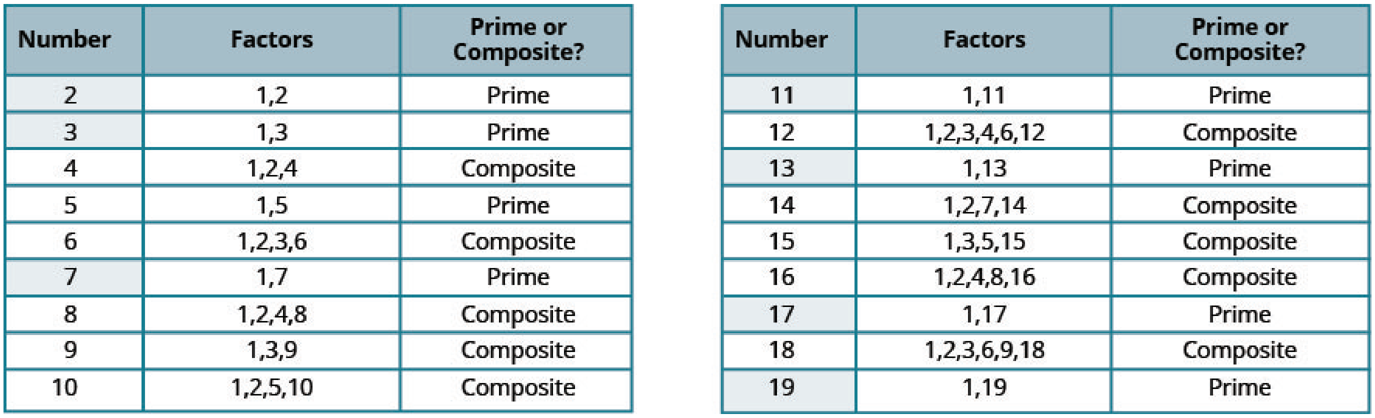
Some numbers, like 72, have many factors. Other numbers have only two factors.

Prime Number and Composite Number

A **prime number** is a counting number greater than 1, whose only factors are 1 and itself.

A composite number is a counting number that is not prime. A composite number has factors other than 1 and itself.

The counting numbers from 2 to 19 are listed in Figure 4, with their factors. Make sure to agree with the “prime” or “composite” label for each!



Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite

Number	Factors	Prime or Composite?
11	1,11	Prime
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime

Figure 4

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2

A composite number can be written as a unique product of primes. This is called the prime factorization of the number. Finding the prime factorization of a composite number will be useful later in this course.

Prime Factorization

The prime factorization of a number is the product of prime numbers that equals the number.

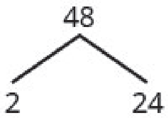
To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime!

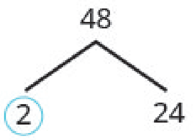
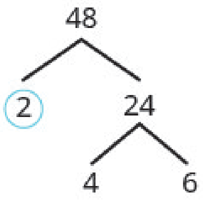
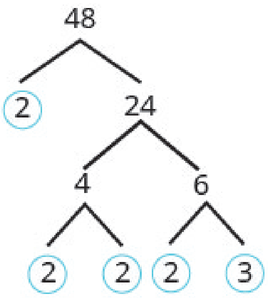
If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

EXAMPLE 7

Factor 48.

Solution

Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	$48 = 2 \cdot 24$	

Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	
Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors. 4 and 6 are not prime. Break them each into two factors. 2 and 3 are prime, so circle them.	 
Step 4. Write the composite number as the product of all the circled primes.		$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer!

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

TRY IT 7.1

Find the prime factorization of 80.

Show answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

TRY IT 7.2

Find the prime factorization of 60.

Show answer
$2 \cdot 2 \cdot 3 \cdot 5$

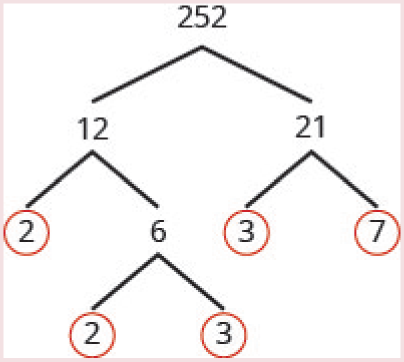
HOW TO: Find the Prime Factorization of a Composite Number.

1. Find two factors whose product is the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
3. If a factor is not prime, write it as the product of two factors and continue the process.
4. Write the composite number as the product of all the circled primes.

EXAMPLE 8

Find the prime factorization of 252

Solution

Step 1. Find two factors whose product is 252. 12 and 21 are not prime. Break 12 and 21 into two more factors. Continue until all primes are factored.	
Step 2. Write 252 as the product of all the circled primes.	$252 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

TRY IT 8.1

Find the prime factorization of 126

Show answer
$2 \cdot 3 \cdot 3 \cdot 7$

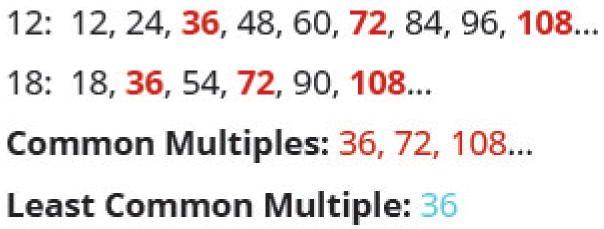
TRY IT 8.2

Find the prime factorization of 294

Show answer
$2 \cdot 3 \cdot 7 \cdot 7$

One of the reasons we look at multiples and primes is to use these techniques to find the least common multiple of two numbers. This will be useful when we add and subtract fractions with different denominators. Two methods are used most often to find the least common multiple and we will look at both of them.

The first method is the Listing Multiples Method. To find the least common multiple of 12 and 18, we list the first few multiples of 12 and 18:



12: 12, 24, **36**, 48, 60, **72**, 84, 96, **108**…

18: 18, **36**, 54, **72**, 90, **108**…

Common Multiples: 36, 72, 108…

Least Common Multiple: 36

Notice that some numbers appear in both lists. They are the **common multiples** of 12 and 18

We see that the first few common multiples of 12 and 18 are 36, 72, and 108. Since 36 is the smallest of the common multiples, we call it the *least common multiple.* We often use the abbreviation LCM.

Least Common Multiple

The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers.

The procedure box lists the steps to take to find the LCM using the prime factors method we used above for 12 and 18

HOW TO: Find the Least Common Multiple by Listing Multiples.

1. List several multiples of each number.
2. Look for the smallest number that appears on both lists.
3. This number is the LCM.

EXAMPLE 9

Find the least common multiple of 15 and 20 by listing multiples.

Solution

Make lists of the first few multiples of 15 and of 20, and use them to find the least common multiple.	 **15:** 15, 30, 45, 60, 75, 90, 105, 120 **20:** 20, 40, 60, 80, 100, 120, 140, 160
Look for the smallest number that appears in both lists.	The first number to appear on both lists is 60, so 60 is the least common multiple of 15 and 20.

Notice that 120 is in both lists, too. It is a common multiple, but it is not the *least* common multiple.

TRY IT 9.1

Find the least common multiple by listing multiples: 9 and 12

Show answer
36

TRY IT 9.2

Find the least common multiple by listing multiples: 18 and 24

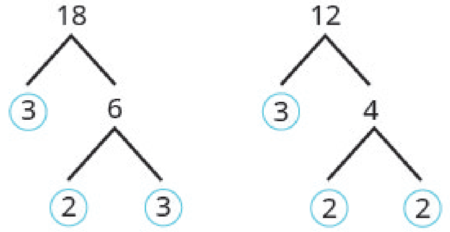
Show answer
72

Our second method to find the least common multiple of two numbers is to use The Prime Factors Method. Let’s find the LCM of 12 and 18 again, this time using their prime factors.

EXAMPLE 10

Find the Least Common Multiple (LCM) of 12 and 18 using the prime factors method.

Solution

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		$\text{LCM} = 36$

Notice that the prime factors of 12 $(2 \cdot 2 \cdot 3)$ and the prime factors of 18 $(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

TRY IT 10.1

Find the LCM using the prime factors method: 9 and 12

Show answer
36

TRY IT 10.2

Find the LCM using the prime factors method: 18 and 24

Show answer
72

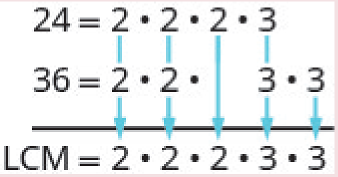
HOW TO: Find the Least Common Multiple Using the Prime Factors Method.

1. Write each number as a product of primes.
2. List the primes of each number. Match primes vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

EXAMPLE 11

Find the Least Common Multiple (LCM) of 24 and 36 using the prime factors method.

Solution

Find the primes of 24 and 36. Match primes vertically when possible.Bring down all columns.	 $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $36 = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$
Multiply the factors.	$\text{LCM} = 72$
	The LCM of 24 and 36 is 72.

TRY IT 11.1

Find the LCM using the prime factors method: 21 and 28

Show answer
84

TRY IT 11.2

Find the LCM using the prime factors method: 24 and 32

Show answer
96

Key Concepts

- **Place Value** as in Figure 2.
- **Name a Whole Number in Words**
    1. Start at the left and name the number in each period, followed by the period name.
    2. Put commas in the number to separate the periods.
    3. Do not name the ones period.
- **Write a Whole Number Using Digits**
    1. Identify the words that indicate periods. (Remember the ones period is never named.)
    2. Draw 3 blanks to indicate the number of places needed in each period. Separate the periods by commas.
    3. Name the number in each period and place the digits in the correct place value position.
- **Round Whole Numbers**
    1. Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
    2. Underline the digit to the right of the given place value.
    3. Is this digit greater than or equal to 5?
        - Yes—add 1 to the digit in the given place value.
        - No—do <u>not</u> change the digit in the given place value.
    4. Replace all digits to the right of the given place value with zeros.
- **Divisibility Tests:** A number is divisible by:
    - 2 if the last digit is 0, 2, 4, 6, or 8.
    - 3 if the sum of the digits is divisible by 3.
    - 5 if the last digit is 5 or 0.
    - 6 if it is divisible by both 2 and 3.

    - 10 if it ends with 0.

- **Find the Prime Factorization of a Composite Number**

    1. Find two factors whose product is the given number, and use these numbers to create two branches.
    2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
    3. If a factor is not prime, write it as the product of two factors and continue the process.
    4. Write the composite number as the product of all the circled primes.

- **Find the Least Common Multiple by Listing Multiples**

    1. List several multiples of each number.
    2. Look for the smallest number that appears on both lists.
    3. This number is the LCM.

- **Find the Least Common Multiple Using the Prime Factors Method**

    1. Write each number as a product of primes.
    2. List the primes of each number. Match primes vertically when possible.
    3. Bring down the columns.
    4. Multiply the factors.

Glossary

composite number

A composite number is a counting number that is not prime. A composite number has factors other than 1 and itself.

counting numbers

The counting numbers are the numbers 1, 2, 3, …

divisible by a number

If a number m is a multiple of n, then m is divisible by n. (If 6 is a multiple of 3, then 6 is divisible by 3.)

factors

If $ab = m$, then $a \text{ and } b$ are factors of m. Since 3 · 4 = 12, then 3 and 4 are factors of 12.

least common multiple

The least common multiple of two numbers is the smallest number that is a multiple of both numbers.

multiple of a number

A number is a multiple of n if it is the product of a counting number and n.

number line

A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

origin

The origin is the point labeled 0 on a number line.

prime factorization

The prime factorization of a number is the product of prime numbers that equals the number.

prime number

A prime number is a counting number greater than 1, whose only factors are 1 and itself.

whole numbers

The whole numbers are the numbers 0, 1, 2, 3, ….

Practice Makes Perfect

Use Place Value with Whole Numbers

In the following exercises, find the place value of each digit in the given numbers.

1. 51,493 a) 1 b) 4 c) 9 d) 5 e) 3	2. 87,210 a) 2 b) 8 c) 0 d) 7 e) 1
3. 164,285 a) 5 b) 6 c) 1 d) 8 e) 2	4. 395,076 a) 5 b) 3 c) 7 d) 0 e) 9
5. 93,285,170 a) 9 b) 8 c) 7 d) 5 e) 3	6. 36,084,215 a) 8 b) 6 c) 5 d) 4 e) 3
7. 7,284,915,860,132 a) 7 b) 4 c) 5 d) 3 e) 0	8. 2,850,361,159,433 a) 9 b) 8 c) 6 d) 4 e) 2

In the following exercises, name each number using words.

9. 1,078	10. 5,902
11. 364,510	12. 146,023
13. 5,846,103	14. 1,458,398
15. 37,889,005	16. 62,008,465

In the following exercises, write each number as a whole number using digits.

17. four hundred twelve	18. two hundred fifty-three
19. thirty-five thousand, nine hundred seventy-five	20. sixty-one thousand, four hundred fifteen
21. eleven million, forty-four thousand, one hundred sixty-seven	22. eighteen million, one hundred two thousand, seven hundred eighty-three
23. three billion, two hundred twenty-six million, five hundred twelve thousand, seventeen	24. eleven billion, four hundred seventy-one million, thirty-six thousand, one hundred six

In the following, round to the indicated place value.

25. Round to the nearest ten. a) 386 b) 2,931	26. Round to the nearest ten. a) 792 b) 5,647
27. Round to the nearest hundred. a) 13,748 b) 391,794	28. Round to the nearest hundred. a) 28,166 b) 481,628
29. Round to the nearest ten. a) 1,492 b) 1,497	30. Round to the nearest ten. a) 2,791 b) 2,795
31. Round to the nearest hundred. a) 63,994 b) 63,940	32. Round to the nearest hundred. a) 49,584 b) 49,548

In the following exercises, round each number to the nearest a) hundred, b) thousand, c) ten thousand.

33. 392,546	34. 619,348
35. 2,586,991	36. 4,287,965

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, 3, 5, 6, and 10

37. 84	38. 9,696
39. 75	40. 78
41. 900	42. 800
43. 986	44. 942
45. 350	46. 550
47. 22,335	48. 39,075

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

49. 86	50. 78
51. 132	52. 455
53. 693	54. 400
55. 432	56. 627
57. 2,160	58. 2,520

In the following exercises, find the least common multiple of the each pair of numbers using the multiples method.

59. 8, 12	60. 4, 3
61. 12, 16	62. 30, 40
63. 20, 30	64. 44, 55

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.

65. 8, 12	66. 12, 16
67. 28, 40	68. 84, 90
69. 55, 88	70. 60, 72

Everyday Math

71. **Writing a Check** Jorge bought a car for $24,493. He paid for the car with a check. Write the purchase price in words.

72. **Writing a Check** Marissa’s kitchen remodeling cost $18,549. She wrote a check to the contractor. Write the amount paid in words.

73. **Buying a Car** Jorge bought a car for $24,493. Round the price to the nearest a) ten b) hundred c) thousand; and d) ten-thousand.

74. **Remodeling a Kitchen** Marissa’s kitchen remodeling cost $18,549, Round the cost to the nearest a) ten b) hundred c) thousand and d) ten-thousand.

75. **Population** The population of China was 1,339,724,852 on November 1, 2010. Round the population to the nearest a) billion b) hundred-million; and c) million.

76. **Astronomy** The average distance between Earth and the sun is 149,597,888 kilometres. Round the distance to the nearest a) hundred-million b) ten-million; and c) million.

77. **Grocery Shopping** Hot dogs are sold in packages of 10, but hot dog buns come in packs of eight. What is the smallest number that makes the hot dogs and buns come out even?

78. **Grocery Shopping** Paper plates are sold in packages of 12 and party cups come in packs of eight. What is the smallest number that makes the plates and cups come out even?

Writing Exercises

79. What is the difference between prime numbers and composite numbers?

80. Give an everyday example where it helps to round numbers.

81. Explain in your own words how to find the prime factorization of a composite number, using any method you prefer.

Answers

1. a) thousands b) hundreds c) tens d) ten thousands e) ones	3. a) ones b) ten thousands c) hundred thousands d) tens e) hundreds	5. a) ten millions b) ten thousands c) tens d) thousands e) millions
7. a) trillions b) billions c) millions d) tens e) thousands	9. one thousand, seventy-eight	11. three hundred sixty-four thousand, five hundred ten
13. five million, eight hundred forty-six thousand, one hundred three	15. thirty-seven million, eight hundred eighty-nine thousand, five	17. 412
19. 35,975	21. 11,044,167	23. 3,226,512,017
25. a) 390 b) 2,930	27. a) 13,700 b) 391,800	29. a) 1,490 b) 1,500
31. a) 64,000 b) 63,900	33. a) 392,500 b) 393,000 c) 390,000	35. a) 2,587,000 b) 2,587,000 c) 2,590,000
37. divisible by 2, 3, and 6	39. divisible by 3 and 5	41. divisible by 2, 3, 5, 6, and 10
43. divisible by 2	45. divisible by 2, 5, and 10	47. divisible by 3 and 5
49. $2 \cdot 43$	51. $2 \cdot 2 \cdot 3 \cdot 11$	53. $3 \cdot 3 \cdot 7 \cdot 11$
55. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$	57. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5$	59. 24
61. 48	63. 60	65. 24
67. 420	69. 440	71. twenty-four thousand, four hundred ninety-three dollars
73. a) $24,490 b) $24,500 c) $24,000 d) $20,000	75. a) 1,000,000,000 b) 1,300,000,000 c) 1,340,000,000	77. 40
79. Answers may vary.	81. Answers may vary.	

Attributions

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1.2 Use the Language of Algebra

Learning Objectives

By the end of this section, you will be able to:

- Use variables and algebraic symbols
- Identify expressions and equations
- Simplify expressions with exponents
- Simplify expressions using the order of operations

Use Variables and Algebraic Symbols

Greg and Alex have the same birthday, but they were born in different years. This year Greg is 20 years old and Alex is 23, so Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right?

In the language of algebra, we say that Greg's age and Alex's age are variable and the three is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same, so the age difference is the constant.

In algebra, letters of the alphabet are used to represent variables. Suppose we call Greg's age g. Then we could use $g+3$ to represent Alex's age. See the table below.

Greg's age	**Alex's age**
12	15
20	23
35	38
g	$g+3$

Letters are used to represent variables. Letters often used for variables are $x, y, a, b,$ and c.

Variables and Constants

A variable is a letter that represents a number or quantity whose value may change.

A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. In 1.1 Whole Numbers, we introduced the symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Operation	**Notation**	**Say:**	**The result is…**
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a\cdot b, (a)\,(b)\,, (a)\,b, a\,(b)$	a times b	The product of a and b
Division	$a\div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

In algebra, the cross symbol, ×, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3\ \times\ y$ (three times y) or $3\cdot x\cdot y$ (three times x times y)? To make it clear, use • or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5+3$.
- The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9-2$.
- The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4\cdot 8$.
- The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20\div 5$.

EXAMPLE 1

Translate from algebra to words:

a. $12+14$
b. $(30)\,(5)$
c. $64\div 8$

d. $x - y$

Solution

a.
$12 + 14$
12 plus 14
the sum of twelve and fourteen

b.
$(30)\,(5)$
30 times 5
the product of thirty and five

c.
$64 \div 8$
64 divided by 8
the quotient of sixty-four and eight

d.
$x - y$
x minus y
the difference of x and y

TRY IT 1.1

Translate from algebra to words.

a. $18 + 11$
b. $(27)\,(9)$
c. $84 \div 7$
d. $p - q$

Show Answer

a. 18 plus 11; the sum of eighteen and eleven
b. 27 times 9; the product of twenty-seven and nine
c. 84 divided by 7; the quotient of eighty-four and seven
d. *p* minus *q*; the difference of *p* and *q*

TRY IT 1.2

Translate from algebra to words.

a. $47-19$
b. $72\div 9$
c. $m+n$
d. $(13)\,(7)$

Show Answer

a. 47 minus 19; the difference of forty-seven and nineteen
b. 72 divided by 9; the quotient of seventy-two and nine
c. *m* plus *n*; the sum of *m* and *n*
d. 13 times 7; the product of thirteen and seven

When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

Equality Symbol

$a=b$ is read a is equal to b

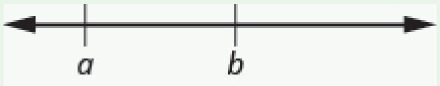
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols < and > for inequalities.

Inequality

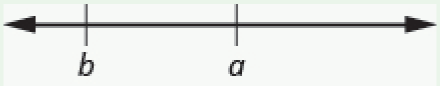
$a < b$ is read a is less than b

a is to the left of b on the number line



$a > b$ is read a is greater than b

a is to the right of b on the number line



The expressions $a < b$ and $a > b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general,

$$a < b \text{ is equivalent to } b > a. \qquad \text{For example, } 7 < 11 \text{ is equivalent to } 11 > 7.$$
$$a > b \text{ is equivalent to } b < a. \qquad \text{For example, } 17 > 4 \text{ is equivalent to } 4 < 17.$$

When we write an inequality symbol with a line under it, such as $a \leq b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\neq$ it means not equal.

We summarize the symbols of equality and inequality in the table below.

Algebraic Notation	**Say**
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

Symbols < and >

The symbols < and > each have a smaller side and a larger side.

smaller side < larger side
larger side > smaller side

The smaller side of the symbol faces the smaller number and the larger faces the larger number.

EXAMPLE 2

Translate from algebra to words:

a. $20 \leq 35$
b. $11 \neq 15 - 3$
c. $9 > 10 \div 2$
d. $x + 2 < 10$

Solution

a.
$20 \leq 35$
20 is less than or equal to 35

b.
$11 \neq 15 - 3$
11 is not equal to 15 minus 3

c.
$9 > 10 \div 2$
9 is greater than 10 divided by 2

d.
$x + 2 < 10$
x plus 2 is less than 10

TRY IT 2.1

Translate from algebra to words.

a. $14 \leq 27$
b. $19 - 2 \neq 8$
c. $12 > 4 \div 2$
d. $x - 7 < 1$

Show Answer

a. fourteen is less than or equal to twenty-seven
b. nineteen minus two is not equal to eight
c. twelve is greater than four divided by two
d. *x* minus seven is less than one

TRY IT 2.2

Translate from algebra to words.

a. $19 \geq 15$
b. $7 = 12 - 5$
c. $15 \div 3 < 8$
d. $y - 3 > 6$

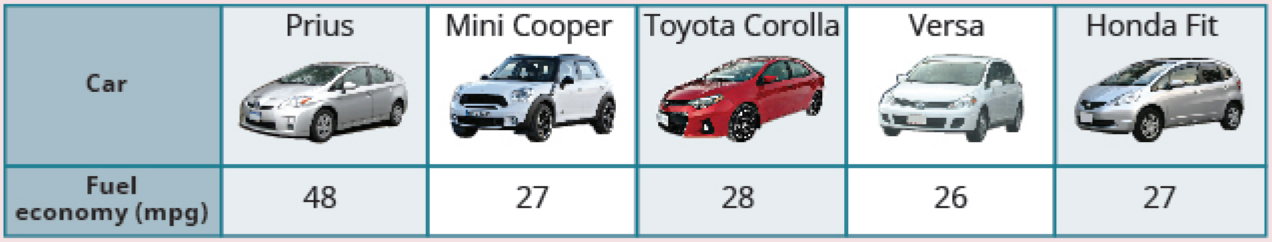
Show Answer

a. nineteen is greater than or equal to fifteen
b. seven is equal to twelve minus five
c. fifteen divided by three is less than eight
d. *y* minus three is greater than six

EXAMPLE 3

The information in (Figure 1) compares the fuel economy in miles-per-gallon (mpg) of several cars. Write the appropriate symbol $\text{symbol}, =,$ < , or >. in each expression to compare the fuel economy of the cars.

(credit: modification of work by Bernard Goldbach, Wikimedia Commons)



Car	Prius	Mini Cooper	Toyota Corolla	Versa	Honda Fit
Fuel economy (mpg)	48	27	28	26	27

Figure 1

a. MPG of Prius_____ MPG of Mini Cooper
b. MPG of Versa_____ MPG of Fit
c. MPG of Mini Cooper_____ MPG of Fit
d. MPG of Corolla_____ MPG of Versa
e. MPG of Corolla_____ MPG of Prius

Solution

a.	
	MPG of Prius____MPG of Mini Cooper
Find the values in the chart.	48____27
Compare.	48 > 27
	MPG of Prius > MPG of Mini Cooper

b.	
	MPG of Versa____MPG of Fit
Find the values in the chart.	26____27
Compare.	26 < 27
	MPG of Versa < MPG of Fit

c.	
	MPG of Mini Cooper____MPG of Fit
Find the values in the chart.	27____27
Compare.	27 = 27
	MPG of Mini Cooper = MPG of Fit

d.	
	MPG of Corolla____MPG of Versa
Find the values in the chart.	28____26
Compare.	$28 > 26$
	MPG of Corolla > MPG of Versa

e.	
	MPG of Corolla____MPG of Prius
Find the values in the chart.	28____48
Compare.	$28 < 48$
	MPG of Corolla < MPG of Prius

TRY IT 3.1

Use Figure 1 to fill in the appropriate symbol, $=$, $<$, or $>$.

a. MPG of Prius_____MPG of Versa
b. MPG of Mini Cooper_____ MPG of Corolla

Show Answer

a. >
b. <

TRY IT 3.2

Use Figure 1 to fill in the appropriate symbol, $=$, $<$, or $>$.

a. MPG of Fit_____ MPG of Prius
b. MPG of Corolla _____ MPG of Fit

Show Answer

a. <

b. $<$

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. The table below lists three of the most commonly used grouping symbols in algebra.

Common Grouping Symbols

parentheses	( )
brackets	[ ]
braces	$\{\ \}$

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$8\left(14-8\right)$ $\quad$ $21-3\left[2+4\left(9-8\right)\right]$ $\quad$ $24\div\left\{13-2\left[1\left(6-5\right)+4\right]\right\}$

Identify Expressions and Equations

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb.

In algebra, we have *expressions* and *equations*. An expression is like a phrase. Here are some examples of expressions and how they relate to word phrases:

Expression	**Words**	**Phrase**
$3+5$	$3\text{ plus }5$	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6\cdot7$	$6\text{ times }7$	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the phrases do not form a complete sentence because the phrase does not have a verb. An equation is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. Here are some examples of equations:

Equation	**Sentence**
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Expressions and Equations

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

An equation is made up of two expressions connected by an equal sign.

EXAMPLE 4

Determine if each is an expression or an equation:

a. $16-6=10$
b. $4\cdot 2+1$
c. $x\div 25$
d. $y+8=40$

Solution

a. $16-6=10$	This is an equation—two expressions are connected with an equal sign.
b. $4\cdot 2+1$	This is an expression—no equal sign.
c. $x\div 25$	This is an expression—no equal sign.
d. $y+8=40$	This is an equation—two expressions are connected with an equal sign.

TRY IT 4.1

Determine if each is an expression or an equation:

a. $23 + 6 = 29$
b. $7 \cdot 3 - 7$

Show Answer

a. equation
b. expression

TRY IT 4.2

Determine if each is an expression or an equation:

a. $y \div 14$
b. $x - 6 = 21$

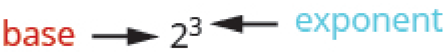
Show Answer

a. expression
b. equation

Simplify Expressions with Exponents

To simplify a numerical expression means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we'd first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:
$4 \cdot 2 + 1$
$8 + 1$
9

Suppose we have the expression $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. We could write this more compactly using exponential notation. Exponential notation is used in algebra to represent a quantity multiplied by itself several times. We write $2 \cdot 2 \cdot 2$ as 2^3 and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the base and the 3 is called the exponent. The exponent tells us how many factors of the base we have to multiply.



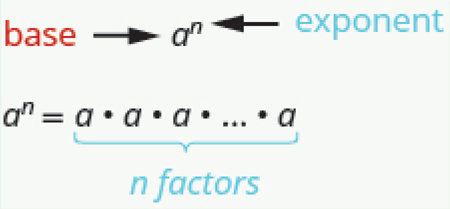
$\text{means multiply three factors of } 2$

We say 2^3 is in exponential notation and $2 \cdot 2 \cdot 2$ is in expanded notation.

Exponential Notation

For any expression a^n, a is a factor multiplied by itself n times if n is a positive integer.

a^nmeans multiply n factors of a



The expression a^n is read a to the n^{th} power.

For powers of $n = 2$ and $n = 3$, we have special names.
a^2 is read as ”a squared”
a^3 is read as ”a cubed”

The table below lists some examples of expressions written in exponential notation.

Exponential Notation	**In Words**
7^2	7 to the second power, or 7 squared
5^3	5 to the third power, or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

EXAMPLE 5

Write each expression in exponential form:

a. $16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16$
b. $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$
c. $x \cdot x \cdot x \cdot x$
d. $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

Solution

a. The base 16 is a factor 7 times.	16^7
b. The base 9 is a factor 5 times.	9^5
c. The base x is a factor 4 times.	x^4
d. The base a is a factor 8 times.	a^8

TRY IT 5.1

Write each expression in exponential form:

$41 \cdot 41 \cdot 41 \cdot 41 \cdot 41$

Show Answer

41^5

TRY IT 5.2

Write each expression in exponential form:

$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$

Show Answer

7^9

EXAMPLE 6

Write each exponential expression in expanded form:

a. 8^6

b. x^5

Solution

a. The base is 8 and the exponent is 6, so 8^6 means $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

b. The base is x and the exponent is 5, so x^5 means $x \cdot x \cdot x \cdot x \cdot x$

TRY IT 6.1

Write each exponential expression in expanded form:

a. 4^8
b. a^7

Show Answer

a. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
b. $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

TRY IT 6.2

Write each exponential expression in expanded form:

a. 8^8
b. b^6

Show Answer

a. $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$
b. $b \cdot b \cdot b \cdot b \cdot b \cdot b$

To simplify an exponential expression without using a calculator, we write it in expanded form and then multiply the factors.

EXAMPLE 7

Simplify: 3^4.

Solution

	3^4
Expand the expression.	$3 \cdot 3 \cdot 3 \cdot 3$
Multiply left to right.	$9 \cdot 3 \cdot 3$
	$27 \cdot 3$
Multiply.	81

TRY IT 7.1

Simplify:

a. 5^3
b. 1^7

Show Answer

a. 125
b. 1

TRY IT 7.2

Simplify:

a. 7^2
b. 0^5

Show Answer

a. 49
b. 0

Simplify Expressions Using the Order of Operations

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression:

$4 + 3 \cdot 7$

Some students say it simplifies to 49.

	$4 + 3 \cdot 7$
Since $4 + 3$ gives 7.	$7 \cdot 7$
And $7 \cdot 7$ is 49.	49

Some students say it simplifies to 25.

	$4 + 3 \cdot 7$
Since $3 \cdot 7$ is 21.	$4 + 21$
And $21 + 4$ makes 25.	25

Imagine the confusion that could result if every problem had several different correct answers. The same expression should give the same result. So mathematicians established some guidelines called the order of operations, which outlines the order in which parts of an expression must be simplified.

Order of Operations

When simplifying mathematical expressions perform the operations in the following order:

1. **P**arentheses and other Grouping Symbols
   - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. **E**xponents
   - Simplify all expressions with exponents.
3. **M**ultiplication and **D**ivision
   - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. **A**ddition and **S**ubtraction
   - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, “How will I remember the order?” Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase.

Please **E**xcuse **M**y **D**ear **A**unt **S**ally.

Please	**P**arentheses
Excuse	**E**xponents
My **D**ear	**M**ultiplication and **D**ivision
Aunt **S**ally	**A**ddition and **S**ubtraction

It’s good that ‘**M**y **D**ear’ goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority.

We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, '**A**unt **S**ally' goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

EXAMPLE 8

Simplify the expressions:

a. $4 + 3 \cdot 7$
b. $(4 + 3) \cdot 7$

Solution

a.	
	$4 + 3 \cdot 7$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply first.	$4 + 3 \cdot 7$
Add.	$4 + 21$
	25

b.	
	$(4 + 3) \cdot 7$
Are there any **p**arentheses? Yes.	$(4 + 3) \cdot 7$
Simplify inside the parentheses.	$(7)7$
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply.	49

TRY IT 8.1

Simplify the expressions:

a. $12 - 5 \cdot 2$
b. $(12 - 5) \cdot 2$

Show Answer

a. 2
b. 14

TRY IT 8.2

Simplify the expressions:

a. $8 + 3 \cdot 9$
b. $(8 + 3) \cdot 9$

Show Answer

a. 35
b. 99

EXAMPLE 9

Simplify:

a. $18 \div 9 \cdot 2$
b. $18 \cdot 9 \div 2$

Solution

a.	
	$18 \div 9 \cdot 2$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply and divide from left to right. Divide.	$2 \cdot 2$
Multiply.	4

b.	
	$18 \cdot 9 \div 2$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply and divide from left to right.	
Multiply.	$162 \div 2$
Divide.	81

TRY IT 9.1

Simplify:

$$42 \div 7 \cdot 3$$

Show Answer

18

TRY IT 9.2

Simplify:

$$12 \cdot 3 \div 4$$

Show Answer

9

EXAMPLE 10

Simplify: $18 \div 6 + 4\left(5 - 2\right)$.

Solution

	$18 \div 6 + 4(5 - 2)$
Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	15

TRY IT 10.1

Simplify:

$$30 \div 5 + 10\left(3 - 2\right)$$

Show Answer
16

TRY IT 10.2

Simplify:

$$70 \div 10 + 4\left(6 - 2\right)$$

Show Answer
23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

EXAMPLE 11

Simplify: $5+2^3+3\left[6-3\left(4-2\right)\right]$.

Solution

	$5+2^3+3[6-3(4-2)]$
Are there any parentheses (or other grouping symbol)? Yes.	
Focus on the parentheses that are inside the brackets.	$5+2^3+3[6-3(4-2)]$
Subtract.	$5+2^3+3[6-3(2)]$
Continue inside the brackets and multiply.	$5+2^3+3[6-6]$
Continue inside the brackets and subtract.	$5+2^3+3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	
Simplify exponents.	$5+2^3+3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5+8+3[0]$
Is there any addition or subtraction? Yes.	
Add.	$5+8+0$
Add.	$13+0$
	13

TRY IT 11.1

Simplify:

$$9+5^3-\left[4\left(9+3\right)\right]$$

Show Answer

86

TRY IT 11.2

Simplify:

$7^2 - 2\left[4\left(5+1\right)\right]$

Show Answer

1

EXAMPLE 12

Simplify: $2^3 + 3^4 \div 3 - 5^2$.

Solution

	$2^3 + 3^4 \div 3 - 5^2$
If an expression has several exponents, they may be simplified in the same step.	
Simplify exponents.	$2^3 + 3^4 \div 3 - 5^2$
Divide.	$8 + 81 \div 3 - 25$
Add.	$8 + 27 - 25$
Subtract.	$35 - 25$
	10

TRY IT 12.1

Simplify:

$3^2 + 2^4 \div 2 + 4^3$

Show Answer

81

TRY IT 12.2

Simplify:

$6^2 - 5^3 \div 5 + 8^2$

Show Answer
75

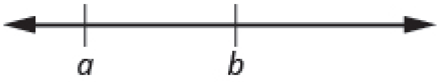
ACCESS ADDITIONAL ONLINE RESOURCES

- Order of Operations
- Order of Operations – The Basics
- Ex: Evaluate an Expression Using the Order of Operations
- Example 3: Evaluate an Expression Using The Order of Operations

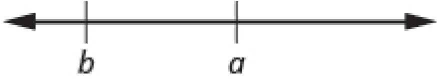
Key Concepts

Operation	**Notation**	**Say:**	**The result is…**
Addition	$a + b$	a plus b	the sum of a and b
Multiplication	$a \cdot b, (a)(b), (a)b, a(b)$	a times b	The product of a and b
Subtraction	$a - b$	a minus b	the difference of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

- **Equality Symbol**
    - $a = b$ is read as a is equal to b
    - The symbol $=$ is called the equal sign.
- **Inequality**
    - $a < b$ is read a is less than b
    - a is to the left of b on the number line



    - $a > b$ is read a is greater than b
    - a is to the right of b on the number line



Algebraic Notation	**Say**
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

- **Exponential Notation**
    - For any expression a^n is a factor multiplied by itself n times, if n is a positive integer.
    - a^n means multiply n factors of a

      $$\text{base} \longrightarrow a^n \longleftarrow \text{exponent}$$

      $$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

    - The expression of a^n is read a to the $n\mathrm{th}$ power.

Order of Operations When simplifying mathematical expressions perform the operations in the following order:

- Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
- Exponents: Simplify all expressions with exponents.
- Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
- Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

Glossary

expressions

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

equation

An equation is made up of two expressions connected by an equal sign.

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebraic notation to words.

1. $16-9$	2. $25-7$
3. $5\cdot 6$	4. $3\cdot 9$
5. $28\div 4$	6. $45\div 5$
7. $x+8$	8. $x+11$
9. $(2)(7)$	10. $(4)(8)$
11. $14<21$	12. $17<35$
13. $36\geq 19$	14. $42\geq 27$
15. $3n=24$	16. $6n=36$
17. $y-1>6$	18. $y-4>8$
19. $2\leq 18\div 6$	20. $3\leq 20\div 4$
21. $a\neq 7\cdot 4$	22. $a\neq 1\cdot 12$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or an equation.

23. $9\cdot 6=54$	24. $7\cdot 9=63$
25. $5\cdot 4+3$	26. $6\cdot 3+5$
27. $x+7$	28. $x+9$
29. $y-5=25$	30. $y-8=32$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

31. $3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3$	32. $4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4$
33. $x\cdot x\cdot x\cdot x\cdot x$	34. $y\cdot y\cdot y\cdot y\cdot y\cdot y$

In the following exercises, write in expanded form.

35. 5^3	36. 8^3
37. 2^8	38. 10^5

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

39. a.$3 + 8 \cdot 5$ b.$(3+8) \cdot 5$	40. a.$2 + 6 \cdot 3$ b.$(2+6) \cdot 3$
41. $2^3 - 12 \div (9 - 5)$	42. $3^2 - 18 \div (11 - 5)$
43. $3 \cdot 8 + 5 \cdot 2$	44. $4 \cdot 7 + 3 \cdot 5$
45. $2 + 8\,(6 + 1)$	46. $4 + 6\,(3 + 6)$
47. $4 \cdot 12/8$	48. $2 \cdot 36/6$
49. $6 + 10/2 + 2$	50. $9 + 12/3 + 4$
51. $(6 + 10) \div (2 + 2)$	52. $(9 + 12) \div (3 + 4)$
53. $20 \div 4 + 6 \cdot 5$	54. $33 \div 3 + 8 \cdot 2$
55. $20 \div (4 + 6) \cdot 5$	56. $33 \div (3 + 8) \cdot 2$
57. $4^2 + 5^2$	58. $3^2 + 7^2$
59. $(4 + 5)^2$	60. $(3 + 7)^2$
61. $3\,(1 + 9 \cdot 6) - 4^2$	62. $5\,(2 + 8 \cdot 4) - 7^2$
63. $2\,[1 + 3\,(10 - 2)]$	64. $5\,[2 + 4\,(3 - 2)]$

Everyday Math

65. Basketball In the 2014 NBA playoffs, the San Antonio Spurs beat the Miami Heat. The table below shows the heights of the starters on each team. Use this table to fill in the appropriate symbol (= ,<, >).

Spurs	**Height**		**Heat**	**Height**
Tim Duncan	$83''$		Rashard Lewis	$82''$
Boris Diaw	$80''$		LeBron James	$80''$
Kawhi Leonard	$79''$		Chris Bosh	$83''$
Tony Parker	$74''$		Dwyane Wade	$76''$
Danny Green	$78''$		Ray Allen	$77''$

a. Height of Tim Duncan____Height of Rashard Lewis

b. Height of Boris Diaw____Height of LeBron James

c. Height of Kawhi Leonard____Height of Chris Bosh

d. Height of Tony Parker____Height of Dwyane Wade

e. Height of Danny Green____Height of Ray Allen

66. Elevation In Colorado there are more than 50 mountains with an elevation of over $14,000\text{ feet}$. The table shows the ten tallest. Use this table to fill in the appropriate inequality symbol.

Mountain	**Elevation**
Mt. Elbert	$14,433'$
Mt. Massive	$14,421'$
Mt. Harvard	$14,420'$
Blanca Peak	$14,345'$
La Plata Peak	$14,336'$
Uncompahgre Peak	$14,309'$
Crestone Peak	$14,294'$
Mt. Lincoln	$14,286'$
Grays Peak	$14,270'$
Mt. Antero	$14,269'$

a. Elevation of La Plata Peak____Elevation of Mt. Antero

b. Elevation of Blanca Peak____Elevation of Mt. Elbert

c. Elevation of Gray’s Peak____Elevation of Mt. Lincoln

d. Elevation of Mt. Massive____Elevation of Crestone Peak

e. Elevation of Mt. Harvard____Elevation of Uncompahgre Peak

Writing Exercises

67.Explain the difference between an expression and an equation.

68. Why is it important to use the order of operations to simplify an expression?

Answers

1. 16 minus 9, the difference of sixteen and nine	3. 5 times 6, the product of five and six	5. 28 divided by 4, the quotient of twenty-eight and four
7. x plus 8, the sum of x and eight	9. 2 times 7, the product of two and seven	11. fourteen is less than twenty-one
13. thirty-six is greater than or equal to nineteen	15. 3 times n equals 24, the product of three and n equals twenty-four	17. y minus 1 is greater than 6, the difference of y and one is greater than six
19. 2 is less than or equal to 18 divided by 6; 2 is less than or equal to the quotient of eighteen and six	21. a is not equal to 7 times 4, a is not equal to the product of seven and four	23. equation
25. expression	27. expression	29. equation
31. 3^7	33. x^5	35. 125
37. 256	39. a. 43 b. 55	41. 5
43. 34	45. 58	47. 6
49. 13	51. 4	53. 35
55. 10	57. 41	59. 81
61. 149	63. 50	65. a. > b. = c. < d. < e. >
67. Answer may vary.		

Attributions

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1.3 Evaluate, Simplify, and Translate Expressions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate algebraic expressions
- Identify terms, coefficients, and like terms
- Simplify expressions by combining like terms
- Translate word phrases to algebraic expressions

Evaluate Algebraic Expressions

In the last section, we simplified expressions using the order of operations. In this section, we'll evaluate expressions—again following the order of operations.

To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number. To evaluate an expression, we substitute the given number for the variable in the expression and then simplify the expression using the order of operations.

EXAMPLE 1

Evaluate $x+7$ when

a. $x=3$
b. $x=12$

Solution
a. To evaluate, substitute 3 for x in the expression, and then simplify.

	$x+7$
Substitute.	$3+7$
Add.	10

When $x=3$, the expression $x+7$ has a value of 10.

b. To evaluate, substitute 12 for x in the expression, and then simplify.

	$x+7$
Substitute.	$12+7$
Add.	19

When $x = 12$, the expression $x + 7$ has a value of 19.

Notice that we got different results for parts a) and b) even though we started with the same expression. This is because the values used for x were different. When we evaluate an expression, the value varies depending on the value used for the variable.

TRY IT 1.1

Evaluate:

$y + 4$ when

a. $y = 6$
b. $y = 15$

Show Answer

a. 10
b. 19

TRY IT 1.2

Evaluate:

$a - 5$ when

a. $a = 9$
b. $a = 17$

Show Answer

a. 4
b. 12

EXAMPLE 2

Evaluate $9x - 2$, when

a. $x = 5$
b. $x = 1$

Solution
Remember ab means a times b, so $9x$ means 9 times x.

a. To evaluate the expression when $x = 5$, we substitute 5 for x, and then simplify.

	$9x - 2$
Substitute 5 for x.	$9 \cdot 5 - 2$
Multiply.	$45 - 2$
Subtract.	43

b. To evaluate the expression when $x = 1$, we substitute 1 for x, and then simplify.

	$9x - 2$
Substitute 1 for x.	$9(1) - 2$
Multiply.	$9 - 2$
Subtract.	7

Notice that in part a) that we wrote $9 \cdot 5$ and in part b) we wrote $9\,(1)$. Both the dot and the parentheses tell us to multiply.

TRY IT 2.1

Evaluate:

$8x - 3$, when

a. $x = 2$
b. $x = 1$

Show Answer

a. 13
b. 5

TRY IT 2.2

Evaluate:

$4y - 4$, when

- a. $y = 3$
- b. $y = 5$

Show Answer

- a. 8
- b. 16

EXAMPLE 3

Evaluate x^2 when $x = 10$.

Solution

We substitute 10 for x, and then simplify the expression.

	x^2
Substitute 10 for x.	10^2
Use the definition of exponent.	$10 \cdot 10$
Multiply.	100

When $x = 10$, the expression x^2 has a value of 100.

TRY IT 3.1

Evaluate:

x^2 when $x = 8$.

Show Answer

64

TRY IT 3.2

Evaluate:

x^3 when $x = 6$.

Show Answer
216

EXAMPLE 4

Evaluate 2^x when $x = 5$.

Solution
In this expression, the variable is an exponent.

	2^x
Substitute 5 for x.	2^5
Use the definition of exponent.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Multiply.	32

When $x = 5$, the expression 2^x has a value of 32.

TRY IT 4.1

Evaluate:

2^x when $x = 6$.

Show Answer
64

TRY IT 4.2

Evaluate:

3^x when $x = 4$.

Show Answer
81

EXAMPLE 5

Evaluate $3x + 4y - 6$ when $x = 10$ and $y = 2$.

Solution
This expression contains two variables, so we must make two substitutions.

	$3x + 4y - 6$
Substitute 10 for *x* and 2 for *y*.	$3(10) + 4(2) - 6$
Multiply.	$30 + 8 - 6$
Add and subtract left to right.	32

When $x = 10$ and $y = 2$, the expression $3x + 4y - 6$ has a value of 32.

TRY IT 5.1

Evaluate:

$2x + 5y - 4$ when $x = 11$ and $y = 3$

Show Answer
33

TRY IT 5.2

Evaluate:

$5x - 2y - 9$ when $x = 7$ and $y = 8$

Show Answer
10

EXAMPLE 6

Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Solution

We need to be careful when an expression has a variable with an exponent. In this expression, $2x^2$ means $2 \cdot x \cdot x$ and is different from the expression $(2x)^2$, which means $2x \cdot 2x$.

	$2x^2 + 3x + 8$
Substitute 4 for each x.	$2(4)^2 + 3(4) + 8$
Simplify 4^2.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

TRY IT 6.1

Evaluate:

$3x^2 + 4x + 1$ when $x = 3$.

Show Answer

40

TRY IT 6.2

Evaluate:

$6x^2 - 4x - 7$ when $x = 2$.

Show Answer

9

Identify Terms, Coefficients, and Like Terms

Algebraic expressions are made up of *terms*. A term is a constant or the product of a constant and one or more variables. Some examples of terms are $7, y, 5x^2, 9a$, and $13xy$.

The constant that multiplies the variable(s) in a term is called the coefficient. We can think of the coefficient as the number *in front of* the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$. The table below gives the coefficients for each of the terms in the left column.

Term	**Coefficient**
7	7
$9a$	9
y	1
$5x^2$	5

An algebraic expression may consist of one or more terms added or subtracted. In this chapter, we will only work with terms that are added together. The table below gives some examples of algebraic expressions with various numbers of terms. Notice that we include the operation before a term with it.

Expression	**Terms**
7	7
y	y
$x + 7$	$x, 7$
$2x + 7y + 4$	$2x, 7y, 4$
$3x^2 + 4x^2 + 5y + 3$	$3x^2, 4x^2, 5y, 3$

EXAMPLE 7

Identify each term in the expression $9b + 15x^2 + a + 6$. Then identify the coefficient of each term.

Solution

The expression has four terms. They are $9b$, $15x^2$, a, and 6.

The coefficient of $9b$ is 9.

The coefficient of $15x^2$ is 15.

Remember that if no number is written before a variable, the coefficient is 1. So the coefficient of a is 1.

The coefficient of a constant is the constant, so the coefficient of 6 is 6.

TRY IT 7.1

Identify all terms in the given expression, and their coefficients:

$4x+3b+2$

Show Answer

The terms are 4*x*, 3*b*, and 2. The coefficients are 4, 3, and 2

TRY IT 7.2

Identify all terms in the given expression, and their coefficients:

$9a+13a^2+a^3$

Show Answer

The terms are 9*a*, $13a^2$, and a^3, The coefficients are 9, 13, and 1

Some terms share common traits. Look at the following terms. Which ones seem to have traits in common?

$5x, 7, n^2, 4, 3x, 9n^2$

Which of these terms are like terms?

- The terms 7 and 4 are both constant terms.
- The terms $5x$ and $3x$ are both terms with x.
- The terms n^2 and $9n^2$ both have n^2.

Terms are called like terms if they have the same variables and exponents. All constant terms are also like terms. So among the terms $5x, 7, n^2, 4, 3x, 9n^2$,
$7 \text{ and } 4 \text{ are like terms.}$
$5x \text{ and } 3x \text{ are like terms.}$
$n^2 \text{ and } 9n^2 \text{ are like terms.}$

Like Terms

Terms that are either constants or have the same variables with the same exponents are like terms.

EXAMPLE 8

Identify the like terms:

a. $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$
b. $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Solution

a. $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$

Look at the variables and exponents. The expression contains y^3, x^2, x, and constants.

The terms y^3 and $4y^3$ are like terms because they both have y^3.

The terms $7x^2$ and $5x^2$ are like terms because they both have x^2.

The terms 14 and 23 are like terms because they are both constants.

The term $9x$ does not have any like terms in this list since no other terms have the variable x raised to the power of 1.

b. $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Look at the variables and exponents. The expression contains the terms $4x^2, 2x, 5x^2, 6x, 40x, \text{and } 8xy$

The terms $4x^2$ and $5x^2$ are like terms because they both have x^2.

The terms $2x, 6x, \text{and } 40x$ are like terms because they all have x.

The term $8xy$ has no like terms in the given expression because no other terms contain the two variables xy.

TRY IT 8.1

Identify the like terms in the list or the expression:

$9, 2x^3, y^2, 8x^3, 15, 9y, 11y^2$

Show Answer

9, 15; $2x^3$ and $8x^3$, y^2, and $11y^2$

TRY IT 8.2

Identify the like terms in the list or the expression:

$4x^3 + 8x^2 + 19 + 3x^2 + 24 + 6x^3$

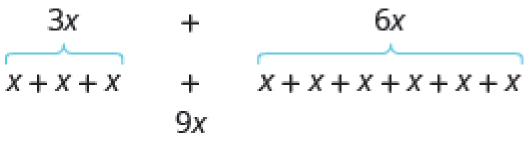
Show Answer

$4x^3$ and $6x^3$; $8x^2$ and $3x^2$; 19 and 24

Simplify Expressions by Combining Like Terms

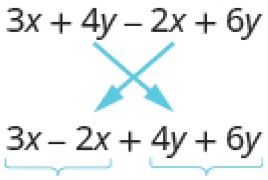
We can simplify an expression by combining the like terms. What do you think $3x + 6x$ would simplify to? If you thought $9x$, you would be right!

We can see why this works by writing both terms as addition problems.



Add the coefficients and keep the same variable. It doesn't matter what x is. If you have 3 of something and add 6 more of the same thing, the result is 9 of them. For example, 3 oranges plus 6 oranges is 9 oranges. We will discuss the mathematical properties behind this later.

The expression $3x + 6x$ has only two terms. When an expression contains more terms, it may be helpful to rearrange the terms so that like terms are together. The Commutative Property of Addition says that we can change the order of addends without changing the sum. So we could rearrange the following expression before combining like terms.



Now it is easier to see the like terms to be combined.

HOW TO: Combine like terms

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add the coefficients of the like terms.

EXAMPLE 9

Simplify the expression: $3x + 7 + 4x + 5$.

Solution

	$3x+7+4x+5$
Identify the like terms.	$3x+7+4x+5$
Rearrange the expression, so the like terms are together.	$3x+4x+7+5$
Add the coefficients of the like terms.	$\underbrace{3x+4x}_{7x}+\underbrace{7+5}_{12}$
The original expression is simplified to…	$7x+12$

TRY IT 9.1

Simplify:

$$7x+9+9x+8$$

Show Answer

$16x+17$

TRY IT 9.2

Simplify:

$$5y+2+8y+4y+5$$

Show Answer

$17y+7$

EXAMPLE 10

Simplify the expression: $7x^2+8x+x^2+4x$.

Solution

	$7x^2+8x+x^2+4x$
Identify the like terms.	$7x^2+8x+x^2+4x$
Rearrange the expression so like terms are together.	$7x^2+x^2+8x+4x$
Add the coefficients of the like terms.	$8x^2+12x$

These are not like terms and cannot be combined. So $8x^2+12x$ is in simplest form.

TRY IT 10.1

Simplify:

$3x^2+9x+x^2+5x$

Show Answer

$4x^2 + 14x$

TRY IT 10.2

Simplify:

$11y^2+8y+y^2+7y$

Show Answer

$12y^2 + 15y$

Translate Words to Algebraic Expressions

In the previous section, we listed many operation symbols that are used in algebra, and then we translated expressions and equations into word phrases and sentences. Now we'll reverse the process and translate word phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. They are summarized in the table below.

Operation	**Phrase**	**Expression**
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b b subtracted from a a decreased by b b less than a	$a-b$
Multiplication	a times b the product of a and b	$a \cdot b, ab, a\left(b\right), \left(a\right)\left(b\right)$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$

Look closely at these phrases using the four operations:

- the sum *of* a *and* b
- the difference *of* a *and* b
- the product *of* a *and* b
- the quotient *of* a *and* b

Each phrase tells you to operate on two numbers. Look for the words ***of*** and ***and*** to find the numbers.

EXAMPLE 11

Translate each word phrase into an algebraic expression:

a. the difference of 20 and 4
b. the quotient of $10x$ and 3

Solution

a. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

$$\text{the difference of } 20 \textit{ and } 4$$
$$20 \text{ minus } 4$$
$$20-4$$

b. The key word is *quotient*, which tells us the operation is division.

the quotient of $10x$ and 3
divide $10x$ by 3
$10x \div 3$
This can also be written as $10x/3$ or $\frac{10x}{3}$

TRY IT 11.1

Translate the given word phrase into an algebraic expression:

- a. the difference of 47 and 41
- b. the quotient of $5x$ and 2

Show Answer

- a. $47 - 41$
- b. $5x \div 2$

TRY IT 11.2

Translate the given word phrase into an algebraic expression:

- a. the sum of 17 and 19
- b. the product of 7 and x

Show Answer

- a. $17 + 19$
- b. $7x$

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight *more than* means eight added to your present age.

How old were you seven years ago? This is seven years less than your age now. You subtract 7 from your present age. Seven *less than* means seven subtracted from your present age.

EXAMPLE 12

Translate each word phrase into an algebraic expression:

a. Eight more than y
b. Seven less than $9z$

Solution
a. The key words are *more than*. They tell us the operation is addition. *More than* means “added to”.

$\text{Eight more than } y$
$\text{Eight added to } y$
$y + 8$

b. The key words are *less than*. They tell us the operation is subtraction. *Less than* means “subtracted from”.

$\text{Seven less than } 9z$
$\text{Seven subtracted from } 9z$
$9z - 7$

TRY IT 12.1

Translate each word phrase into an algebraic expression:

a. Eleven more than x
b. Fourteen less than $11a$

Show Answer

a. $x + 11$
b. $11a - 14$

TRY IT 12.2

Translate each word phrase into an algebraic expression:

a. 19 more than j
b. 21 less than $2x$

Show Answer

a. $j + 19$

b. $2x - 21$

EXAMPLE 13

Translate each word phrase into an algebraic expression:

a. five times the sum of m and n
b. the sum of five times m and n

Solution
a. There are two operation words: *times* tells us to multiply and *sum* tells us to add. Because we are multiplying 5 times the sum, we need parentheses around the sum of m and n.

five times the sum of m and n

$$5\left(m+n\right)$$

b. To take a sum, we look for the words *of* and *and* to see what is being added. Here we are taking the sum *of* five times m and n.

the sum of five times m and n

$$5m+n$$

Notice how the use of parentheses changes the result. In part a), we add first and in part b), we multiply first.

TRY IT 13.1

Translate the word phrase into an algebraic expression:

a. four times the sum of p and q
b. the sum of four times p and q

Show Answer

a. $4(p + q)$
b. $4p + q$

TRY IT 13.2

Translate the word phrase into an algebraic expression:

a. the difference of two times x and 8
b. two times the difference of x and 8

Show Answer

a. $2x - 8$
b. $2(x - 8)$

Later in this course, we'll apply our skills in algebra to solving equations. We'll usually start by translating a word phrase to an algebraic expression. We'll need to be clear about what the expression will represent. We'll see how to do this in the next two examples.

EXAMPLE 14

The height of a rectangular window is 6 inches less than the width. Let w represent the width of the window. Write an expression for the height of the window.

Solution

Write a phrase about the height.	6 less than the width
Substitute w for the width.	6 less than w
Rewrite 'less than' as 'subtracted from'.	6 subtracted from w
Translate the phrase into algebra.	$w - 6$

TRY IT 14.1

The length of a rectangle is 5 inches less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Show Answer
$w - 5$

TRY IT 14.2

The width of a rectangle is 2 metres greater than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Show Answer

$l + 2$

EXAMPLE 15

Blanca has dimes and quarters in her purse. The number of dimes is 2 less than 5 times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

Write a phrase about the number of dimes.	two less than five times the number of quarters
Substitute q for the number of quarters.	2 less than five times q
Translate 5 *times* q.	2 less than $5q$
Translate the phrase into algebra.	$5q - 2$

TRY IT 15.1

Geoffrey has dimes and quarters in his pocket. The number of dimes is seven less than six times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Show Answer

$6q - 7$

TRY IT 15.2

Lauren has dimes and nickels in her purse. The number of dimes is eight more than four times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Show Answer

$4n + 8$

ACCESS ADDITIONAL ONLINE RESOURCES

- Algebraic Expression Vocabulary

Key Concepts

- **Combine like terms.**
    1. Identify like terms.
    2. Rearrange the expression so like terms are together.
    3. Add the coefficients of the like terms

Glossary

term
A term is a constant or the product of a constant and one or more variables.

coefficient
The constant that multiplies the variable(s) in a term is called the coefficient.

like terms
Terms that are either constants or have the same variables with the same exponents are like terms.

evaluate
To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.

Practice Makes Perfect

Evaluate Algebraic Expressions

In the following exercises, evaluate the expression for the given value.

1. $7x+8$ when $x=2$	2. $9x+7$ when $x=3$
3. $5x-4$ when $x=6$	4. $8x-6$ when $x=7$
5. x^2 when $x=12$	6. x^3 when $x=5$
7. x^5 when $x=2$	8. x^4 when $x=3$
9. 3^x when $x=3$	10. 4^x when $x=2$
11. x^2+3x-7 when $x=4$	12. x^2+5x-8 when $x=6$
13. $2x+4y-5$ when $x=7, y=8$	14. $6x+3y-9$ when $x=6, y=9$
15. $(x-y)^2$ when $x=10, y=7$	16. $(x+y)^2$ when $x=6, y=9$
17. a^2+b^2 when $a=3, b=8$	18. r^2-s^2 when $r=12, s=5$
19. $2l+2w$ when $l=15, w=12$	20. $2l+2w$ when $l=18, w=14$

Identify Terms, Coefficients, and Like Terms

In the following exercises, list the terms in the given expression.

21. $15x^2+6x+2$	22. $11x^2+8x+5$
23. $10y^3+y+2$	24. $9y^3+y+5$

In the following exercises, identify the coefficient of the given term.

25. $8a$	26. $13m$
27. $5r^2$	28. $6x^3$

In the following exercises, identify all sets of like terms.

29. $x^3, 8x, 14, 8y, 5, 8x^3$	30. $6z, 3w^2, 1, 6z^2, 4z, w^2$
31. $9a, a^2, 16ab, 16b^2, 4ab, 9b^2$	32. $3, 25r^2, 10s, 10r, 4r^2, 3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the given expression by combining like terms.

33. $10x + 3x$	34. $15x + 4x$
35. $17a + 9a$	36. $18z + 9z$
37. $4c + 2c + c$	38. $6y + 4y + y$
39. $9x + 3x + 8$	40. $8a + 5a + 9$
41. $7u + 2 + 3u + 1$	42. $8d + 6 + 2d + 5$
43. $7p + 6 + 5p + 4$	44. $8x + 7 + 4x - 5$
45. $10a + 7 + 5a - 2 + 7a - 4$	46. $7c + 4 + 6c - 3 + 9c - 1$
47. $3x^2 + 12x + 11 + 14x^2 + 8x + 5$	48. $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translate English Phrases into Algebraic Expressions

In the following exercises, translate the given word phrase into an algebraic expression.

49. The sum of 8 and 12	50. The sum of 9 and 1
51. The difference of 14 and 9	52. 8 less than 19
53. The product of 9 and 7	54. The product of 8 and 7
55. The quotient of 36 and 9	56. The quotient of 42 and 7
57. The difference of x and 4	58. 3 less than x
59. The product of 6 and y	60. The product of 9 and y
61. The sum of $8x$ and $3x$	62. The sum of $13x$ and $3x$
63. The quotient of y and 3	64. The quotient of y and 8
65. Eight times the difference of y and nine	66. Seven times the difference of y and one
67. Five times the sum of x and y	68. times five less than twice x

In the following exercises, write an algebraic expression.

69. Adele bought a skirt and a blouse. The skirt cost $15 more than the blouse. Let b represent the cost of the blouse. Write an expression for the cost of the skirt.

70. Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let c represent the number of classical CDs. Write an expression for the number of rock CDs.

71. The number of girls in a second-grade class is 4 less than the number of boys. Let b represent the number of boys. Write an expression for the number of girls.

72. Marcella has 6 fewer male cousins than female cousins. Let f represent the number of female cousins. Write an expression for the number of boy cousins.

73. Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

74. Jeannette has $5 and $10 bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Everyday Math

In the following exercises, use algebraic expressions to solve the problem.

75. **Car insurance** Justin's car insurance has a $750 deductible per incident. This means that he pays $750 and his insurance company will pay all costs beyond $750. If Justin files a claim for $2,100, how much will he pay, and how much will his insurance company pay?

76. **Home insurance** Pam and Armando's home insurance has a $2,500 deductible per incident. This means that they pay $2,500 and their insurance company will pay all costs beyond $2,500. If Pam and Armando file a claim for $19,400, how much will they pay, and how much will their insurance company pay?

Writing Exercises

77. Explain why "the sum of *x* and *y*" is the same as "the sum of *y* and *x*," but "the difference of *x* and *y*" is not the same as "the difference of *y* and *x*." Try substituting two random numbers for x and y to help you explain.

78. Explain the difference between "4 times the sum of x and y" and "the sum of 4 times x and y."

Answers

1. 22	3. 26	5. 144
7. 32	9. 27	11. 21
13. 41	15. 9	17. 73
19. 54	21. $15x^2$, $6x$, 2	23. $10y^3$, y, 2
25. 8	27. 5	29. x^3, $8x^3$ and 14, 5
31. $16ab$ and $4ab$; $16b^2$ and $9b^2$	33. $13x$	35. $26a$
37. $7c$	39. $12x + 8$	41. $10u + 3$
43. $12p + 10$	45. $22a + 1$	47. $17x^2 + 20x + 16$
49. $8 + 12$	51. $14 - 9$	53. $9 \cdot 7$
55. $36 \div 9$	57. $x - 4$	59. $6y$
61. $8x + 3x$	63. $\frac{y}{3}$	65. $8\,(y - 9)$
67. $5\,(x + y)$	69. $b + 15$	71. $b - 4$
73. $2n - 7$	75. He will pay $750. His insurance company will pay $1350.	77. Answers will vary.

Attributions

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1.4 Add and Subtract Integers

Learning Objectives

By the end of this section, you will be able to:

- Use negatives and opposites
- Simplify: expressions with absolute value
- Add integers
- Subtract integers

Use Negatives and Opposites

Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The negative numbers are to the left of zero on the number line. See Figure 1.

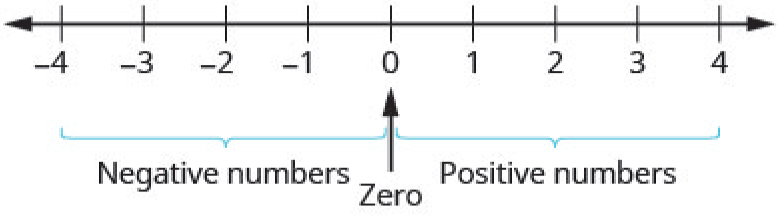


Figure 1 The number line shows the location of positive and negative numbers.

The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See Figure 2.

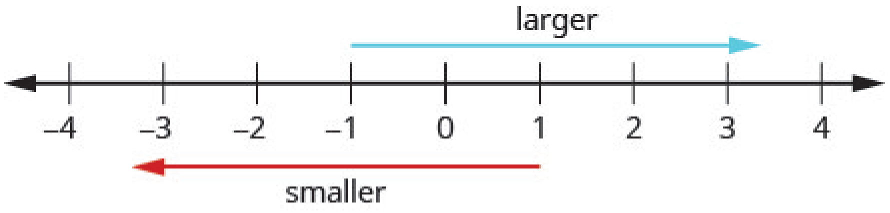


Figure 2 The numbers on a number line increase in value going from left to right and decrease in value going from right to left.

Remember that we use the notation:

a < *b* (read “a is less than b”) when a is to the left of b on the number line.

a > *b* (read “*a* is greater than *b*”) when *a* is to the right of *b* on the number line.

Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in Figure 3 are called the integers. The integers are the numbers $-3, -2, -1, 0, 1, 2, 3$

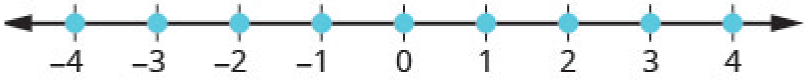
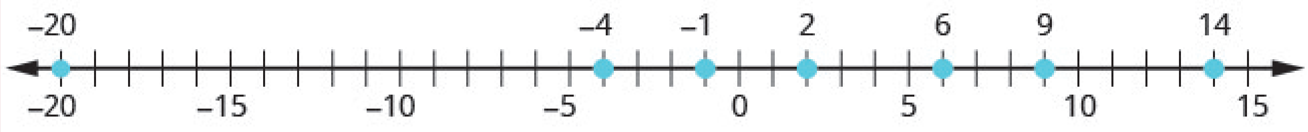


Figure 3 All the marked numbers are called integers.

EXAMPLE 1

Order each of the following pairs of numbers, using < or >: a) $14\_\_\_6$ b) $-1\_\_\_9$ c) $-1\_\_\_-4$ d) $2\_\_\_-20$.

Solution
It may be helpful to refer to the number line shown.



a) 14 is to the right of 6 on the number line.	$14\_\_6$ $14 > 6$
b) −1 is to the left of 9 on the number line.	$-1\_\_9$ $-1 < 9$
c) −1 is to the right of −4 on the number line.	$-1\_\_-4$
d) 2 is to the right of −20 on the number line.	$2\_\_-20$ $2 > -20$

TRY IT 1.1

Order each of the following pairs of numbers, using < or > : a) $15\_\_\_7$ b) $-2\_\_\_5$ c) $-3\_\_\_-7$
d) $5\_\_\_-17$.

Show answer
a) > b) < c) > d) >

TRY IT 1.2

Order each of the following pairs of numbers, using < or > : a) $8\_\_\_13$ b) $3\_\_\_-4$ c) $-5\_\_\_-2$
d) $9\_\_\_-21$.

Show answer
a) < b) > c) < d) >

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called opposite**s**. The opposite of 2 is -2, and the opposite of -2 is 2

Opposite

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

(Figure 4) illustrates the definition.

The opposite of 3 is -3.

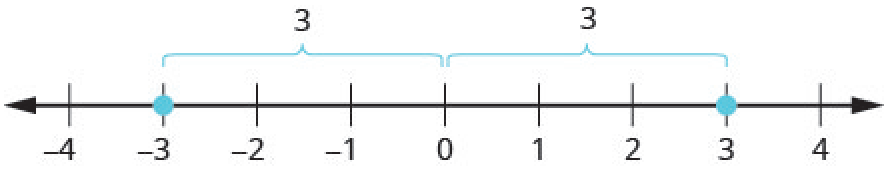


Figure 4

Sometimes in algebra the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol “−” used in three different ways.

$10-4$	Between two numbers, it indicates the operation of subtraction. We read $10-4$ as "10 minus 4."
-8	In front of a number, it indicates a negative number. We read -8 as "negative eight."
$-x$	In front of a variable, it indicates the opposite. We read $-x$ as "the opposite of x."
$-(-2)$	Here there are two "$-$" signs. The one in the parentheses tells us the number is negative 2. The one outside the parentheses tells us to take the opposite of -2. We read $-(-2)$ as "the opposite of negative two."

$10-4$	Between two numbers, it indicates the operation of *subtraction*. We read $10-4$ as "10 minus 4."
-8	In front of a number, it indicates a *negative* number. We read −8 as "negative eight."
$-x$	In front of a variable, it indicates the *opposite*. We read $-x$ as "the opposite of x."
$-(-2)$	Here there are two "−" signs. The one in the parentheses tells us the number is negative 2. The one outside the parentheses tells us to take the *opposite* of −2. We read $-(-2)$ as "the opposite of negative two."

Opposite Notation

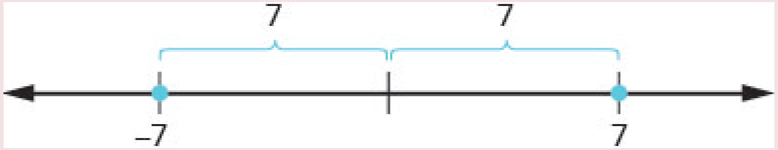
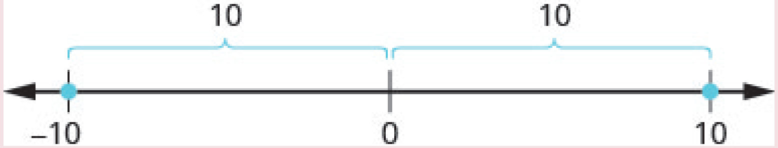
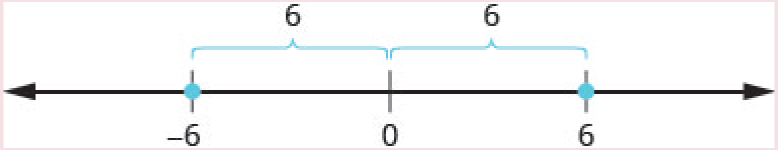
$-a$ means the opposite of the number *a*.

The notation $-a$ is read as "the opposite of *a*."

EXAMPLE 2

Find: a) the opposite of 7 b) the opposite of -10 c) $-(-6)$.

Solution

a) −7 is the same distance from 0 as 7, but on the opposite side of 0.	 The opposite of 7 is −7.
b) 10 is the same distance from 0 as −10, but on the opposite side of 0.	 The opposite of −10 is 10.
c) −(−6)	 The opposite of −(−6) is −6.

TRY IT 2.1

Find: a) the opposite of 4 b) the opposite of -3 c) $-\left(-1\right)$.

Show answer
a)-4 b) 3 c) 1

TRY IT 2.2

Find: a) the opposite of 8 b) the opposite of -5 c) $-\left(-5\right)$.

Show answer
a)-8 b) 5 c) 5

Our work with opposites gives us a way to define the integers.The whole numbers and their opposites are called the integers. The integers are the numbers $-3, -2, -1, 0, 1, 2, 3$

Integers

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

$$-3, -2, -1, 0, 1, 2, 3$$

When evaluating the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether $-x$ is positive or negative. We can see this in Example 3.

EXAMPLE 3

Evaluate a) $-x$, when $x = 8$ b) $-x$, when $x = -8$.

Solution

a.

To evaluate when $x = 8$ means to substitute 8 for x.	
	$-x$
Substitute 8 for x.	$-(8)$
Write the opposite of 8.	-8

b.

To evaluate when $x = -8$ means to substitute -8 for $-x$.	
	$-x$
Substitute -8 for x.	$-(-8)$
Write the opposite of -8.	8

TRY IT 3.1

Evaluate $-n$, when a) $n = 4$ b) $n = -4$.

Show answer

a)-4 b) 4

TRY IT 3.2

Evaluate $-m$, when a) $m = 11$ b) $m = -11$.

Show answer
a)-11 b) 11

Simplify: Expressions with Absolute Value

We saw that numbers such as $2 \text{ and } -2$ are opposites because they are the same distance from 0 on the number line. They are both two units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Absolute Value

The absolute value of a number is its distance from 0 on the number line.

The absolute value of a number *n* is written as $|n|$.

For example,

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure 5 illustrates this idea.

The integers $5 \text{ and } \text{are} \, 5$ units away from 0.

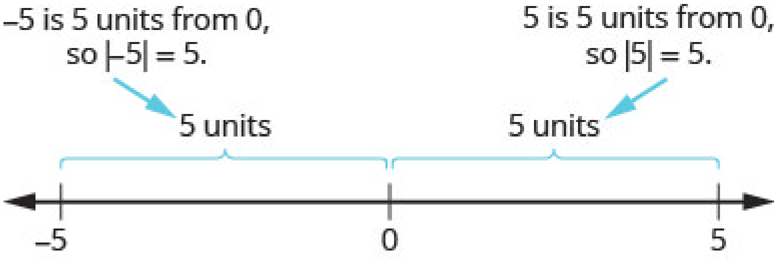


Figure 5

The absolute value of a number is never negative (because distance cannot be negative). The only number with absolute value equal to zero is the number zero itself, because the distance from $0 \text{ to } 0$ on the number line is zero units.

Property of Absolute Value

$|n| \geq 0$ for all numbers

Absolute values are always greater than or equal to zero!

Mathematicians say it more precisely, “absolute values are always non-negative.” Non-negative means greater than or equal to zero.

EXAMPLE 4

Simplify: a) $|3|$ b) $|-44|$ c) $|0|$.

Solution

The absolute value of a number is the distance between the number and zero. Distance is never negative, so the absolute value is never negative.

a) $|3|$

3

b) $|-44|$

44

c) $|0|$

0

TRY IT 4.1

Simplify: a) $|4|$ b) $|-28|$ c) $|0|$.

Show answer

a) 4 b) 28 c) 0

TRY IT 4.2

Simplify: a) $|-13|$ b) $|47|$ c) $|0|$.

Show answer

a) 13 b) 47 c) 0

In the next example, we’ll order expressions with absolute values. Remember, positive numbers are always greater than negative numbers!

EXAMPLE 5

Fill in <, >, or $=$ for each of the following pairs of numbers:

a) $|-5|$___$-|-5|$ b) 8___$-|-8|$c) -9___$-|-9|$ d) –-16___$-|-16|$

Solution

	$\|-5\|$___–$\|-5\|$
a) Simplify. Order.	5 ___ -5
	5 > -5
	$\|-5\|$ > –$\|-5\|$
b) Simplify. Order.	8 ___–$\|-8\|$
	8 ___ -8
	8 > -8
	8 > –$\|-8\|$
c) Simplify. Order.	9 ___ –$\|-9\|$ -9 ___ -9 -9 = -9 -9 = –$\|-9\|$
d) Simplify. Order.	–(-16) ___–$\|-16\|$ 16 ____ -16 16 > -16 –(-16) > –$\|-16\|$

TRY IT 5.1

Fill in <, >, or = for each of the following pairs of numbers: a) $|-9|$___-$|-9|$ b) 2___-$|-2|$ c) -8___ $|-8|$
d) $(-9$___$-|-9|$.

Show answer
a) > b) > c) < d) >

TRY IT 5.2

Fill in <, >, or $=$ for each of the following pairs of numbers: a) 7___-$|-7|$ b) $(-10$___-$|-10|$ c) $|-4|$___$-|-4|$ d) -1___$|-1|$.

Show answer
a) > b) > c) > d) <

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Grouping Symbols

Parentheses	( )
Brackets	[ ]
Braces	{ }
Absolute value	| |

In the next example, we simplify the expressions inside absolute value bars first, just like we do with parentheses.

EXAMPLE 6

Simplify: $24-|19-3\left(6-2\right)|$.

Solution

	$24-|19-3\left(6-2\right)|$
Work inside parentheses first: subtract 2 from 6.	$24-|19-3\left(4\right)|$
Multiply 3(4).	$24-|19-12|$
Subtract inside the absolute value bars.	$24-|7|$
Take the absolute value.	$24-7$
Subtract.	17

TRY IT 6.1

Simplify: $19-\left|11-4\left(3-1\right)\right|$.

Show answer
16

TRY IT 6.2

Simplify: $9-\left|8-4\left(7-5\right)\right|$.

Show answer
9

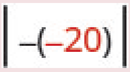
EXAMPLE 7

Evaluate: a) $|x|$ when $x=-35$ b) $|-y|$ when $y=-20$ c) $-|u|$ when $u=12$ d) $-|p|$ when $p=-14$.

Solution
a)$|x|$ when $x=-35$

	$\|x\|$
 Substitute −35 for x.	 $\|-35\|$
Take the absolute value.	35

b)$|-y|$ when $y=-20$

	$\|-y\|$
 Substitute −20 for y.	 $\|-(-20)\|$
Simplify.	$\|20\|$
Take the absolute value.	20

c)$-|u|$ when $u=12$

	$-\lvert u\rvert$
 Substitute 12 for *u*.	 –|12|
Take the absolute value.	-12

d)$-\lvert p\rvert$ when $p=-14$

	$-\lvert p\rvert$
 Substitute –14 for *p*.	 –|–14|
Take the absolute value.	-14

TRY IT 7.1

Evaluate: a) $\lvert x\rvert$ when $x=-17$ b) $\lvert -y\rvert$ when $y=-39$ c) $-\lvert m\rvert$ when $m=22$ d) $-\lvert p\rvert$ when $p=-11$.

Show answer
a) 17 b) 39 c) -22 d) -11

TRY IT 7.2

Evaluate: a) $\lvert y\rvert$ when $y=-23$ b) $\lvert -y\rvert$ when $y=-21$ c) $-\lvert n\rvert$ when $n=37$ d) $-\lvert q\rvert$ when $q=-49$.

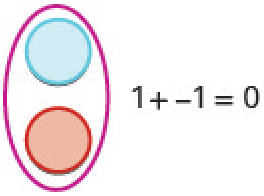
Show answer
a) 23 b) 21 c) -37 d) -49

Add Integers

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

We will use two colour counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

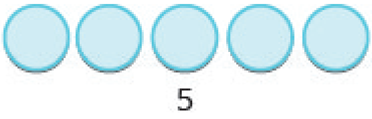
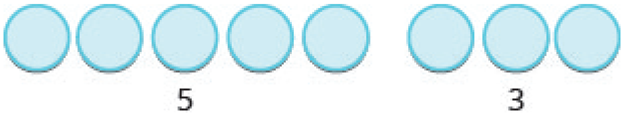
We let one colour (blue) represent positive. The other colour (red) will represent the negatives. If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



We will use the counters to show how to add the four addition facts using the numbers $5, -5$ and $3, -3$.

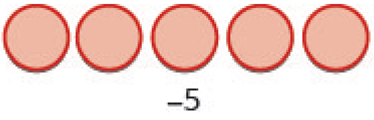
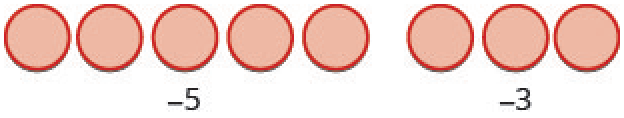
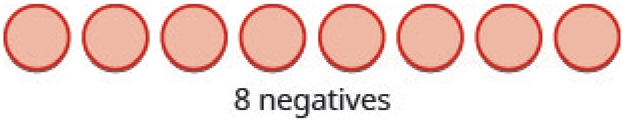
$5+3$ $\quad$ $-5+(-3)$ $\quad$ $-5+3$ $\quad$ $5+(-3)$

To add $5+3$, we realize that $5+3$ means the sum of 5 and 3

We start with 5 positives.	
And then we add 3 positives.	
We now have 8 positives. The sum of 5 and 3 is 8.	

Now we will add $-5+(-3)$. Watch for similarities to the last example $5+3=8$.

To add $-5+(-3)$, we realize this means the sum of $-5 \text{ and } -3$.

We start with 5 negatives.	
And then we add 3 negatives.	
We now have 8 negatives. The sum of −5 and −3 is −8.	

In what ways were these first two examples similar?

- The first example adds 5 positives and 3 positives—both positives.

- The second example adds 5 negatives and 3 negatives—both negatives.

In each case we got 8—either 8 positives or 8 negatives.

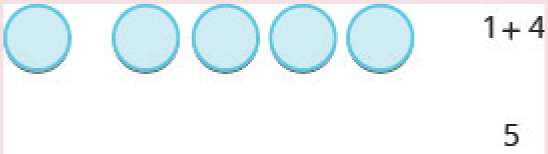
When the signs were the same, the counters were all the same color, and so we added them.

EXAMPLE 8

Add: a) $1+4$ b) $-1+(-4)$.

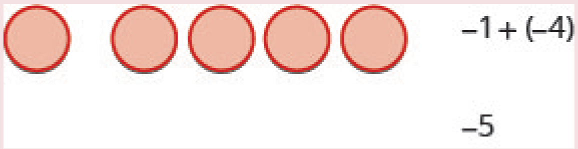
Solution

a)



1 positive plus 4 positives is 5 positives.

b)



1 negative plus 4 negatives is 5 negatives.

TRY IT 8.1

Add: a) $2+4$ b) $-2+(-4)$.

Show answer

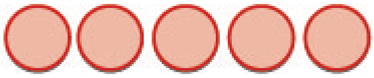
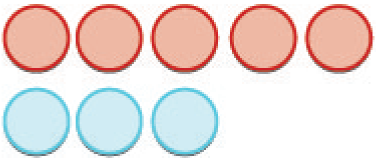
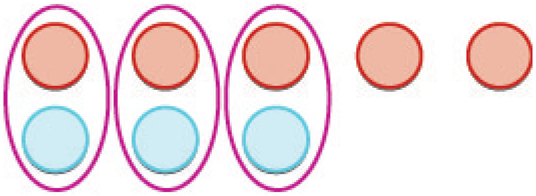
a) 6 b) -6

TRY IT 8.2

Add: a) $2 + 5$ b) $-2 + (-5)$.

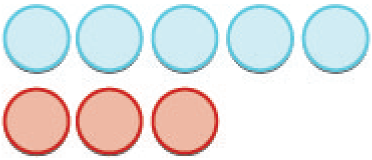
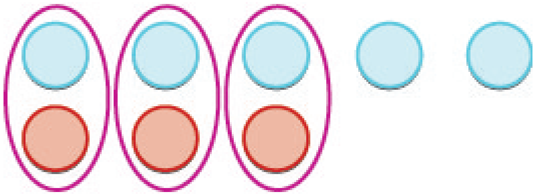
Show answer
a) 7 b) -7

So what happens when the signs are different? Let’s add $-5 + 3$. We realize this means the sum of -5 and 3. When the counters were the same color, we put them in a row. When the counters are a different color, we line them up under each other.

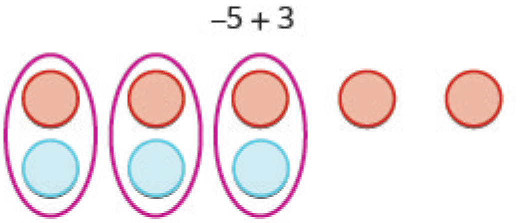
	−5 + 3 means the sum of −5 and 3.
We start with 5 negatives.	
And then we add 3 positives.	
We remove any neutral pairs.	
We have 2 negatives left.	
The sum of −5 and 3 is −2.	−5 + 3 = −2

Notice that there were more negatives than positives, so the result was negative.

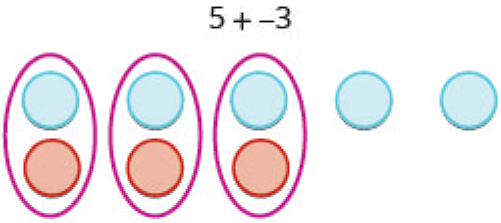
Let’s now add the last combination, $5 + (-3)$.

	5 + (−3) means the sum of 5 and −3.
We start with 5 positives.	
And then we add 3 negatives.	
We remove any neutral pairs.	
We have 2 positives left.	
The sum of 5 and −3 is 2.	5 + (−3) = 2

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.



More negatives – the sum is negative.



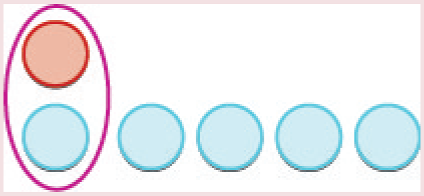
More positives – the sum is positive.

EXAMPLE 9

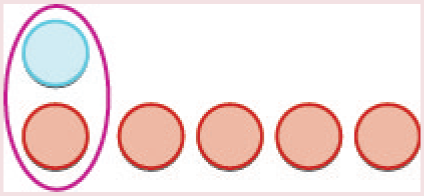
Add: a) $-1+5$ b) $1+(-5)$.

Solution

a)

	−1 + 5
	
There are more positives, so the sum is positive.	4

b)

	1 + (−5)
	
There are more negatives, so the sum is negative.	−4

TRY IT 9.1

Add: a) $-2+4$ b) $2+(-4)$.

Show answer
a) 2 b) -2

TRY IT 9.2

Add: a) $-2+5$ b) $2+(-5)$.

Show answer
a) 3 b) -3

Now that we have added small positive and negative integers with a model, we can visualize the model in our minds to simplify problems with any numbers.

When you need to add numbers such as $37+(-53)$, you really don't want to have to count out 37 blue counters and 53 red counters. With the model in your mind, can you visualize what you would do to solve the problem?

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more red (negative) counters than blue (positive) counters, the sum would be *negative*. How many more red counters would there be? Because $53-37=16$, there are 16 more red counters.

Therefore, the sum of $37+(-53)$ is -16.
$37+(-53)=-16$

Let's try another one. We'll add $-74+(-27)$. Again, imagine 74 red counters and 27 more red counters, so we'd have 101 red counters. This means the sum is -101.

$$-74+(-27)=-101$$

Let's look again at the results of adding the different combinations of $5, -5$ and $3, -3$.

Addition of Positive and Negative Integers

$$\begin{array}{cc} 5+3 & -5+(-3) \\ 8 & -8 \\ \text{both positive, sum positive} & \text{both negative, sum negative} \end{array}$$

When the signs are the same, the counters would be all the same color, so add them.

$$\begin{array}{cc} -5+3 & 5+(-3) \\ -2 & 2 \\ \text{different signs, more negatives, sum negative} & \text{different signs, more positives, sum positive} \end{array}$$

When the signs are different, some of the counters would make neutral pairs, so subtract to see how many are left.

Visualize the model as you simplify the expressions in the following examples.

EXAMPLE 10

Simplify: a) $19+(-47)$ b) $-14+(-36)$.

Solution

a. Since the signs are different, we subtract $19 \text{ from } 47$. The answer will be negative because there are more negatives than positives.

$$\begin{array}{ll} & 19+(-47) \\ \text{Add.} & -28 \end{array}$$

b. Since the signs are the same, we add. The answer will be negative because there are only negatives.

	$-14+(-36)$
Add.	-50

TRY IT 10.1

Simplify: a) $-31+(-19)$ b) $15+(-32)$.

Show answer
a)-50 b)-17

TRY IT 10.2

Simplify: a) $-42+(-28)$ b) $25+(-61)$.

Show answer
a)-70 b)-36

The techniques used up to now extend to more complicated problems, like the ones we've seen before. Remember to follow the order of operations!

EXAMPLE 11

Simplify: $-5+3\left(-2+7\right)$.

Solution

	$-5+3\left(-2+7\right)$
Simplify inside the parentheses.	$-5+3\left(5\right)$
Multiply.	$-5+15$
Add left to right.	10

TRY IT 11.1

Simplify: $-2+5\left(-4+7\right)$.

Show answer
13

TRY IT 11.2

Simplify: $-4+2\left(-3+5\right)$.

Show answer
0

Subtract Integers

We will continue to use counters to model the subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read “$5-3$” as “5 take away 3.” When you use counters, you can think of subtraction the same way!

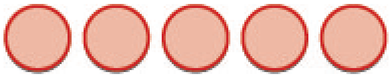
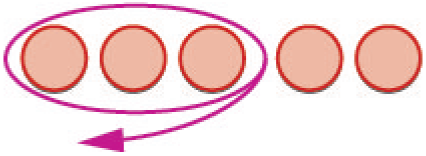
We will model the four subtraction facts using the numbers 5 and 3.
$5-3$ $-5-\left(-3\right)$ $-5-3$ $5-\left(-3\right)$

To subtract $5-3$, we restate the problem as “5 take away 3.”

We start with 5 positives.	
We ‘take away’ 3 positives.	
We have 2 positives left.	
The difference of 5 and 3 is 2.	2

Now we will subtract $-5-\left(-3\right)$. Watch for similarities to the last example $5-3=2$.

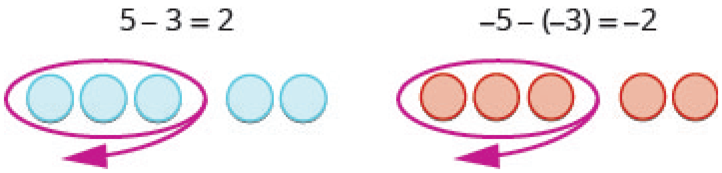
To subtract $-5-\left(-3\right)$, we restate this as “-5 take away -3”

We start with 5 negatives.	
We ‘take away’ 3 negatives.	
We have 2 negatives left.	
The difference of −5 and −3 is −2.	−2

Notice that these two examples are much alike: The first example, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the “take away” model of subtraction was easy to apply.



EXAMPLE 12

Subtract: a) $7-5$ b) $-7-(-5)$.

Solution

a) Take 5 positive from 7 positives and get 2 positives.	$7-5$ 2
b) Take 5 negatives from 7 negatives and get 2 negatives.	$-7-(-5)$ -2

TRY IT 12.1

Subtract: a) $6-4$ b) $-6-(-4)$.

Show answer

a) 2 b) -2

TRY IT 12.2

Subtract: a) $7 - 4$ b) $-7 - (-4)$.

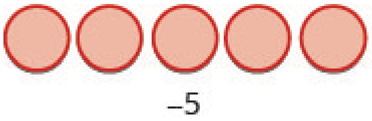
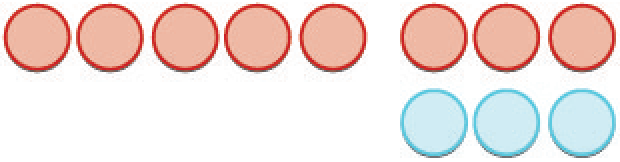
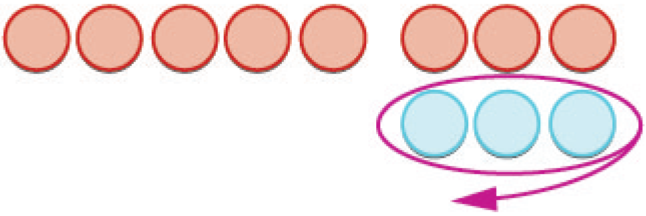
Show answer
a) 3 b) -3

What happens when we have to subtract one positive and one negative number? We'll need to use both white and red counters as well as some neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

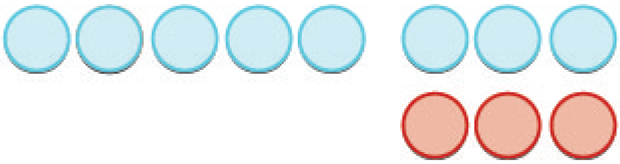
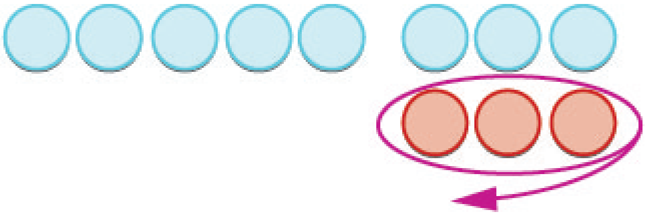
- To subtract $-5 - 3$, we restate it as -5 take away 3.

We start with 5 negatives. We need to take away 3 positives, but we do not have any positives to take away.

Remember, a neutral pair has value zero. If we add 0 to 5 its value is still 5. We add neutral pairs to the 5 negatives until we get 3 positives to take away.

	−5 − 3 means −5 take away 3.
We start with 5 negatives.	
We now add the neutrals needed to get 3 positives.	
We remove the 3 positives.	
We are left with 8 negatives.	
The difference of −5 and 3 is −8.	−5 − 3 = −8

And now, the fourth case, $5-(-3)$. We start with 5 positives. We need to take away 3 negatives, but there are no negatives to take away. So we add neutral pairs until we have 3 negatives to take away.

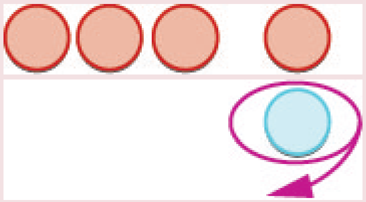
	5 − (−3) means 5 take away −3.
We start with 5 positives.	
We now add the needed neutrals pairs.	
We remove the 3 negatives.	
We are left with 8 positives.	
The difference of 5 and −3 is 8.	5 − (−3) = 8

EXAMPLE 13

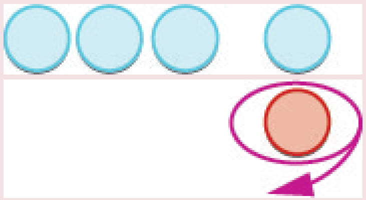
Subtract: a) $-3-1$ b) $3-(-1)$.

Solution

a)

Take 1 positive from the one added neutral pair.		−3 − 1 −4

b)

Take 1 negative from the one added neutral pair.		3 − (−1) 4

TRY IT 13.1

Subtract: a) $-6 - 4$ b) $6 - (-4)$.

Show answer
a)-10 b) 10

TRY IT 13.2

Subtract: a) $-7 - 4$ b) $7 - (-4)$.

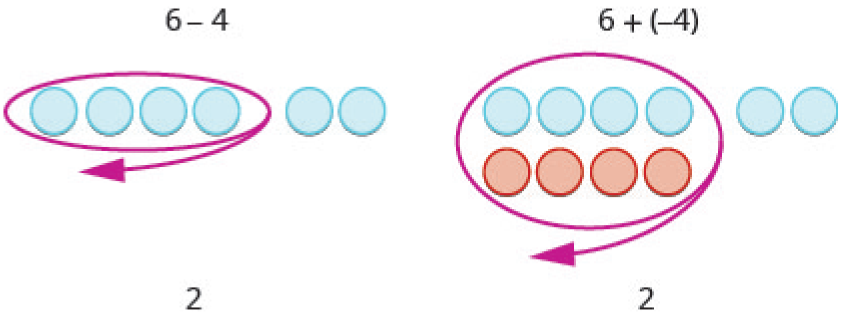
Show answer
a)-11 b) 11

Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In Example 13, $-3 - 1$ is the same as $-3 + (-1)$ and $3 - (-1)$ is the same as $3 + 1$. You will often see this idea, the subtraction property, written as follows:

Subtraction Property

$a - b = a + (-b)$
Subtracting a number is the same as adding its opposite.

Look at these two examples.



$6 - 4$ gives the same answer as $6 + (-4)$.

Of course, when you have a subtraction problem that has only positive numbers, like $6 - 4$, you just do the subtraction. You already knew how to subtract $6 - 4$ long ago. But *knowing* that $6 - 4$ gives the same answer as $6 + (-4)$ helps when you are subtracting negative numbers. Make sure that you understand how $6 - 4$ and $6 + (-4)$ give the same results!

EXAMPLE 14

Simplify: a) $13-8$ and $13+(-8)$ b) $-17-9$ and $-17+(-9)$.

Solution

a) Subtract.	$13-8$ 5	$13+(-8)$ 5
b) Subtract.	$-17-9$ -29	$-17+(-9)$ -26

TRY IT 14.1

Simplify: a) $21-13$ and $21+(-13)$ b) $-11-7$ and $-11+(-7)$.

Show answer

a) 8 b) -18

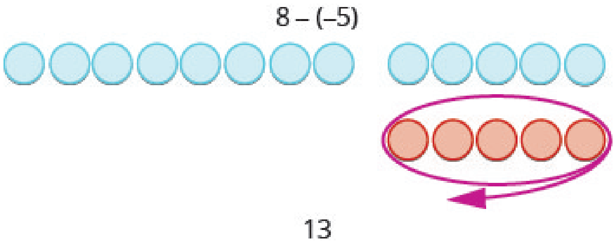
TRY IT 14.2

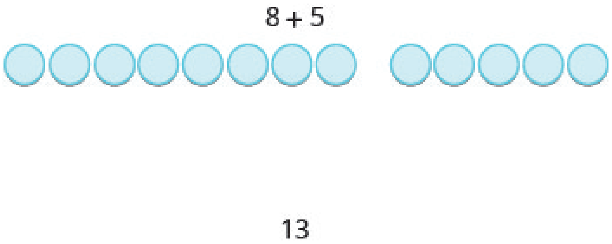
Simplify: a) $15-7$ and $15+(-7)$ b) $-14-8$ and $-14+(-8)$.

Show answer

a) 8 b)-22

Look at what happens when we subtract a negative.





$8-(-5)$ gives the same answer as $8+5$

Subtracting a negative number is like adding a positive!

You will often see this written as $a-(-b)=a+b$.

Does that work for other numbers, too? Let's do the following example and see.

EXAMPLE 15

Simplify: a) $9-(-15)$ and $9+15$ b) $-7-(-4)$ and $-7+4$.

Solution

a)

$$\begin{array}{lcc} & 9-(-15) & 9+15 \\ \text{Subtract.} & 24 & 24 \end{array}$$

b)

$$\begin{array}{lcc} & -7-(-4) & -7+4 \\ \text{Subtract.} & -3 & -3 \end{array}$$

a) Subtract.	$9-(-15)$ 24	$9+15$ 24
b) Subtract.	$-7-(-4)$ -3	$-7+4$ -3

TRY IT 15.1

Simplify: a) $6-(-13)$ and $6+13$ b) $-5-(-1)$ and $-5+1$.

Show answer

a)19 b)-4

TRY IT 15.2

Simplify: a) $4-(-19)$ and $4+19$ b) $-4-(-7)$ and $-4+7$.

Show answer

a) 23 b) 3

Let's look again at the results of subtracting the different combinations of $5, -5$ and $3, -3$.

Subtraction of Integers

$$\begin{array}{cc} 5-3 & -5-(-3) \\ 2 & -2 \\ 5\text{ positives take away }3\text{ positives} & 5\text{ negatives take away }3\text{ negatives} \\ 2\text{ positives} & 2\text{ negatives} \end{array}$$

When there would be enough counters of the colour to take away, subtract.

$$\begin{array}{cc} -5-3 & 5-(-3) \\ -8 & 8 \\ 5\text{ negatives, want to take away }3\text{ positives} & 5\text{ positives, want to take away }3\text{ negatives} \\ \text{need neutral pairs} & \text{need neutral pairs} \end{array}$$

When there would be not enough counters of the colour to take away, add.

What happens when there are more than three integers? We just use the order of operations as usual.

EXAMPLE 16

Simplify: $7-(-4-3)-9$.

Solution

	$7-(-4-3)-9$
Simplify inside the parentheses first.	$7-(-7)-9$
Subtract left to right.	$14-9$
Subtract.	5

TRY IT 16.1

Simplify: $8-(-3-1)-9$.

Show answer

3

TRY IT 16.2

Simplify: $12-(-9-6)-14$.

Show answer
13

Access these online resources for additional instruction and practice with adding and subtracting integers. You will need to enable Java in your web browser to use the applications.

- Add Colored Chip
- Subtract Colored Chip

Key Concepts

- **Addition of Positive and Negative Integers**

$$\begin{array}{ll} 5+3 & -5+(-3) \\ \quad 8 & \quad -8 \\ \text{both positive,} & \text{both negative,} \\ \text{sum positive} & \text{sum negative} \end{array}$$

$$\begin{array}{ll} -5+3 & 5+(-3) \\ \quad -2 & \quad 2 \\ \text{different signs,} & \text{different signs,} \\ \text{more negatives} & \text{more positives} \\ \text{sum negative} & \text{sum positive} \end{array}$$

- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!
- **Subtraction of Integers**

$5-3$ 2 5 positives take away 3 positives 2 positives	$-5-(-3)$ -2 5 negatives take away 3 negatives 2 negatives
$-5-3$ -8 5 negatives, want to subtract 3 positives need neutral pairs	$5-(-3)$ 8 5 positives, want to subtract 3 negatives need neutral pairs

- **Subtraction Property:** Subtracting a number is the same as adding its opposite.

Glossary

absolute value

The absolute value of a number is its distance from 0 on the number line. The absolute value of a number n is written as $|n|$.

integers

The whole numbers and their opposites are called the integers: …−3, −2, −1, 0, 1, 2, 3…

opposite

The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero: $-a$ means the opposite of the number. The notation $-a$ is read “the opposite of a.”

Practice Makes Perfect

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using < or >.

1. a) 9___4 b) -3___6 c) -8___-2 d) 1___-10	2. a) -7___3 b) -10___-5 c) 2___-6 d) 8___9

In the following exercises, find the opposite of each number.

3. a) 2 b) -6	4. a) 9 b) -4

In the following exercises, simplify.

5. $-(-4)$	6. $-(-8)$
7. $-(-15)$	8. $-(-11)$

In the following exercises, evaluate.

9. $-c$ when a) $c = 12$ b) $c = -12$	10. $-d$ when a) $d = 21$ b) $d = -21$

Simplify Expressions with Absolute Value

In the following exercises, simplify.

11. a) $\|-32\|$ b) $\|0\|$ c) $\|16\|$	12. a) $\|0\|$ b) $\|-40\|$ c) $\|22\|$

In the following exercises, fill in <, >, or = for each of the following pairs of numbers.

13. a) -6___$\|-6\|$ b) $-\|-3\|$___-3	14. a) $\|-5\|$___$-\|-5\|$ b) 9___$-\|-9\|$

In the following exercises, simplify.

15. $-(-5)$ and $-\|-5\|$	16. $-\|-9\|$ and $-(-9)$
17. $8\|-7\|$	18. $5\|-5\|$
19. $\|15-7\|-\|14-6\|$	20. $\|17-8\|-\|13-4\|$
21. $18-\|2(8-3)\|$	22. $18-\|3(8-5)\|$

In the following exercises, evaluate.

23. a) $-\|p\|$ when $p=19$ b) $-\|q\|$ when $q=-33$	24. a) $-\|a\|$ when $a=60$ b) $-\|b\|$ when $b=-12$

Add Integers

In the following exercises, simplify each expression.

25. $-21+(-59)$	26. $-35+(-47)$
27. $48+(-16)$	28. $34+(-19)$
29. $-14+(-12)+4$	30. $-17+(-18)+6$
31. $135+(-110)+83$	32. 6 $-38+27+(-8)+126$
33. $19+2(-3+8)$	34. $24+3(-5+9)$

Subtract Integers

In the following exercises, simplify.

35. $8-2$	36. $-6-(-4)$
37. $-5-4$	38. $-7-2$
39. $8-(-4)$	40. $7-(-3)$
41. a) $44-28$ b) $44+(-28)$	42. a) $35-16$ b) $35+(-16)$
43. a) $27-(-18)$ b) $27+18$	44. a) $46-(-37)$ b) $46+37$

In the following exercises, simplify each expression.

45. $15-(-12)$	46. $14-(-11)$
47. $48-87$	48. $45-69$
49. $-17-42$	50. $-19-46$
51. $-103-(-52)$	52. $-105-(-68)$
53. $-45-(54)$	54. $-58-(-67)$
55. $8-3-7$	56. $9-6-5$
57. $-5-4+7$	58. $-3-8+4$
59. $-14-(-27)+9$	60. $64+(-17)-9$
61. $(2-7)-(3-8)\,(2)$	62. $(1-8)-(2-9)$
63. $-(6-8)-(2-4)$	64. $-(4-5)-(7-8)$
65. $25-[10-(3-12)]$	66. $32-[5-(15-20)]$
67. $6.3-4.3-7.2$	68. $5.7-8.2-4.9$
69. 5^2-6^2	70. 6^2-7^2

Everyday Math

71. **Elevation** The highest elevation in North America is Mount McKinley, Alaska, at 20,320 feet above sea level. The lowest elevation is Death Valley, California, at 282 feet below sea level.

Use integers to write the elevation of:

a) Mount McKinley.
b) Death Valley.

72. **Extreme temperatures** The highest recorded temperature on Earth was 58° Celsius, recorded in the Sahara Desert in 1922. The lowest recorded temperature was 90° below 0° Celsius, recorded in Antarctica in 1983

Use integers to write the:

a) highest recorded temperature.

b) lowest recorded temperature.

73. **Provincial budgets** For 2019 the province of Quebec estimated it would have a budget surplus of $5.6 million. That same year, Alberta estimated it would have a budget deficit of $7.5 million.

Use integers to write the budget of:

a) Quebec.
b) Alberta.

74. **University enrolments**The number of international students enrolled in Canadian postsecondary institutions has been on the rise for two decades, with their numbers increasing at a higher rate than that of Canadian students. Enrolments of international students rose by 24,315 from 2015 to 2017. Meanwhile, there was a slight decline in the number of Canadian students, by 912 for the same fiscal years.

Use integers to write the change:

a) in International Student enrolment from Fall 2015 to Fall 2017.

b) in Canadian student enrolment from Fall 2015 to Fall 2017.

75. **Stock Market** The week of September 15, 2008 was one of the most volatile weeks ever for the US stock market. The closing numbers of the Dow Jones Industrial Average each day were:

Monday	-504
Tuesday	$+142$
Wednesday	-449
Thursday	$+410$
Friday	$+369$

What was the overall change for the week? Was it positive or negative?

76. **Stock Market** During the week of June 22, 2009, the closing numbers of the Dow Jones Industrial Average each day were:

Monday	-201
Tuesday	-16
Wednesday	-23
Thursday	$+172$
Friday	-34

What was the overall change for the week? Was it positive or negative?

Writing Exercises

77. Give an example of a negative number from your life experience.	78. What are the three uses of the “ $-$ ” sign in algebra? Explain how they differ.
79. Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 is positive.	80. Give an example from your life experience of adding two negative numbers.

Answers

1. a) > b) < c) < d) >	3. a)-2b) 6	5. 4
7. 15	9. a)-12b) 12	11. a) 32 b) 0 c) 16
13. a) < b) $=$	15. $5, -5$	17. 56
19. 0	21. 8	23. a) -19 b) -33
25. -80	27. 32	29. -22
31. 108	33. 29	35. 6
37. -9	39. 12	41. a) 16 b) 16
43. a) 45 b) 45	45. 27	47. -39
49. -59	51. -51	53. -99
55. -2	57. -2	59. 22
61. -15	63. 0	65. 6
67. -5.2	69. -11	71. a) 20,320 b) -282
73. a) $5.6 million b) $-\$7.5$ million	75. -32	77. Answers may vary
79. Answers may vary		

Attributions

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1.5 Multiply and Divide Integers

Learning Objectives

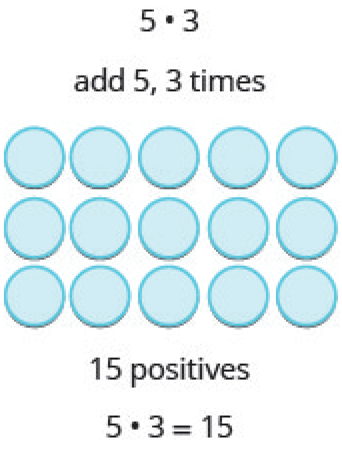
By the end of this section, you will be able to:

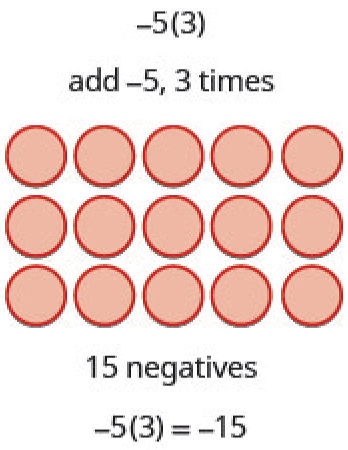
- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate English phrases to algebraic expressions
- Use integers in applications

Multiply Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let’s look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

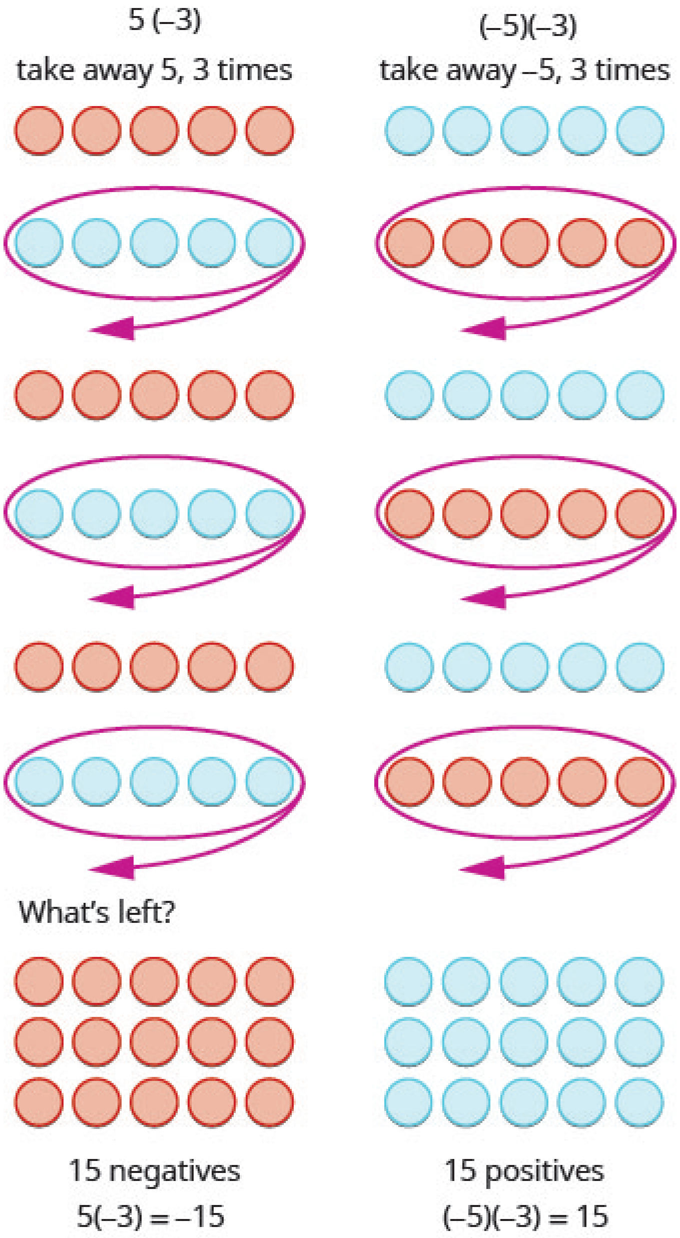
We remember that $a \cdot b$ means add *a*, *b* times. Here, we are using the model just to help us discover the pattern.





The next two examples are more interesting.

What does it mean to multiply 5 by -3? It means subtract 5, 3 times. Looking at subtraction as “taking away," it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace. Then we take away 5 three times.



In summary:

$$\begin{aligned} 5\cdot 3 &= 15 & \qquad -5\,(3) &= -15 \\ 5\,(-3) &= -15 & \qquad (-5)\,(-3) &= 15 \end{aligned}$$

Notice that for multiplication of two signed numbers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below

Multiplication of Signed Numbers

For multiplication of two signed numbers:

Same signs	**Product**	**Example**
Two positives Two negatives	Positive Positive	$7 \cdot 4 = 28$ $-8(-6) = 48$

Different signs	**Product**	**Example**
Positive \cdot negative Negative \cdot positive	Negative Negative	$7(-9) = -63$ $-5 \cdot 10 = -50$

EXAMPLE 1

Multiply: a) $-9 \cdot 3$ b) $-2(-5)$ c) $4(-8)$ d) $7 \cdot 6$.

Solution

a) Multiply, noting that the signs are different so the product is negative.	$-9 \cdot 3$ -27
b) Multiply, noting that the signs are the same so the product is positive.	$-2(-5)$ 10
c) Multiply, with different signs.	$4(-8)$ -32
d) Multiply, with same signs.	$7 \cdot 6$ 42

TRY IT 1.1

Multiply: a) $-6 \cdot 8$ b) $-4(-7)$ c) $9(-7)$ d) $5 \cdot 12$.

Show answer

a)-48 b) 28 c) -63 d) 60

TRY IT 1.2

Multiply: a) $-8 \cdot 7$ b) $-6\left(-9\right)$ c) $7\left(-4\right)$ d) $3 \cdot 13$.

Show answer
a)-56 b) 54 c) -28 d) 39

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

$$\begin{array}{lcc} & -1 \cdot 4 & -1\left(-3\right) \\ \text{Multiply.} & -4 & 3 \\ & -4\,\text{is the opposite of}\,4. & 3\,\text{is the opposite of}\,-3. \end{array}$$

Each time we multiply a number by -1, we get its opposite!

Multiplication by -1

$-1a =$ -a

Multiplying a number by -1 gives its opposite.

EXAMPLE 2

Multiply: a) $-1 \cdot 7$ b) $-1\left(-11\right)$.

Solution

a) Multiply, noting that the signs are different so the product is negative.	$-1 \cdot 7$ -7 $-7\,\text{is the opposite of}\,7.$
b) Multiply, noting that the signs are the same so the product is positive.	$-1\left(-11\right)$ 11 $11\,\text{is the opposite of}\,-11.$

TRY IT 2.1

Multiply: a) $-1 \cdot 9$ b) $-1 \cdot (-17)$.

Show answer
a)-9 b) 17

TRY IT 2.2

Multiply: a) $-1 \cdot 8$ b) $-1 \cdot (-16)$.

Show answer
a)-8 b) 16

Divide Integers

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \cdot 3 = 15$. In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.

$$\begin{array}{rclrcl} 5 \cdot 3 & = & 15 \text{ so } 15 \div 3 & = & 5 & \qquad -5(3) & = & -15 \text{ so } -15 \div 3 & = & -5 \\ (-5)(-3) & = & 15 \text{ so } 15 \div (-3) & = & -5 & \qquad 5(-3) & = & -15 \text{ so } -15 \div (-3) & = & 5 \end{array}$$

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

Multiplication and Division of Signed Numbers

For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Same signs	**Result**
Two positives	Positive
Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	**Result**
Positive and negative	Negative
Negative and positive	Negative

If the signs are different, the result is negative.

EXAMPLE 3

Divide: a) $-27 \div 3$ b) $-100 \div (-4)$.

Solution

a) Divide. With different signs, the quotient is negative.	$-27 \div 3$ -9
b) Divide. With signs that are the same, the quotient is positive.	$-100 \div (-4)$ 25

TRY IT 3.1

Divide: a) $-42 \div 6$ b) $-117 \div (-3)$.

Show answer
a)-7 b) 39

TRY IT 3.2

Divide: a) $-63 \div 7$ b) $-115 \div (-5)$.

Show answer
a)-9 b) 23

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember My Dear Aunt Sally?

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

EXAMPLE 4

Simplify: $7\left(-2\right)+4\left(-7\right)-6$.

Solution

	$7\left(-2\right)+4\left(-7\right)-6$
Multiply first.	$-14+\left(-28\right)-6$
Add.	$-42-6$
Subtract.	-48

TRY IT 4.1

Simplify: $8\left(-3\right)+5\left(-7\right)-4$.

Show answer
-63

TRY IT 4.2

Simplify: $9\left(-3\right)+7\left(-8\right)-1$.

Show answer
-84

EXAMPLE 5

Simplify: a) ${\left(-2\right)}^{4}$ b) -2^{4}.

Solution

a) Write in expanded form. Multiply. Multiply. Multiply.	${\left(-2\right)}^{4}$ $\left(-2\right)\left(-2\right)\left(-2\right)\left(-2\right)$ $4\left(-2\right)\left(-2\right)$ $-8\left(-2\right)$ 16
b) Write in expanded form. We are asked to find the opposite of 2^{4}. Multiply. Multiply. Multiply.	-2^{4} - $\left(2\cdot 2\cdot 2\cdot 2\right)$ - $\left(4\cdot 2\cdot 2\right)$ - $\left(8\cdot 2\right)$ 16

Notice the difference in parts a) and b). In part a), the exponent means to raise what is in the parentheses, the $\left(-2\right)$ to the 4^{th} power. In part b), the exponent means to raise just the 2 to the 4^{th} power and then take the opposite.

TRY IT 5.1

Simplify: a) ${\left(-3\right)}^{4}$ b) -3^{4}.

Show answer
a) 81 b) -81

TRY IT 5.2

Simplify: a) $(-7)^2$ b) -7^2.

Show answer
a) 49 b) -49

The next example reminds us to simplify inside parentheses first.

EXAMPLE 6

Simplify: $12 - 3(9 - 12)$.

Solution

	$12 - 3(9 - 12)$
Subtract in parentheses first.	$12 - 3(-3)$
Multiply.	$12 - (-9)$
Subtract.	21

TRY IT 6.1

Simplify: $17 - 4(8 - 11)$.

Show answer
29

TRY IT 6.2

Simplify: $16 - 6(7 - 13)$.

Show answer
52

EXAMPLE 7

Simplify: $8\left(-9\right)\div{\left(-2\right)}^{3}$.

Solution

	$8\left(-9\right)\div{\left(-2\right)}^{3}$
Exponents first.	$8\left(-9\right)\div\left(-8\right)$
Multiply.	$-72\div\left(-8\right)$
Divide.	9

TRY IT 7.1

Simplify: $12\left(-9\right)\div{\left(-3\right)}^{3}$.

Show answer
4

TRY IT 7.2

Simplify: $18\left(-4\right)\div{\left(-2\right)}^{3}$.

Show answer
9

EXAMPLE 8

Simplify: $-30\div2+\left(-3\right)\left(-7\right)$.

Solution

	$-30 \div 2 + (-3)(-7)$
Multiply and divide left to right, so divide first.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

TRY IT 8.1

Simplify: $-27 \div 3 + (-5)(-6)$.

Show answer
21

TRY IT 8.2

Simplify: $-32 \div 4 + (-2)(-7)$.

Show answer
6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

EXAMPLE 9

When $n = -5$, evaluate: a) $n + 1$ b) -$n + 1$.

Solution
a)

	$n+1$
Substitute -5 for n.	$-5+1$
Simplify.	-4

b)

	$-n+1$
Substitute -5 for n.	$-(-5)+1$
Simplify.	$5+1$
Add.	6

TRY IT 9.1

When $n=-8$, evaluate a) $n+2$ b) -$n+2$.

Show answer

a)-6 b) 10

TRY IT 9.2

When $y=-9$, evaluate a) $y+8$ b) -$y+8$.

Show answer

a)-1 b) 17

EXAMPLE 10

Evaluate $(x+y)^2$ when $x=-18$ and $y=24$.

Solution

	$(x+y)^2$
Substitute –18 for *x* and 24 for *y*.	$(-18+24)^2$
Add inside parenthesis.	$(6)^2$
Simplify.	36

TRY IT 10.1

Evaluate $(x+y)^2$ when $x=-15$ and $y=29$.

Show answer

196

TRY IT 10.2

Evaluate $(x+y)^3$ when $x=-8$ and $y=10$.

Show answer

8

EXAMPLE 11

Evaluate $20-z$ when a) $z=12$ and b) $z=-12$.

Solution

a)

	$20-z$
Substitute 12 for *z*.	$20-12$
Subtract.	8

b)

	$20 - z$
Substitute -12 for z.	$20 - (-12)$
Subtract.	32

TRY IT 11.1

Evaluate: $17 - k$ when a) $k = 19$ and b) $k = -19$.

Show answer
a)-2 b) 36

TRY IT 11.2

Evaluate: $-5 - b$ when a) $b = 14$ and b) $b = -14$.

Show answer
a)-19 b) 9

EXAMPLE 12

Evaluate: $2x^2 + 3x + 8$ when $x = 4$.

Solution
Substitute 4 for x. Use parentheses to show multiplication.

	$2x^2 + 3x + 8$
Substitute.	$2(4)^2 + 3(4) + 8$
Evaluate exponents.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

TRY IT 12.1

Evaluate: $3x^2 - 2x + 6$ when $x = -3$.

Show answer

39

TRY IT 12.2

Evaluate: $4x^2 - x - 5$ when $x = -2$.

Show answer

13

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

EXAMPLE 13

Translate and simplify: the sum of 8 and -12, increased by 3

Solution

	the **sum** of 8 and -12, increased by 3.
Translate.	$[8 + (-12)] + 3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4) + 3$
Add.	-1

TRY IT 13.1

Translate and simplify the sum of 9 and -16, increased by 4

Show answer

$(9+(-16))+4-3$

TRY IT 13.2

Translate and simplify the sum of -8 and -12, increased by 7

Show answer
$(-8+(-12))+7-13$

When we first introduced the operation symbols, we saw that the expression may be read in several ways. They are listed in the chart below.

$a-b$
a minus b the difference of a and b b subtracted from a b less than a

Be careful to get *a* and *b* in the right order!

EXAMPLE 14

Translate and then simplify a) the difference of 13 and -21 b) subtract 24 from -19.

Solution

a) Translate. Simplify.	the difference of 13 and -21 $13-(-21)$ 34
b) Translate. Remember, “subtract b from a means $a-b$. Simplify.	subtract 24 from -19 $-19-24$ -43

TRY IT 14.1

Translate and simplify a) the difference of 14 and -23 b) subtract 21 from -17.

Show answer

a)$14-\left(-23\right);37$ b)$-17-21;-38$

TRY IT 14.2

Translate and simplify a) the difference of 11 and -19 b) subtract 18 from -11.

Show answer

a)$11-\left(-19\right);30$ b)$-11-18;-29$

Once again, our prior work translating English to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the key word for multiplication is “product” and for division is “quotient.”

EXAMPLE 15

Translate to an algebraic expression and simplify if possible: the product of -2 and 14

Solution

	$\text{the product of }-2\text{ and }14$
Translate.	$\left(-2\right)\left(14\right)$
Simplify.	-28

TRY IT 15.1

Translate to an algebraic expression and simplify if possible: the product of -5 and 12

Show answer

$-5\left(12\right);-60$

TRY IT 15.2

Translate to an algebraic expression and simplify if possible: the product of 8 and -13.

Show answer
$-8\left(13\right);-104$

EXAMPLE 16

Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution

	$\text{the quotient of }-56\text{ and }-7$
Translate.	$-56\div\left(-7\right)$
Simplify.	8

TRY IT 16.1

Translate to an algebraic expression and simplify if possible: the quotient of -63 and -9.

Show answer
$-63\div\left(-9\right);7$

TRY IT 16.2

Translate to an algebraic expression and simplify if possible: the quotient of -72 and -9.

Show answer
$-72\div\left(-9\right);8$

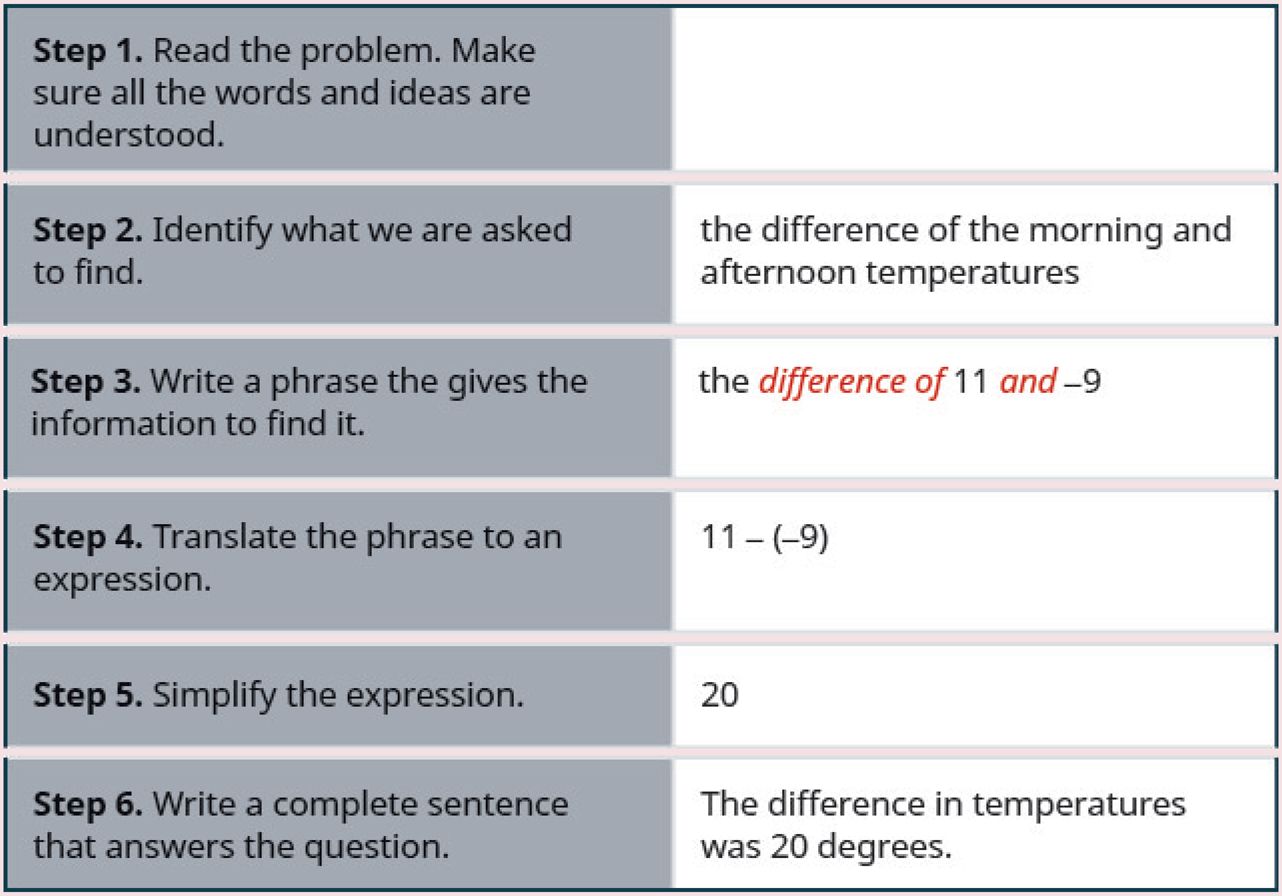
Use Integers in Applications

We’ll outline a plan to solve applications. It’s hard to find something if we don’t know what we’re looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we’ll write a phrase that gives the information to find it. We’ll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

EXAMPLE 17

The temperature in Sparwood, British Columbia, one morning was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference of the morning and afternoon temperatures?

Solution



Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the difference of the morning and afternoon temperatures
Step 3. Write a phrase the gives the information to find it.	the *difference of* 11 *and* –9
Step 4. Translate the phrase to an expression.	11 – (–9)
Step 5. Simplify the expression.	20
Step 6. Write a complete sentence that answers the question.	The difference in temperatures was 20 degrees.

TRY IT 17.1

The temperature in Whitehorse, Yukon, one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

Show answer

The difference in temperatures was 45 degrees.

TRY IT 17.2

The temperature in Quesnel, BC, was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?

Show answer
The difference in temperatures was 9 degrees.

HOW TO: Apply a Strategy to Solve Applications with Integers

1. Read the problem. Make sure all the words and ideas are understood
2. Identify what we are asked to find.
3. Write a phrase that gives the information to find it.
4. Translate the phrase to an expression.
5. Simplify the expression.
6. Answer the question with a complete sentence.

EXAMPLE 18

The Mustangs football team received three penalties in the third quarter. Each penalty gave them a loss of fifteen yards. What is the number of yards lost?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the number of yards lost
Step 3. Write a phrase that gives the information to find it.	three times a 15-yard penalty
Step 4. Translate the phrase to an expression.	$3\left(-15\right)$
Step 5. Simplify the expression.	-45
Step 6. Answer the question with a complete sentence.	The team lost 45 yards.

TRY IT 18.1

The Bears played poorly and had seven penalties in the game. Each penalty resulted in a loss of 15 yards. What is the number of yards lost due to penalties?

Show answer
The Bears lost 105 yards.

TRY IT 18.2

Bill uses the ATM on campus because it is convenient. However, each time he uses it he is charged a $2 fee. Last month he used the ATM eight times. How much was his total fee for using the ATM?

Show answer
A $16 fee was deducted from his checking account.

Key Concepts

- **Multiplication and Division of Two Signed Numbers**
    - Same signs—Product is positive
    - Different signs—Product is negative
- **Strategy for Applications**
    1. Identify what you are asked to find.
    2. Write a phrase that gives the information to find it.
    3. Translate the phrase to an expression.
    4. Simplify the expression.
    5. Answer the question with a complete sentence.

Practice Makes Perfect

Multiply Integers

In the following exercises, multiply.

1. $-4 \cdot 8$	2. $-3 \cdot 9$
3. $9(-7)$	4. $13(-5)$
5. $-1 \cdot 6$	6. $-1 \cdot 3$
7. $-1(-14)$	8. $-1(-19)$

Divide Integers

In the following exercises, divide.

9. $-24 \div 6$	10. $35 \div (-7)$
11. $-52 \div (-4)$	12. $-84 \div (-6)$
13. $-180 \div 15$	14. $-192 \div 12$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

15. $5(-6) + 7(-2) - 3$	16. $8(-4) + 5(-4) - 6$
17. $(-2)^6$	18. $(-3)^5$
19. -4^2	20. -6^2
21. $-3(-5)(6)$	22. $-4(-6)(3)$
23. $(8-11)(9-12)$	24. $(6-11)(8-13)$
25. $26 - 3(2-7)$	26. $23 - 2(4-6)$
27. $65 \div (-5) + (-28) \div (-7)$	28. $52 \div (-4) + (-32) \div (-8)$
29. $9 - 2[3 - 8(-2)]$	30. $11 - 3[7 - 4(-2)]$
31. $(-3)^2 - 24 \div (8-2)$	32. $(-4)^2 - 32 \div (12-4)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

33. $y+(-14)$ when a) $y=-33$ b) $y=30$	34. $x+(-21)$ when a) $x=-27$ b) $x=44$
35. a)$a+3$ when $a=-7$ b) -$a+3$ when $a=-7$	36. a)$d+(-9)$ when $d=-8$ b) -$d+(-9)$ when $d=-8$
37. $m+n$ when $m=-15, n=7$	38. $p+q$ when $p=-9, q=17$
39. $r+s$ when $r=-9, s=-7$	40. $t+u$ when $t=-6, u=-5$
41. $(x+y)^2$ when $x=-3, y=14$	42. $(y+z)^2$ when $y=-3, z=15$
43. $-2x+17$ when a) $x=8$ b) $x=-8$	44. $-5y+14$ when a) $y=9$ b) $y=-9$
45. $10-3m$ when a) $m=5$ b) $m=-5$	46. $18-4n$ when a) $n=3$ b) $n=-3$
47. $2w^2-3w+7$ when $w=-2$	48. $3u^2-4u+5$ when $u=-3$
49. $9a-2b-8$ when $a=-6 \text{ and } b=-3$	50. $7m-4n-2$ when $m=-4 \text{ and } n=-9$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

51. the sum of 3 and -15, increased by 7	52. the sum of -8 and -9, increased by 23
53. the difference of 10 and -18	54. subtract 11 from -25
55. the difference of -5 and -30	56. subtract -6 from -13
57. the product of -3 and 15	58. the product of -4 and 16
59. the quotient of -60 and -20	60. the quotient of -40 and -20
61. the quotient of -6 and the sum of *a* and *b*	62. the quotient of -7 and the sum of *m* and *n*
63. the product of -10 and the difference of $p \text{ and } q$	64. the product of -13 and the difference of $c \text{ and } d$

Use Integers in Applications

In the following exercises, solve.

65. **Temperature** On January 15, the high temperature in Lytton, British Columbia, was 84° . That same day, the high temperature in Fort Nelson, British Columbia was -12°. What was the difference between the temperature in Lytton and the temperature in Embarrass?	66. **Temperature** On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire was -31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?
67. **Football** At the first down, the Chargers had the ball on their 25 yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?	68. **Football** At the first down, the Steelers had the ball on their 30 yard line. On the next three downs, they gained 9 yards, lost 14 yards, and lost 2 yards. What was the yard line at the end of the fourth down?
69. **Checking Account** Ester has \$124 in her checking account. She writes a check for \$152. What is the new balance in her checking account?	70. **Checking Account** Selina has \$165 in her checking account. She writes a check for \$207. What is the new balance in her checking account?
71. **Checking Account** Kevin has a balance of $-\$38$ in his checking account. He deposits \$225 to the account. What is the new balance?	72. **Checking Account** Reymonte has a balance of $-\$49$ in his checking account. He deposits \$281 to the account. What is the new balance?

Everyday Math

73. **Stock market** Javier owns 300 shares of stock in one company. On Tuesday, the stock price dropped \$12 per share. What was the total effect on Javier's portfolio?	74. **Weight loss** In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Writing Exercises

75. In your own words, state the rules for multiplying integers.	76. In your own words, state the rules for dividing integers.
77. Why is $-2^4 \neq (-2)^4$?	78. Why is $-4^3 = (-4)^3$?

Answers

1. -32	3. -63	5. -6
7. 14	9. -4	11. 13
13. -12	15. -47	17. 64
19. -16	21. 90	23. 9
25. 41	27. -9	29. -29
31. 5	33. a)-47 b) 16	35. a)-4 b) 10
37. -8	39. -16	41. 121
43. a) 1 b) 33	45. a)-5b) 25	47. 21
49. -56	51. $(3+(-15))+7; -5$	53. $10-(-18)\,;28$
55. $-5-(-30)\,;25$	57. $-3\cdot 15; -45$	59. $-60\div(-20)\,;3$
61. $\frac{-6}{a+b}$	63. $-10\,(p-q)$	65. $96°$
67. 21	69. -$\$28$	71. $187
73. -$\$3600$	75. Answers may vary	77. Answers may vary

Attributions

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1.6 Chapter Review

Review Exercises

Use Place Value with Whole Number

In the following exercises find the place value of each digit.

1. 26,915 a) 1 b) 2 c) 9 d) 5 e) 6	2. 359,417 a) 9 b) 3 c) 4 d) 7 e) 1
3. 58,129,304 a) 5 b) 0 c) 1 d) 8 e) 2	4. 9,430,286,157 a) 6 b) 4 c) 9 d) 0 e) 5

In the following exercises, name each number.

5. 6,104	6. 493,068
7. 3,975,284	8. 85,620,435

In the following exercises, write each number as a whole number using digits.

9. three hundred fifteen	10. sixty-five thousand, nine hundred twelve
11. ninety million, four hundred twenty-five thousand, sixteen	12. one billion, forty-three million, nine hundred twenty-two thousand, three hundred eleven

In the following exercises, round to the indicated place value.

Round to the nearest ten. 13. a) 407 b) 8,564	Round to the nearest hundred. 14. a) 25,846 b) 25,864

In the following exercises, round each number to the nearest a) hundred b) thousand c) ten thousand.

15. 864,951	16. 3,972,849

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10

17. 168	18. 264
19. 375	20. 750
21. 1430	22. 1080

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

23. 420	24. 115
25. 225	26. 2475
27. 1560	28. 56
29. 72	30. 168
31. 252	32. 391

In the following exercises, find the least common multiple of the following numbers using the multiples method.

33. 6,15	34. 60, 75

In the following exercises, find the least common multiple of the following numbers using the prime factors method.

35. 24, 30	36. 70, 84

Use Variables and Algebraic Symbols

In the following exercises, translate the following from algebra to English.

37. $25 - 7$	38. $5 \cdot 6$
39. $45 \div 5$	40. $x + 8$
41. $42 \geq 27$	42. $3n = 24$
43. $3 \leq 20 \div 4$	44. $a \neq 7 \cdot 4$

In the following exercises, determine if each is an expression or an equation.

45. $6 \cdot 3 + 5$	46. $y - 8 = 32$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

47. 3^5	48. 10^8

In the following exercises, simplify

49. $6 + 10/2 + 2$	50. $9 + 12/3 + 4$
51. $20 \div (4 + 6) \cdot 5$	52. $33 \cdot (3 + 8) \cdot 2$
53. $4^2 + 5^2$	54. $(4 + 5)^2$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

55. $9x + 7$ when $x = 3$	56. $5x - 4$ when $x = 6$
57. x^4 when $x = 3$	58. 3^x when $x = 3$
59. $x^2 + 5x - 8$ when $x = 6$	60. $2x + 4y - 5$ when $x = 7$, $y = 8$

Simplify Expressions by Combining Like Terms

In the following exercises, identify the coefficient of each term.

61. $12n$	62. $9x^2$

In the following exercises, identify the like terms.

63. $3n$, n^2, 12, $12p^2$, 3, $3n^2$	64. 5, $18r^2$, $9s$, $9r$, $5r^2$, $5s$

In the following exercises, identify the terms in each expression.

65. $11x^2 + 3x + 6$	66. $22y^3 + y + 15$

In the following exercises, simplify the following expressions by combining like terms.

67. $17a + 9a$	68. $18z + 9z$
69. $9x + 3x + 8$	70. $8a + 5a + 9$
71. $7p + 6 + 5p – 4$	72. $8x + 7 + 4x – 5$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the following phrases into algebraic expressions.

73. the sum of 8 and 12	74. the sum of 9 and 1
75. the difference of x and 4	76. the difference of x and 3
77. the product of 6 and y	78. the product of 9 and y
79. Derek bought a skirt and a blouse. The skirt cost $15 more than the blouse. Let b represent the cost of the blouse. Write an expression for the cost of the skirt.	80. Marcella has 6 fewer boy cousins than girl cousins. Let g represent the number of girl cousins. Write an expression for the number of boy cousins.

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using < or >.

81. a) 6___2 b) -7___4 c) -9___-1 d) 9___-3	82. a) -5___1 b) -4___-9 c) 6___10 d) 3___-8

In the following exercises,, find the opposite of each number.

83. a) -8 b) 1	84. a) -2 b) 6

In the following exercises, simplify.

85. (–19)	86. (–53)

In the following exercises, simplify.

87. −m when a) m = 3 b) m=-3	88. −p when a) p = 6 b) p = -6

Simplify Expressions with Absolute Value

In the following exercises,, simplify.

89. a) |7| b) |-25| c) |0|	90. a) |5| b) |0| c) |-19|

In the following exercises, fill in <, >, or = for each of the following pairs of numbers.

91. a) – 8 ___ |–8| b) – |–2|___ –2	92. a) –3|___ – | –3| b) 4 ___ – | –4|

In the following exercises, simplify.

93. |8 – 4|	94. |9 – 6|
95. 8 (14 – 2 |- 2|)	96. 6(13 – 4 |-2|)

In the following exercises, evaluate.

97. a) |x| when x = -28 b) |-x| when x =-15	98. a) |y| when y = -37 b) |-z| when z=-24

Add Integers

In the following exercises, simplify each expression.

99. -200 + 65	100. -150 + 45
101. 2 + (-8) + 6	102. 4 + (-9) + 7
103. 140 + (-75) + 67	104. -32 + 24 + (-6) + 10

Subtract Integers

In the following exercises, simplify.

105. 9 – 3	106. -5 – (-1)
107. a) 15 – 6 b) 15 + (-6)	108. a) 12 – 9 b) 12 + (-9)
109. a) 8 – (-9) b) 8 + 9	110. a) 4 – (-4) b) 4 + 4

In the following exercises, simplify each expression.

111. 10 – (-19)	112. 11 – (-18)
113. 31 – 79	114. 39 – 81
115. -31 – 11	116. -32 – 18
117. -15 – (-28) + 5	118. 71 + (-10) – 8
119. -16 – (-4 + 1) – 7	120. -15 – (-6 + 4) – 3

Multiply Integers

In the following exercises, multiply.

121. -5 (7)	122. -8 (6)
123. -18(-2)	124. -10 (-6)

Divide Integers

In the following exercises, divide.

125. -28 ÷ 7	126. 56 ÷ (-7)
127. -120 ÷ -20)	128. -200 ÷ 25

Simplify Expressions with Integers

In the following exercises, simplify each expression.

129. -8 (-2) -3 (-9)	130. -7 (-4) – 5(-3)
131. (-5)3	132. (-4)3
133. -4 · 2 · 11	134. -5 · 3 · 10
135. -10(-4) ÷ (-8)	136. -8(-6) ÷ (-4)
137. 31 – 4(3-9)	138. 24 – 3(2 – 10)

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

139. x + 8 when a) x = -26 b) x = -95	140. y + 9 when a) y = -29 b) y = -84
141. When b = -11, evaluate: a) b + 6 b) −b + 6	142. When c = -9, evaluate: a) c + (-4) b) −c + (-4)
143. p2 – 5p + 2 when p = -1	144. q2 – 2q + 9 when q = -2
145. 6x – 5y + 15 when x = 3 and y = -1	146. 3p – 2q + 9 when p = 8 and q = -2

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

147. the sum of -4 and -17, increased by 32	148. a) the difference of 15 and -7 b) subtract 15 from -7
149. the quotient of -45 and -9	150. the product of -12 and the difference of c and d.

Use Integers in Applications

In the following exercises, solve.

151. Temperature The high temperature one day in Miami Beach, Florida, was 76° F. That same day, the high temperature in Buffalo, New York was −8° F. What was the difference between the temperature in Miami Beach and the temperature in Buffalo?	152. CheckingAccount Adrianne has a balance of -$22 in her checking account. She deposits $301 to the account. What is the new balance?

Review Exercise Answers

1. a) tens b) ten thousands c) hundreds d) ones e) thousands	3. a) ten millions b) tens c) hundred thousands d) millions e) ten thousands	5. six thousand, one hundred four
7. three million, nine hundred seventy-five thousand, two hundred eighty-four	9. 315	11. 90,425,016
13. a)410b)8,560	15. a)865,000 b)865,000c)860,000	17. by 2,3,6
19. by 3,5	21. by 2,5,10	23. $2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$
25. $3 \cdot 3 \cdot 5 \cdot 5$	27. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 13$	29. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$
31. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$	33. 30	35. 120
37. 25 minus 7, the difference of twenty-five and seven	39. 45 divided by 5, the quotient of forty-five and five	41. forty-two is greater than or equal to twenty-seven
43. 3 is less than or equal to 20 divided by 4, three is less than or equal to the quotient of twenty and four	45. expression	47. 243
49. 13	51. 10	53. 41
55. 34	57. 81	59. 58
61. 12	63. 12 and 3, n2 and 3n2	65. 11×2, 3x, 6
67. 26a	69. 12x + 8	71. 12p + 2
73. 8 + 12	75. x – 4	77. 6y
79. b + 15	81. a) > b) < c) < d) >	83. a) 8 b) -1
85. 19	87. a) -3 b) 3	89. a) 7 b) 25 c) 0
91. a) < b) =	93. 4	95. 80
97. a) 28 b) 15	99. -135	101. 0
103. 132	105. 6	107. a) 9 b) 9
109. a) 17 b) 17	111. 29	113. -48
115. -42	117. 18	119. -20
121. -35	123. 36	125. -4
127. 6	129. 43	131. -125
133. -88	135. -5	137. 55
139. a) -18 b) -87	141. a) -5 b) 17	143. 8
145. 38	147. (-4 + (-17)) + 32; 11	149. $\frac{-45}{-9}$;5

151. 84 degrees F		

Practice Test

1. Write as a whole number using digits: two hundred five thousand, six hundred seventeen.	2. Find the prime factorization of 504.
3. Find the Least Common Multiple of 18 and 24.	4. Combine like terms: 5n + 8 + 2n – 1.

In the following exercises, evaluate.

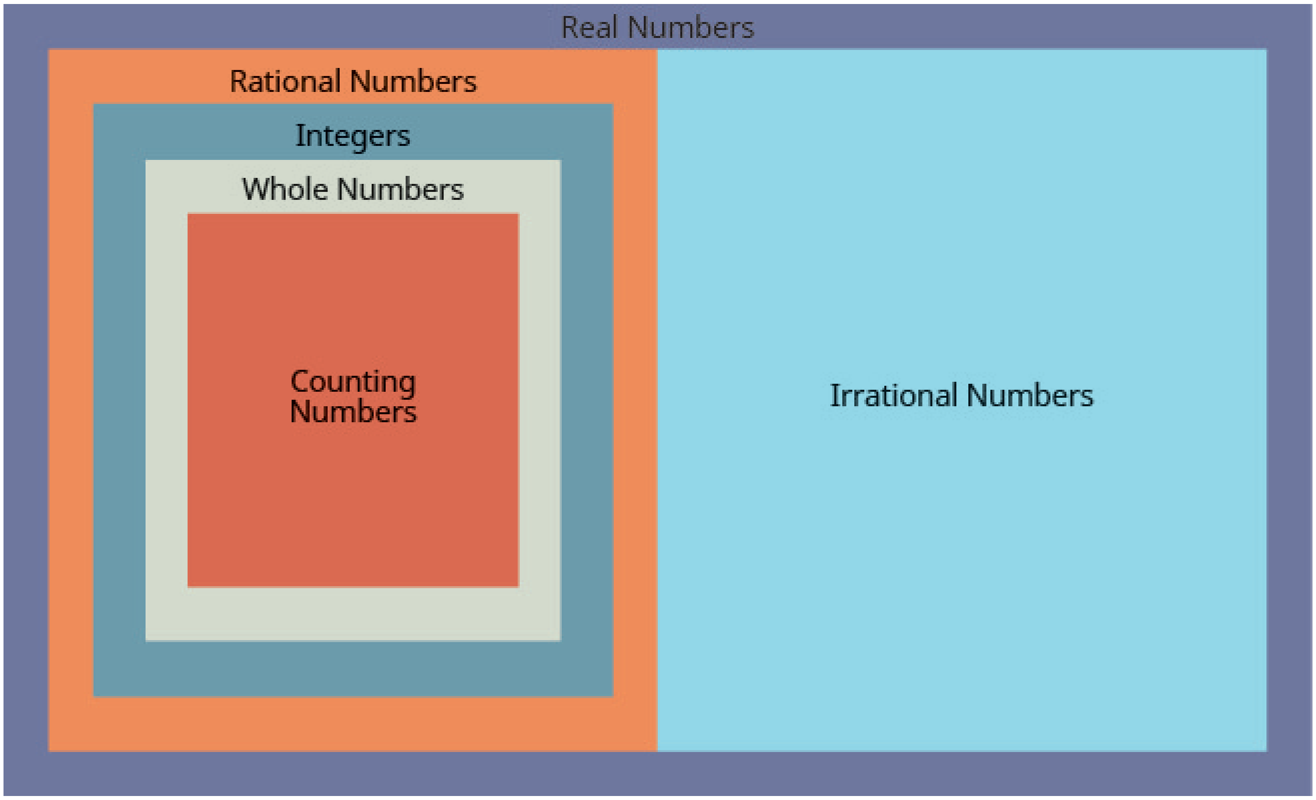
5. $-\lvert x \rvert$ when x = -2	6. 11 – a when a = -3
7. Translate to an algebraic expression and simplify: twenty less than negative 7.	8. Monique has a balance of −\$18 in her checking account. She deposits \$152 to the account. What is the new balance?
9. Round 677.1348 to the nearest hundredth.	10. Simplify expression -6 (-2) – 3 · 4 ÷ (-6)
11. Simplify expression 4(-2) + 4 ·2 – $(-3)^3$	12. Simplify expression -8(-3) ÷ (-6)
13.Simplify expression 21 – 5(2 – 7)	14. Simplify expression 2 + 2(3 – 10) – $(2)^3$

Practice Test Answers

1. 205,617	2. 2 · 2 · 2 · 3 · 3 · 7	3. 72
4. 7n + 7	5. -2	6. 14
7. -7 – 20; -27	8. \$ 134	9. 677.13
10. 10	11. 27	12. -4
13. 46	14. -20	

CHAPTER 2 Operations with Rational Numbers and Introduction to Real Numbers

All the numbers we use in the intermediate algebra course are real numbers. The chart below shows us how the number sets we use in algebra fit together. In this chapter we will work with rational numbers, but you will be also introduced to irrational numbers. The set of rational numbers together with the set of irrational numbers make up the set of real numbers.



2.1 Visualize Fractions

Learning Objectives

By the end of this section, you will be able to:

- Find equivalent fractions
- Simplify fractions
- Multiply fractions
- Divide fractions
- Simplify expressions written with a fraction bar
- Translate phrases to expressions with fractions

Find Equivalent Fractions

Fractions are a way to represent parts of a whole. The fraction $\frac{1}{3}$ means that one whole has been divided into 3 equal parts and each part is one of the three equal parts. See (Figure 1). The fraction $\frac{2}{3}$ represents two of three equal parts. In the fraction $\frac{2}{3}$, the 2 is called the numerator and the 3 is called the denominator.

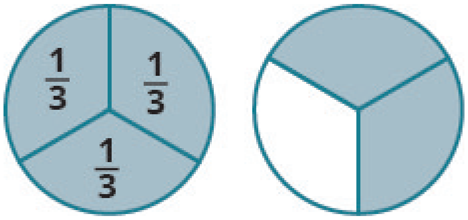


Figure 1.

The circle on the left has been divided into 3 equal parts. Each part is $\frac{1}{3}$ of the 3 equal parts. In the circle on the right, $\frac{2}{3}$ of the circle is shaded (2 of the 3 equal parts).

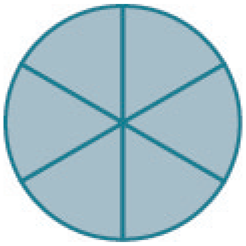
Fraction

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and

- *a* is the **numerator** and *b* is the **denominator**.

A fraction represents parts of a whole. The denominator *b* is the number of equal parts the whole has been divided into, and the numerator *a* indicates how many parts are included.

If a whole pie has been cut into 6 pieces and we eat all 6 pieces, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie.



So $\frac{6}{6} = 1$. This leads us to the property of one that tells us that any number, except zero, divided by itself is 1

Property of One

$\frac{a}{a} = 1 \qquad (a \neq 0)$

Any number, except zero, divided by itself is one.

If a pie was cut in 6 pieces and we ate all 6, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie. If the pie was cut into 8 pieces and we ate all 8, we ate $\frac{8}{8}$ pieces, or one whole pie. We ate the same amount—one whole pie.

The fractions $\frac{6}{6}$ and $\frac{8}{8}$ have the same value, 1, and so they are called equivalent fractions. **Equivalent fractions** are fractions that have the same value.

Let’s think of pizzas this time. (Figure 2) shows two images: a single pizza on the left, cut into two equal pieces, and a second pizza of the same size, cut into eight pieces on the right. This is a way to show that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. In other words, they are equivalent fractions.

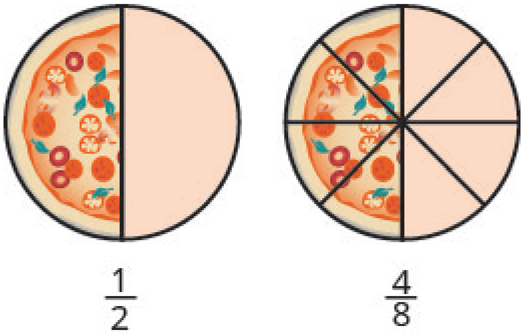


Figure 2.

Since the same amount is of each pizza is shaded, we see that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. They are equivalent fractions.

Equivalent Fractions

Equivalent fractions are fractions that have the same value.

How can we use mathematics to change $\frac{1}{2}$ into $\frac{4}{8}$? How could we take a pizza that is cut into 2 pieces and cut it into 8 pieces? We could cut each of the 2 larger pieces into 4 smaller pieces! The whole pizza would then be cut into 8 pieces instead of just 2. Mathematically, what we've described could be written like this as $\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$. See (Figure 3).

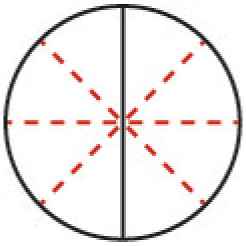


Figure 3.

Cutting each half of the pizza into 4 pieces, gives us pizza cut into 8 pieces: $\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$.

This model leads to the following property:

Equivalent Fractions Property

If a, b, c are numbers where $b \neq 0, c \neq 0$, then
$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$

If we had cut the pizza differently, we could get

$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \quad \text{so} \quad \frac{1}{2} = \frac{2}{4}$$

$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \quad \text{so} \quad \frac{1}{2} = \frac{3}{6}$$

$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \quad \text{so} \quad \frac{1}{2} = \frac{10}{20}$$

So, we say $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \text{ and } \frac{10}{20}$ are equivalent fractions.

EXAMPLE 1

Find three fractions equivalent to $\frac{2}{5}$.

Solution
To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number. We can choose any number, except for zero. Let’s multiply them by 2, 3, and then 5.

$$\frac{2\cdot 2}{5\cdot 2}=\frac{4}{10}\qquad \frac{2\cdot 3}{5\cdot 3}=\frac{6}{15}\qquad \frac{2\cdot 5}{5\cdot 5}=\frac{10}{25}$$

So, $\frac{4}{10}, \frac{6}{15},$ and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

TRY IT 1.1

Find three fractions equivalent to $\frac{3}{5}$.

Show answer
$\frac{6}{10}, \frac{9}{15}, \frac{12}{20}$; answers may vary

TRY IT 1.2

Find three fractions equivalent to $\frac{4}{5}$.

Show answer
$\frac{8}{10}, \frac{12}{15}, \frac{16}{20}$; answers may vary

Simplify Fractions

A fraction is considered **simplified** if there are no common factors, other than 1, in its numerator and denominator.

For example,

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

Simplified Fraction

A fraction is considered simplified if there are no common factors in its numerator and denominator.

The phrase *reduce a fraction* means to simplify the fraction. We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

In Example 1, we used the equivalent fractions property to find equivalent fractions. Now we'll use the equivalent fractions property in reverse to simplify fractions. We can rewrite the property to show both forms together.

Equivalent Fractions Property

If a, b, c are numbers where $b \neq 0, c \neq 0,$

$$\text{then} \quad \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

EXAMPLE 2

Simplify: $-\frac{32}{56}$.

Solution

	$-\frac{32}{56}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 8}{7 \cdot 8}$
Simplify using the equivalent fractions property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

TRY IT 2.1

Simplify: $-\frac{42}{54}$.

Show answer

$-\frac{7}{9}$

TRY IT 2.2

Simplify: $-\frac{45}{81}$.

Show answer

$-\frac{5}{9}$

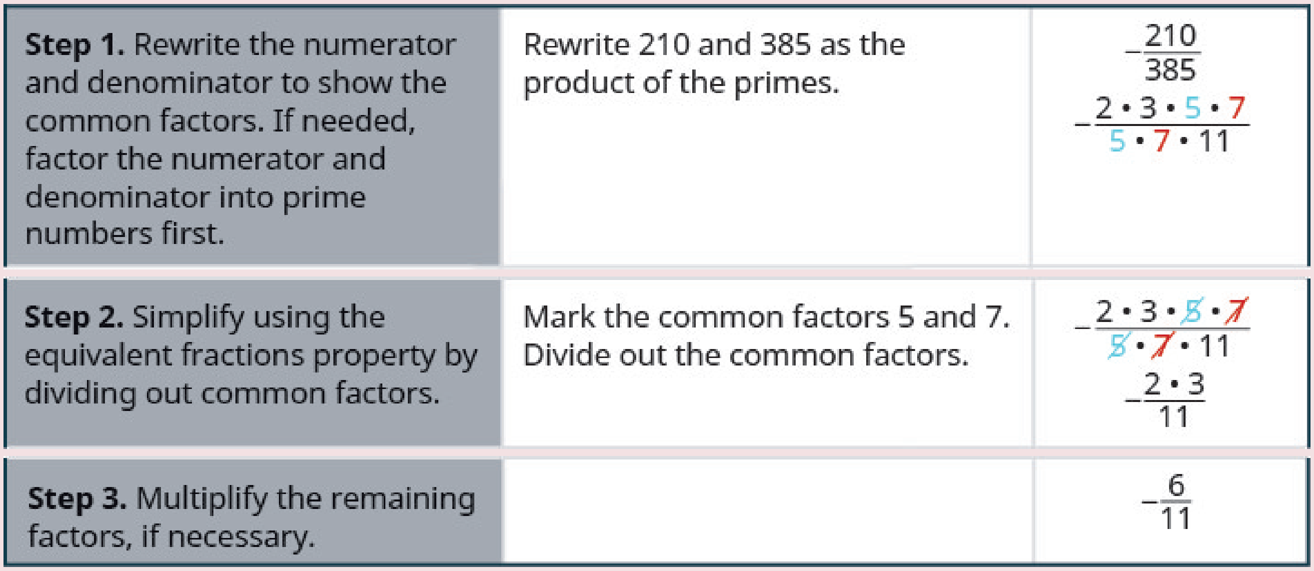
Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the equivalent fractions property.

EXAMPLE 3

How to Simplify a Fraction

Simplify: $-\frac{210}{385}$.

Solution



Step 1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.	Rewrite 210 and 385 as the product of the primes.	$-\frac{210}{385}$ $-\frac{2\cdot 3\cdot 5\cdot 7}{5\cdot 7\cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors.	$-\frac{2\cdot 3\cdot \cancel{5}\cdot \cancel{7}}{\cancel{5}\cdot \cancel{7}\cdot 11}$ $-\frac{2\cdot 3}{11}$
Step 3. Multiplify the remaining factors, if necessary.		$-\frac{6}{11}$

TRY IT 3.1

Simplify: $-\frac{69}{120}$.

Show answer
$-\frac{23}{40}$

TRY IT 3.2

Simplify: $-\frac{120}{192}$.

Show answer
$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

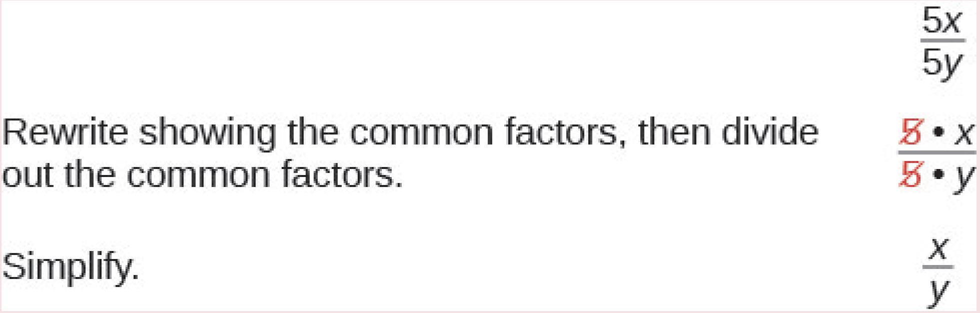
HOW TO: Simplify a Fraction

1. Rewrite the numerator and denominator to show the common factors.
   If needed, factor the numerator and denominator into prime numbers first.
2. Simplify using the equivalent fractions property by dividing out common factors.
3. Multiply any remaining factors, if needed.

EXAMPLE 4

Simplify: $\frac{5x}{5y}$.

Solution

	$\frac{5x}{5y}$
Rewrite showing the common factors, then divide out the common factors.	 $\frac{5x}{5y}$ Rewrite showing the common factors, then divide out the common factors. $\frac{\cancel{5}\cdot x}{\cancel{5}\cdot y}$ Simplify. $\frac{x}{y}$
Simplify.	$\frac{x}{y}$

TRY IT 4.1

Simplify: $\frac{7x}{7y}$.

Show answer

$\frac{x}{y}$

TRY IT 4.2

Simplify: $\frac{7x}{7y}$.

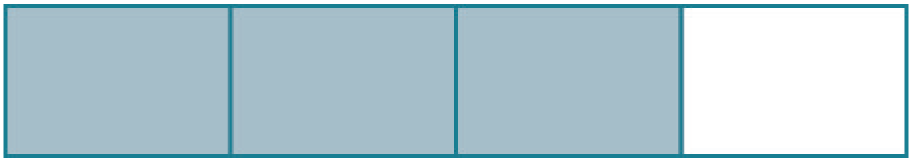
Show answer

$\frac{x}{y}$

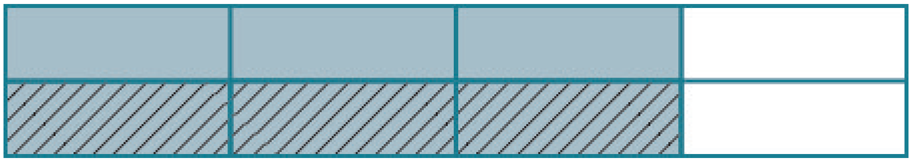
Multiply Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions. So we will start with fraction multiplication.

We'll use a model to show you how to multiply two fractions and to help you remember the procedure. Let's start with $\frac{3}{4}$.



Now we'll take $\frac{1}{2}$ of $\frac{3}{4}$.



Notice that now, the whole is divided into 8 equal parts. So $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

To multiply fractions, we multiply the numerators and multiply the denominators.

Fraction Multiplication

If a, b, c and d are numbers where $b \neq 0$ and $d \neq 0$, then

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In Example 5, we will multiply negative and a positive, so the product will be negative.

EXAMPLE 5

Multiply: $-\frac{11}{12} \cdot \frac{5}{7}$.

Solution

The first step is to find the sign of the product. Since the signs are the different, the product is negative.

	$-\frac{11}{12} \cdot \frac{5}{7}$
Determine the sign of the product; multiply.	$-\frac{11 \cdot 5}{12 \cdot 7}$
Are there any common factors in the numerator and the demonimator? No.	$-\frac{55}{84}$

TRY IT 5.1

Multiply: $-\frac{10}{28} \cdot \frac{8}{15}$.

Show answer

$-\frac{4}{21}$

TRY IT 5.2

Multiply: $-\frac{9}{20} \cdot \frac{5}{12}$.

Show answer

$-\frac{3}{16}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

EXAMPLE 6

Multiply: $-\frac{12}{5} \cdot (-20x)$.

Solution

Determine the sign of the product. The signs are the same, so the product is positive.

	$-\frac{12}{5} \cdot (-20x)$
Write $20x$ as a fraction.	$\frac{12}{5} \cdot \left(\frac{20x}{1}\right)$
Multiply.	
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12 \cdot 4 \cdot \cancel{5}x}{\cancel{5} \cdot 1}$
Simplify.	$48x$

TRY IT 6.1

Multiply: $\frac{11}{3}\cdot(-9a)$.

Show answer
$-33a$

TRY IT 6.2

Multiply: $\frac{13}{7}\cdot(-14b)$.

Show answer
$-2b$

Divide Fractions

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, that we need some vocabulary.

The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Notice that $\frac{2}{3}\cdot\frac{3}{2}=1$. A number and its reciprocal multiply to 1.

To get a product of positive 1 when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

The reciprocal of $-\frac{10}{7}$ is $-\frac{7}{10}$, since $-\frac{10}{7}\cdot\left(-\frac{7}{10}\right)=1$.

Reciprocal

The **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$.

A number and its reciprocal multiply to one $\frac{a}{b}\cdot\frac{b}{a}=1$.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Fraction Division

If a, b, c and d are numbers where $b \neq 0, c \neq 0$ and $d \neq 0$, then
$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

We need to say $b \neq 0, c \neq 0$ and $d \neq 0$ to be sure we don't divide by zero!

EXAMPLE 7

Divide: $-\frac{2}{3} \div \frac{n}{5}$.

Solution

	$-\frac{2}{3} \div \frac{n}{5}$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{2}{3} \cdot \frac{5}{n}$
Multiply.	$-\frac{10}{3n}$

TRY IT 7.1

Divide: $-\frac{3}{5} \div \frac{p}{7}$.

Show answer
$-\frac{21}{5p}$

TRY IT 7.2

Divide: $-\frac{5}{8} \div \frac{q}{3}$.

Show answer
$-\frac{15}{8q}$

EXAMPLE 8

Find the quotient: $-\frac{7}{8} \div \left(-\frac{14}{27}\right)$.

Solution

	$-\frac{7}{18} \div \left(-\frac{14}{27}\right)$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18} \cdot -\frac{27}{14}$
Determine the sign of the product, and then multiply..	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

TRY IT 8.1

Find the quotient: $-\frac{7}{27} \div \left(-\frac{35}{36}\right)$.

Show answer

$\frac{4}{15}$

TRY IT 8.2

Find the quotient: $-\frac{5}{14} \div \left(-\frac{15}{28}\right)$.

Show answer

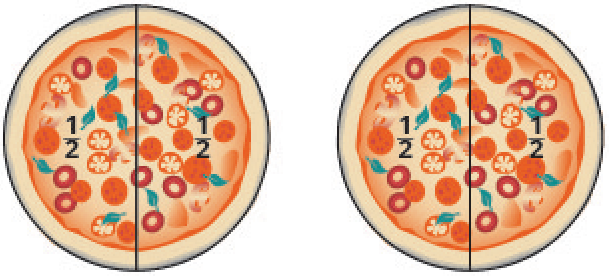
$\frac{2}{3}$

There are several ways to remember which steps to take to multiply or divide fractions. One way is to repeat the call outs to yourself. If you do this each time you do an exercise, you will have the steps memorized.

- “To multiply fractions, multiply the numerators and multiply the denominators.”
- “To divide fractions, multiply the first fraction by the reciprocal of the second.”

Another way is to keep two examples in mind:

One fourth of two pizzas is one half of a pizza.



$$2 \cdot \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{1}{4}$$

$$\frac{2}{4}$$

$$\frac{1}{2}$$

There are eight quarters in $2.00.

$$2 \div \frac{1}{4}$$

$$\frac{2}{1} \div \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{4}{1}$$

$$8$$

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Complex Fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \qquad \frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, we remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

EXAMPLE 9

Simplify: $\frac{\frac{3}{4}}{\frac{5}{8}}$.

Solution

	$\frac{\frac{3}{4}}{\frac{5}{8}}$
Rewrite as division.	$\frac{3}{4} \div \frac{5}{8}$
Multiply the first fraction by the reciprocal of the second.	$\frac{3}{4} \cdot \frac{8}{5}$
Multiply.	$\frac{3 \cdot 8}{4 \cdot 5}$
Look for common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 5}$
Divide out common factors and simplify.	$\frac{6}{5}$

TRY IT 9.1

Simplify: $\frac{\frac{2}{3}}{\frac{5}{6}}$.

Show answer

$\frac{4}{5}$

TRY IT 9.2

Simplify: $\frac{\frac{3}{7}}{\frac{6}{11}}$.

Show answer

$\frac{11}{14}$

EXAMPLE 10

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Solution

	$\frac{\frac{x}{2}}{\frac{xy}{6}}$
Rewrite as division.	$\frac{x}{2} \div \frac{xy}{6}$
Multiply the first fraction by the reciprocal of the second.	$\frac{x}{2} \cdot \frac{6}{xy}$
Multiply.	$\frac{x \cdot 6}{2 \cdot xy}$
Look for common factors.	 $\frac{\cancel{x} \cdot 3 \cdot \cancel{2}}{\cancel{2} \cdot \cancel{x} \cdot y}$
Divide out common factors and simplify.	$\frac{3}{y}$

TRY IT 10.1

Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

Show answer
$\frac{3}{4b}$

TRY IT 10.2

Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$.

Show answer
$\frac{4}{2q}$

Simplify Expressions with a Fraction Bar

The line that separates the numerator from the denominator in a fraction is called a fraction bar. A fraction bar acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

To simplify the expression $\frac{5-3}{7+1}$, we first simplify the numerator and the denominator separately. Then we divide.

$\frac{5-3}{7+1}$

$\frac{2}{8}$

$\frac{1}{4}$

HOW TO: Simplify an Expression with a Fraction Bar

1. Simplify the expression in the numerator. Simplify the expression in the denominator.
2. Simplify the fraction.

EXAMPLE 11

Simplify: $\frac{4-2(3)}{2^2+2}$.

Solution

	$\frac{4-2(3)}{2^2+2}$
Use the order of operations to simpliy the numerator and the denominator.	$\frac{4-6}{4+2}$
Simplify the numerator and the denominator.	$\frac{-2}{6}$
Simplify. A negative divided by a positive is negative.	$-\frac{1}{3}$

TRY IT 11.1

Simplify: $\frac{6-3(5)}{3^2+3}$.

Show answer

$-\frac{3}{4}$

TRY IT 11.2

Simplify: $\frac{4-4(6)}{3^2+3}$.

Show answer
$-\frac{2}{3}$

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$\frac{-1}{3} = -\frac{1}{3}$ $\frac{\text{negative}}{\text{positive}} = \text{negative}$
$\frac{1}{-3} = -\frac{1}{3}$ $\frac{\text{positive}}{\text{negative}} = \text{negative}$

Placement of Negative Sign in a Fraction

For any positive numbers *a* and *b*,

$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$

EXAMPLE 12

Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Solution

	$\frac{4(-3)+6(-2)}{-3(2)-2}$
Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

TRY IT 12.1

Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Show answer
4

TRY IT 12.2

Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Show answer
2

Translate Phrases to Expressions with Fractions

Now that we have done some work with fractions, we are ready to translate phrases that would result in expressions with fractions.

The English words quotient and ratio are often used to describe fractions. Remember that “quotient” means division. The quotient of a and b is the result we get from dividing a by b, or $\frac{a}{b}$.

EXAMPLE 13

Translate the English phrase into an algebraic expression: the quotient of the difference of *m* and *n*, and *p*.

Solution
We are looking for the *quotient of* the difference of *m* and *n*, *and p*. This means we want to divide the difference of m and n by p.

$\frac{m-n}{p}$

TRY IT 13.1

Translate the English phrase into an algebraic expression: the quotient of the difference of *a* and *b*, and *cd*.

Show answer

$\frac{a-b}{cd}$

TRY IT 13.2

Translate the English phrase into an algebraic expression: the quotient of the sum of p and q, and r

Show answer

$\frac{p+q}{r}$

Key Concepts

- **Equivalent Fractions Property:** If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.
- **Fraction Division:** If a, b, c and d are numbers where $b \neq 0, c \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. To divide fractions, multiply the first fraction by the reciprocal of the second.
- **Fraction Multiplication:** If a, b, c and d are numbers where $b \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. To multiply fractions, multiply the numerators and multiply the denominators.
- **Placement of Negative Sign in a Fraction:** For any positive numbers a and b, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **Property of One:**$\frac{a}{a} = 1$; Any number, except zero, divided by itself is one.
- **Simplify a Fraction**
  1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
  2. Simplify using the equivalent fractions property by dividing out common factors.
  3. Multiply any remaining factors.
- **Simplify an Expression with a Fraction Bar**
  1. Simplify the expression in the numerator. Simplify the expression in the denominator.
  2. Simplify the fraction.

Glossary

complex fraction
A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

denominator
The denominator is the value on the bottom part of the fraction that indicates the number of equal parts into which the whole has been divided.

equivalent fractions
Equivalent fractions are fractions that have the same value.

fraction
A fraction is written $\frac{a}{b}$, where $b \neq 0$ a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

numerator
The numerator is the value on the top part of the fraction that indicates how many parts of the whole are included.

reciprocal
The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. A number and its reciprocal multiply to one: $\frac{a}{b} \cdot \frac{b}{a} = 1$.

simplified fraction
A fraction is considered simplified if there are no common factors in its numerator and denominator.

Practice Makes Perfect

Find Equivalent Fractions

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

1. $\frac{3}{8}$	2. $\frac{5}{8}$
3. $\frac{5}{9}$	4. $\frac{1}{8}$

Simplify Fractions

In the following exercises, simplify.

5. $-\frac{40}{88}$	6. $-\frac{63}{99}$
7. $-\frac{108}{63}$	8. $-\frac{104}{48}$
9. $\frac{120}{252}$	10. $\frac{182}{294}$
11. $-\frac{3x}{12y}$	12. $-\frac{4x}{32y}$
13. $\frac{14x^2}{21y}$	14. $\frac{24a}{32b^2}$

Multiply Fractions

In the following exercises, multiply.

15. $\frac{3}{4} \cdot \frac{9}{10}$	16. $\frac{4}{5} \cdot \frac{2}{7}$
17. $-\frac{2}{3} \cdot \left(-\frac{3}{8}\right)$	18. $-\frac{3}{4} \cdot \left(-\frac{4}{9}\right)$
19. $-\frac{5}{9} \cdot \frac{3}{10}$	20. $-\frac{3}{8} \cdot \frac{4}{15}$
21. $\left(-\frac{14}{15}\right) \cdot \left(\frac{9}{20}\right)$	22. $\left(-\frac{9}{10}\right) \cdot \left(\frac{25}{33}\right)$
23. $\left(-\frac{63}{84}\right) \cdot \left(-\frac{44}{90}\right)$	24. $\left(-\frac{63}{60}\right) \cdot \left(-\frac{40}{88}\right)$
25. $4 \cdot \frac{5}{11}$	26. $5 \cdot \frac{8}{3}$
27. $\frac{3}{7} \cdot 21n$	28. $\frac{5}{6} \cdot 30m$
29. $-8 \cdot \left(\frac{17}{4}\right)$	30. $(-1) \cdot \left(-\frac{6}{7}\right)$

Divide Fractions

In the following exercises, divide.

31. $\frac{3}{4} \div \frac{2}{3}$	32. $\frac{4}{5} \div \frac{3}{4}$
33. $-\frac{7}{9} \div \left(-\frac{7}{4}\right)$	34. $-\frac{5}{6} \div \left(-\frac{5}{6}\right)$
35. $\frac{3}{4} \div \frac{x}{11}$	36. $\frac{2}{5} \div \frac{y}{9}$
37. $\frac{5}{18} \div \left(-\frac{15}{24}\right)$	38. $\frac{7}{18} \div \left(-\frac{14}{27}\right)$
39. $\frac{8u}{15} \div \frac{12v}{25}$	40. $\frac{12r}{25} \div \frac{18s}{35}$
41. $-5 \div \frac{1}{2}$	42. $-3 \div \frac{1}{4}$
43. $\frac{3}{4} \div (-12)$	44. $-15 \div \left(-\frac{5}{3}\right)$

In the following exercises, simplify.

45. $\frac{-\frac{8}{21}}{\frac{12}{35}}$	46. $\frac{-\frac{9}{16}}{\frac{33}{40}}$
47. $\frac{-\frac{4}{5}}{2}$	48. $\frac{5}{\frac{3}{10}}$
49. $\frac{\frac{m}{3}}{\frac{n}{2}}$	50. $\frac{-\frac{3}{8}}{-\frac{y}{12}}$

Simplify Expressions Written with a Fraction Bar

In the following exercises, simplify.

51. $\frac{22+3}{10}$	52. $\frac{19-4}{6}$
53. $\frac{48}{24-15}$	54. $\frac{46}{4+4}$
55. $\frac{-6+6}{8+4}$	56. $\frac{-6+3}{17-8}$
57. $\frac{4\cdot 3}{6\cdot 6}$	58. $\frac{6\cdot 6}{9\cdot 2}$
59. $\frac{4^2-1}{25}$	60. $\frac{7^2+1}{60}$
61. $\frac{8\cdot 3+2\cdot 9}{14+3}$	62. $\frac{9\cdot 6-4\cdot 7}{22+3}$
63. $\frac{5\cdot 6-3\cdot 4}{4\cdot 5-2\cdot 3}$	64. $\frac{8\cdot 9-7\cdot 6}{5\cdot 6-9\cdot 2}$
65. $\frac{5^2-3^2}{3-5}$	66. $\frac{6^2-4^2}{4-6}$
67. $\frac{7\cdot 4-2(8-5)}{9\cdot 3-3\cdot 5}$	68. $\frac{9\cdot 7-3(12-8)}{8\cdot 7-6\cdot 6}$
69. $\frac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$	70. $\frac{8(9-2)-4(14-9)}{7(8-3)-3(16-9)}$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

71. the quotient of *r* and the sum of *s* and 10	72. the quotient of *A* and the difference of 3 and *B*
73. the quotient of the difference of x and y, and -3	74. the quotient of the sum of m and n, and $4q$

Everyday Math

75. **Baking.** A recipe for chocolate chip cookies calls for $\frac{3}{4}$ cup brown sugar. Imelda wants to double the recipe. a) How much brown sugar will Imelda need? Show your calculation. b) Measuring cups usually come in sets of $\frac{1}{4}, \frac{1}{3}, \frac{1}{2},$ and 1 cup. Draw a diagram to show two different ways that Imelda could measure the brown sugar needed to double the cookie recipe.	76. **Baking.** Nina is making 4 pans of fudge to serve after a music recital. For each pan, she needs $\frac{2}{3}$ cup of condensed milk. a) How much condensed milk will Nina need? Show your calculation. b) Measuring cups usually come in sets of $\frac{1}{4}, \frac{1}{3}, \frac{1}{2},$ and 1 cup. Draw a diagram to show two different ways that Nina could measure the condensed milk needed for 4 pans of fudge.
77. **Portions** Don purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?	78. **Portions** Kristen has $\frac{3}{4}$ yards of ribbon that she wants to cut into 6 equal parts to make hair ribbons for her daughter's 6 dolls. How long will each doll's hair ribbon be?

Writing Exercises

79. Rafael wanted to order half a medium pizza at a restaurant. The waiter told him that a medium pizza could be cut into 6 or 8 slices. Would he prefer 3 out of 6 slices or 4 out of 8 slices? Rafael replied that since he wasn't very hungry, he would prefer 3 out of 6 slices. Explain what is wrong with Rafael's reasoning.	80. Give an example from everyday life that demonstrates how $\frac{1}{2} \cdot \frac{2}{3}$ is $\frac{1}{3}$.
81. Explain how you find the reciprocal of a fraction.	82. Explain how you find the reciprocal of a negative number.

Answers

1. $\frac{6}{16}, \frac{9}{24}, \frac{12}{32}$ answers may vary	3. $\frac{10}{18}, \frac{15}{27}, \frac{20}{36}$ answers may vary	5. $-\frac{5}{11}$
7. $-\frac{12}{7}$	9. $\frac{10}{21}$	11. $-\frac{x}{4y}$
13. $\frac{2x^2}{3y}$	15. $\frac{27}{40}$	17. $\frac{1}{4}$
19. $-\frac{1}{6}$	21. $-\frac{21}{50}$	23. $\frac{11}{30}$
25. $\frac{20}{11}$	27. 9*n*	29. -34
31. $\frac{9}{8}$	33. 1	35. $\frac{33}{4x}$
37. $-\frac{4}{9}$	39. $\frac{10u}{9v}$	41. -10
43. $-\frac{1}{16}$	45. $-\frac{10}{9}$	47. $-\frac{2}{5}$
49. $\frac{2m}{3n}$	51. $\frac{5}{2}$	53. $\frac{16}{3}$
55. 0	57. $\frac{1}{3}$	59. $\frac{3}{5}$
61. $2\frac{8}{17}$	63. $\frac{3}{5}$	65. -8
67. $\frac{11}{6}$	69. $\frac{5}{2}$	71. $\frac{r}{s+10}$
73. $\frac{x-y}{-3}$	75. a) $1\frac{1}{2}$ cups b) answers will vary	77. 20 bags
79. Answers may vary.	81. Answers may vary.	

Attributions

This chapter has been adapted from "Visualize Fractions" in *Elementary Algebra* (OpenStax) by Lynn

2.2 Add and Subtract Fractions

Learning Objectives

By the end of this section, you will be able to:

- Add or subtract fractions with a common denominator
- Add or subtract fractions with different denominators
- Use the order of operations to simplify complex fractions
- Evaluate variable expressions with fractions

Add or Subtract Fractions with a Common Denominator

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Fraction Addition and Subtraction

If $a, b, \text{ and } c$ are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

EXAMPLE 1

Find the sum: $\frac{x}{3} + \frac{2}{3}$.

Solution

	$\frac{x}{3} + \frac{2}{3}$
Add the numerators and place the sum over the common denominator.	$\frac{x+2}{3}$

TRY IT 1.1

Find the sum: $\frac{x}{4}+\frac{3}{4}$.

Show answer
$\frac{x+3}{4}$

TRY IT 1.2

Find the sum: $\frac{y}{8}+\frac{5}{8}$.

Show answer
$\frac{y+5}{8}$

EXAMPLE 2

Find the difference: $-\frac{23}{24}-\frac{13}{24}$.

Solution

	$-\frac{23}{24}-\frac{13}{24}$
Subtract the numerators and place the difference over the common denominator.	$\frac{-23-13}{24}$
Simplify.	$\frac{-36}{24}$
Simplify. Remember, $-\frac{a}{b}=\frac{-a}{b}$.	$-\frac{3}{2}$

TRY IT 2.1

Find the difference: $-\frac{19}{28}-\frac{7}{28}$.

Show answer
$-\frac{26}{28}$

TRY IT 2.2

Find the difference: $-\frac{27}{32}-\frac{1}{32}$.

Show answer
$-\frac{7}{8}$

EXAMPLE 3

Simplify: $-\frac{10}{x}-\frac{4}{x}$.

Solution

	$-\frac{10}{x}-\frac{4}{x}$
Subtract the numerators and place the difference over the common denominator.	$\frac{-14}{x}$
Rewrite with the sign in front of the fraction.	$-\frac{14}{x}$

TRY IT 3.1

Find the difference: $-\frac{9}{x}-\frac{7}{x}$.

Show answer
$-\frac{16}{x}$

TRY IT 3.2

Find the difference: $-\frac{17}{a}-\frac{5}{a}$.

Show answer
$-\frac{22}{a}$

Now we will do an example that has both addition and subtraction.

EXAMPLE 4

Simplify: $\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$.

Solution

Add and subtract fractions—do they have a common denominator? Yes.	$\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$
Add and subtract the numerators and place the difference over the common denominator.	$\frac{3+(-5)-1}{8}$
Simplify left to right.	$\frac{-2-1}{8}$
Simplify.	$-\frac{3}{8}$

TRY IT 4.1

Simplify: $\frac{2}{5}+\left(-\frac{4}{9}\right)-\frac{7}{9}$.

Show answer

-1

TRY IT 4.2

Simplify: $\frac{5}{9}+\left(-\frac{4}{9}\right)-\frac{7}{9}$.

Show answer

$-\frac{2}{3}$

Add or Subtract Fractions with Different Denominators

As we have seen, to add or subtract fractions, their denominators must be the same. The least common denominator (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Least Common Denominator

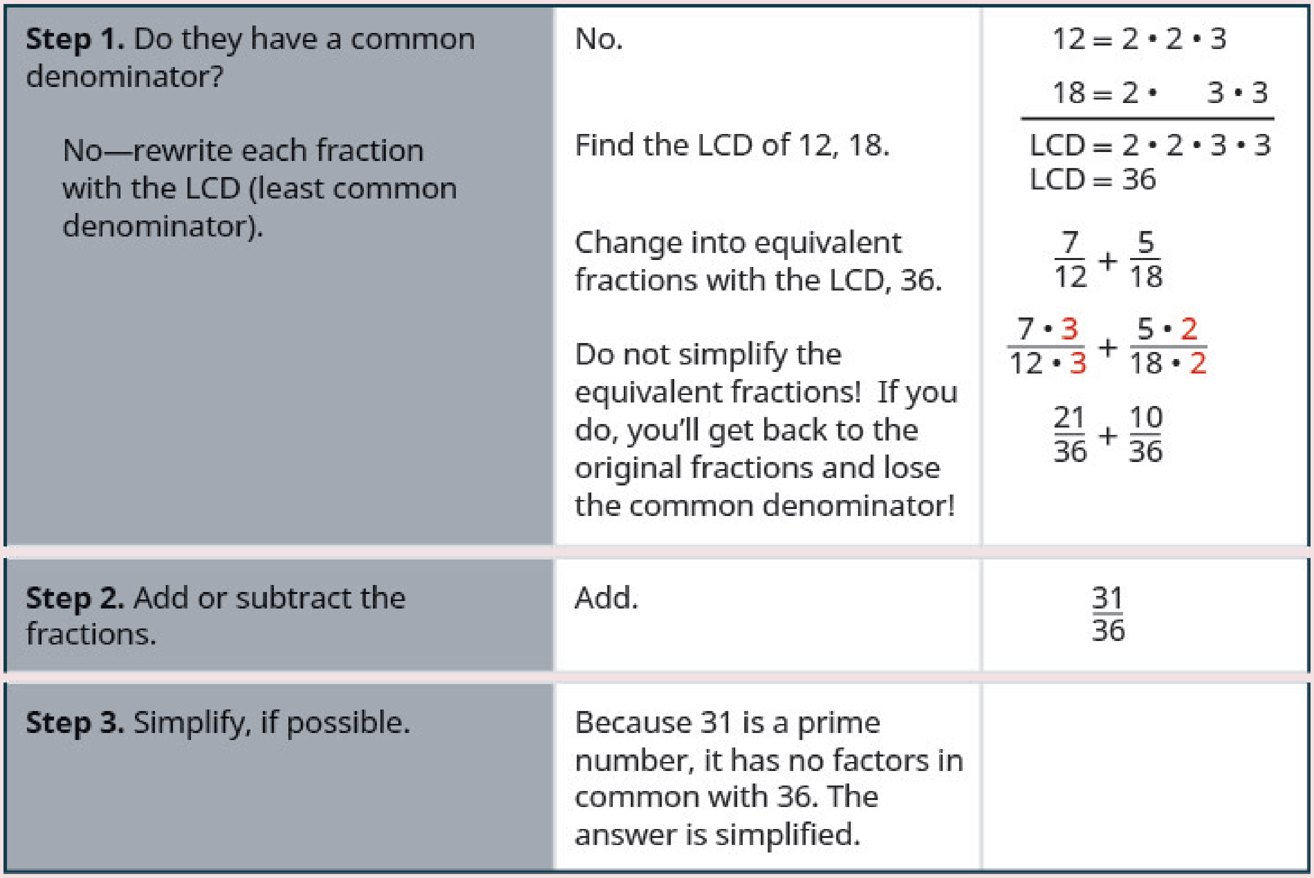
The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

EXAMPLE 5

Add: $\frac{7}{12}+\frac{5}{18}$.

Solution



Step 1. Do they have a common denominator?	No.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
No—rewrite each fraction with the LCD (least common denominator).	Find the LCD of 12, 18.	$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$
	Change into equivalent fractions with the LCD, 36.	$\frac{7}{12}+\frac{5}{18}$ $\frac{7 \cdot 3}{12 \cdot 3}+\frac{5 \cdot 2}{18 \cdot 2}$
	Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$\frac{21}{36}+\frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.	

TRY IT 5.1

Add: $\frac{7}{12}+\frac{11}{15}$.

Show answer

$\frac{79}{60}$

TRY IT 5.2

Add: $\frac{13}{15}+\frac{17}{20}$.

Show answer

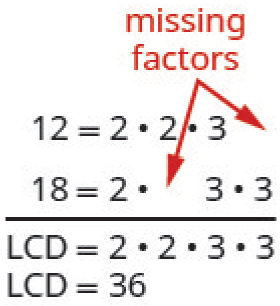
$\frac{103}{60}$

HOW TO: Add or Subtract Fractions

1. Do they have a common denominator?
    - Yes—go to step 2.
    - No—rewrite each fraction with the LCD (least common denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.
2. Add or subtract the fractions.
3. Simplify, if possible.

When finding the equivalent fractions needed to create the common denominators, there is a quick way to find the number we need to multiply both the numerator and denominator. This method works if we found the LCD by factoring into primes.

Look at the factors of the LCD and then at each column above those factors. The “missing” factors of each denominator are the numbers we need.



In (Example 5), the LCD, 36, has two factors of 2 and two factors of 3.

The numerator 12 has two factors of 2 but only one of 3—so it is “missing” one 3—we multiply the numerator and denominator by 3

The numerator 18 is missing one factor of 2—so we multiply the numerator and denominator by 2

We will apply this method as we subtract the fractions in (Example 6).

EXAMPLE 6

Subtract: $\frac{7}{15} - \frac{19}{24}$.

Solution
Do the fractions have a common denominator? No, so we need to find the LCD.

Find the LCD. $\frac{7}{15} - \frac{19}{24}$ $15 = 3 \cdot 5$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ $\text{LCD} = 120$	
Notice, 15 is “missing” three factors of 2 and 24 is “missing” the 5 from the factors of the LCD. So we multiply 8 in the first fraction and 5 in the second fraction to get the LCD.	
Rewrite as equivalent fractions with the LCD.	$\frac{7 \cdot 8}{15 \cdot 8} - \frac{19 \cdot 5}{24 \cdot 5}$
Simplify.	$\frac{56}{120} - \frac{95}{120}$
Subtract.	$-\frac{39}{120}$
Check to see if the answer can be simplified.	$-\frac{13 \cdot 3}{40 \cdot 3}$
Both 39 and 120 have a factor of 3.	
Simplify.	$-\frac{13}{40}$

Do not simplify the equivalent fractions! If you do, you’ll get back to the original fractions and lose the common denominator!

TRY IT 6.1

Subtract: $\frac{13}{24} - \frac{17}{32}$.

Show answer

$\frac{1}{96}$

TRY IT 6.2

Subtract: $\frac{21}{32} - \frac{9}{28}$.

Show answer

$\frac{75}{224}$

In the next example, one of the fractions has a variable in its numerator. Notice that we do the same steps as when both numerators are numbers.

EXAMPLE 7

Add: $\frac{3}{5}+\frac{x}{8}$.

Solution
The fractions have different denominators.

		$\frac{3}{5}+\frac{x}{8}$
Find the LCD. $\begin{array}{l} 5 = \quad\quad\quad\quad 5 \\ 8 = 2 \cdot 2 \cdot 2 \\ \hline \text{LCD} = 2 \cdot 2 \cdot 2 \cdot 5 \\ \text{LCD} = 40 \end{array}$		
Rewrite as equivalent fractions with the LCD.		$\frac{3 \cdot 8}{5 \cdot 8}+\frac{x \cdot 5}{8 \cdot 5}$
Simplify.		$\frac{24}{40}+\frac{5x}{40}$
Add.		$\frac{24+5x}{40}$

TRY IT 7.1

Add: $\frac{y}{6}+\frac{7}{9}$.

Show answer
$\frac{9y+42}{54}$

TRY IT 7.2

Add: $\frac{x}{6}+\frac{7}{15}$.

Show answer

$\frac{15x+42}{135}$

We now have all four operations for fractions. The table below summarizes fraction operations.

Summary of Fraction Operations

Fraction Operation	**Sample Equation**	**What to Do**
Fraction multiplication	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	Multiply the numerators and multiply the denominators
Fraction division	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	Multiply the first fraction by the reciprocal of the second.
Fraction addition	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	Add the numerators and place the sum over the common denominator.
Fraction subtraction	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$	Subtract the numerators and place the difference over the common denominator.

To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.

EXAMPLE 8

Simplify: a) $\frac{5x}{6} - \frac{3}{10}$ b) $\frac{5x}{6} \cdot \frac{3}{10}$.

Solution
First ask, “What is the operation?” Once we identify the operation that will determine whether we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

a) **What is the operation? The operation is subtraction.**	
Do the fractions have a common denominator? No.	$\frac{5x}{6} - \frac{3}{10}$
Rewrite each fraction as an equivalent fraction with the LCD.	$\frac{5x\cdot5}{6\cdot5} - \frac{3\cdot3}{10\cdot3}$ $\frac{25x}{30} - \frac{9}{30}$
Subtract the numerators and place the difference over the common denominators.	$\frac{25x-9}{30}$
Simplify, if possible.	There are no common factors. The fraction is simplified.

b) **What is the operation? Multiplication.**	$\frac{5x}{6}\cdot\frac{3}{10}$
To multiply fractions, multiply the numerators and multiply the denominators.	$\frac{5x\cdot3}{6\cdot10}$
Rewrite, showing common factors. Remove common factors.	$\frac{\cancel{5}x\cdot\cancel{3}}{2\cdot\cancel{3}\cdot2\cdot\cancel{5}}$
Simplify.	$\frac{x}{4}$

Notice we needed an LCD to add $\frac{5x}{6}-\frac{3}{10}$, but not to multiply $\frac{5x}{6}\cdot\frac{3}{10}$.

TRY IT 8.1

Simplify. a) $\frac{27a-32}{36}$ b) $\frac{2a}{3}$

Show answer
a) $\frac{27a-32}{36}$ b) $\frac{2a}{3}$

TRY IT 8.2

Simplify: a) $\frac{4k}{5}-\frac{1}{6}$ b) $\frac{4k}{5}\cdot\frac{1}{6}$.

Show answer
a) $\frac{24k-5}{30}$ b) $\frac{2k}{15}$

Use the Order of Operations to Simplify Complex Fractions

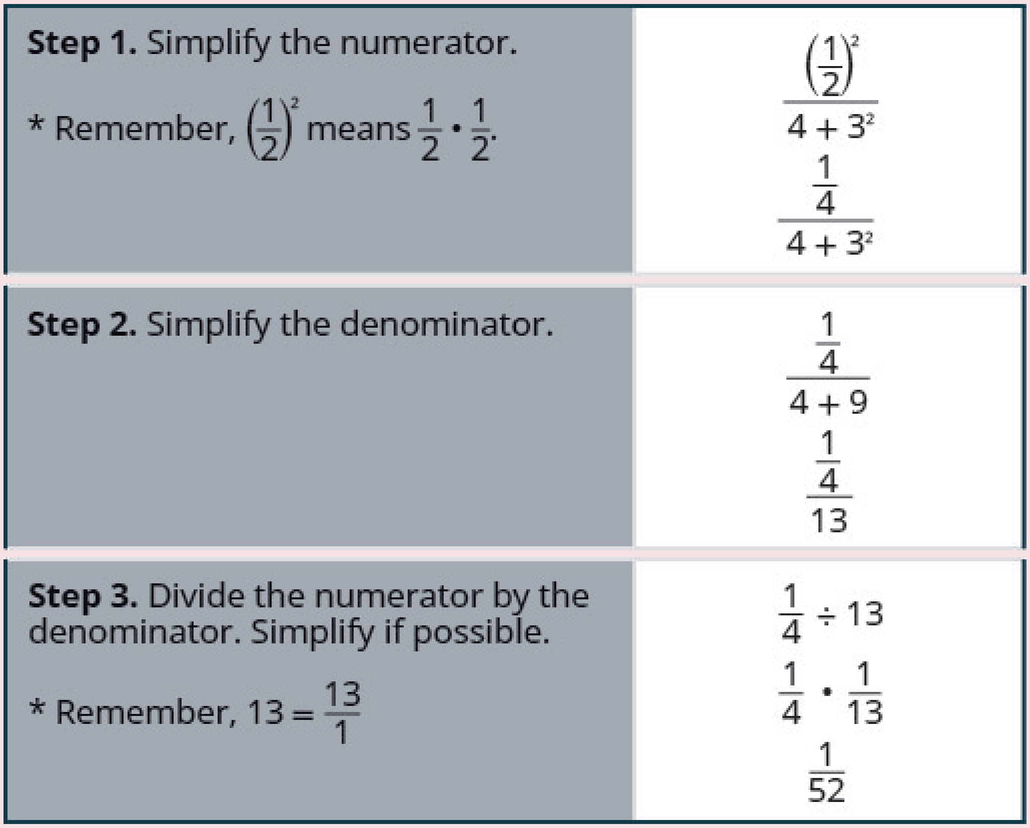
We have seen that a complex fraction is a fraction in which the numerator or denominator contains a fraction. The fraction bar indicates division. We simplified the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ by dividing $\frac{3}{4}$ by $\frac{5}{8}$.

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator.

EXAMPLE 9

Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Solution



Step 1. Simplify the numerator. * Remember, $\left(\frac{1}{2}\right)^2$ means $\frac{1}{2}\cdot\frac{1}{2}$.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$
Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$
Step 3. Divide the numerator by the denominator. Simplify if possible. * Remember, $13=\frac{13}{1}$	$\frac{1}{4}\div 13$ $\frac{1}{4}\cdot\frac{1}{13}$ $\frac{1}{52}$

TRY IT 9.1

Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Show answer

$\frac{1}{90}$

TRY IT 9.2

Simplify: $\frac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

Show answer

272

HOW TO: Simplify Complex Fractions

1. Simplify the numerator.
2. Simplify the denominator.
3. Divide the numerator by the denominator. Simplify if possible.

EXAMPLE 10

Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution

It may help to put parentheses around the numerator and the denominator.

	$\frac{\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{4}-\frac{1}{6}\right)}$
Simplify the numerator (LCD = 6) and simplify the denominator (LCD = 12).	$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$
Simplify.	$\frac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$
Divide out common factors.	$\frac{7 \cdot 6 \cdot 2}{6 \cdot 7}$
Simplify.	2

TRY IT 10.1

Simplify: $\frac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Show answer

2

TRY IT 10.2

Simplify: $\frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Show answer

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

EXAMPLE 11

Evaluate $x+\frac{1}{3}$ when a) $x=-\frac{1}{3}$ b) $x=-\frac{3}{4}$.

Solution

a. To evaluate $x+\frac{1}{3}$ when $x=-\frac{1}{3}$, substitute $-\frac{1}{3}$ for x in the expression.

	$x+\frac{1}{3}$
Substitute $-\frac{1}{3}$ for x.	$-\frac{1}{3}+\frac{1}{3}$
Simplify.	0

b. To evaluate $x+\frac{1}{3}$ when $x=-\frac{3}{4}$, we substitute $-\frac{3}{4}$ for x in the expression.

	$x+\frac{1}{3}$
Substitute $-\frac{3}{4}$ for x.	$-\frac{3}{4}+\frac{1}{3}$
Rewrite as equivalent fractions with the LCD, 12.	$-\frac{3\cdot3}{4\cdot3}+\frac{1\cdot4}{3\cdot4}$
Simplify.	$-\frac{9}{12}+\frac{4}{12}$
Add.	$-\frac{5}{12}$

TRY IT 11.1

Evaluate $x+\frac{3}{4}$ when a) $x=-\frac{7}{4}$ b) $x=-\frac{5}{4}$.

Show answer

a) -1 b) $-\frac{1}{2}$

TRY IT 11.2

Evaluate $y+\frac{1}{2}$ when a) $y=\frac{2}{3}$ b) $y=-\frac{3}{4}$.

Show answer

a) $\frac{7}{6}$ b) $-\frac{1}{12}$

EXAMPLE 12

Evaluate $-\frac{5}{6}-y$ when $y=-\frac{2}{3}$.

Solution

	$-\frac{5}{6}-y$
Substitute $-\frac{2}{3}$ for y.	$-\frac{5}{6}-\left(-\frac{2}{3}\right)$
Rewrite as equivalent fractions with the LCD, 6.	$-\frac{5}{6}-\left(-\frac{4}{6}\right)$
Subtract.	$\frac{-5-(-4)}{6}$
Simplify.	$-\frac{1}{6}$

TRY IT 12.1

Evaluate $-\frac{1}{2}-y$ when $y=-\frac{1}{4}$.

Show answer

$-\frac{1}{4}$

TRY IT 12.2

Evaluate $-\frac{3}{8}-y$ when $x=-\frac{5}{2}$.

Show answer

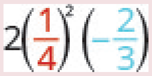
$-\frac{17}{8}$

EXAMPLE 13

Evaluate $2x^2y$ when $x=\frac{1}{4}$ and $y=-\frac{2}{3}$.

Solution

Substitute the values into the expression.

	$2x^2y$
 Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	 $2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply. Divide out the common factors. Notice we write 16 as $2 \cdot 2 \cdot 4$ to make it easy to remove common factors.	$-\frac{\cancel{2}\cdot 1\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot 4\cdot 3}$
Simplify.	$-\frac{1}{12}$

TRY IT 13.1

Evaluate $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$.

Show answer

$-\frac{1}{2}$

TRY IT 13.2

Evaluate $4c^3d$ when $c = -\frac{1}{2}$ and $d = -\frac{4}{3}$.

Show answer

$\frac{2}{3}$

The next example will have only variables, no constants.

EXAMPLE 14

Evaluate $\frac{p+q}{r}$ when $p = -4, q = -2,$ and $r = 8$.

Solution

To evaluate $\frac{p+q}{r}$ when $p = -4, q = -2,$ and $r = 8$, we substitute the values into the expression.

	$\frac{p+q}{r}$
Substitute -4 for p, -2 for q and 8 for r.	$\frac{-4+(-2)}{8}$
Add in the numerator first.	$\frac{-6}{8}$
Simplify.	$-\frac{3}{4}$

TRY IT 14.1

Evaluate $\frac{a+b}{c}$ when $a=-8, b=-7, \text{ and } c=6.$

Show answer
$-\frac{5}{2}$

TRY IT 14.2

Evaluate $\frac{x+y}{z}$ when $x=9, y=-18, \text{ and } z=-6.$

Show answer
$\frac{3}{2}$

Key Concepts

- **Fraction Addition and Subtraction:** If $a, b, \text{ and } c$ are numbers where $c \neq 0$, then $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ and $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **Strategy for Adding or Subtracting Fractions**
  1. Do they have a common denominator?
     Yes—go to step 2.
     No—Rewrite each fraction with the LCD (Least Common Denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.

2. Add or subtract the fractions.
3. Simplify, if possible. To multiply or divide fractions, an LCD IS NOT needed. To add or subtract fractions, an LCD IS needed.

- **Simplify Complex Fractions**
    1. Simplify the numerator.
    2. Simplify the denominator.
    3. Divide the numerator by the denominator. Simplify if possible.

Glossary

least common denominator
The least common denominator (LCD) of two fractions is the Least common multiple (LCM) of their denominators.

Practice Makes Perfect

Add and Subtract Fractions with a Common Denominator

In the following exercises, add.

1. $\frac{6}{13}+\frac{5}{13}$	2. $\frac{4}{15}+\frac{7}{15}$
3. $\frac{x}{4}+\frac{3}{4}$	4. $\frac{8}{q}+\frac{6}{q}$
5. $-\frac{3}{16}+\left(-\frac{7}{16}\right)$	6. $-\frac{5}{16}+\left(-\frac{9}{16}\right)$
7. $-\frac{8}{17}+\frac{15}{17}$	8. $-\frac{9}{19}+\frac{17}{19}$
9. $\frac{6}{13}+\left(-\frac{10}{13}\right)+\left(-\frac{12}{13}\right)$	10. $\frac{5}{12}+\left(-\frac{7}{12}\right)+\left(-\frac{11}{12}\right)$

In the following exercises, subtract.

11. $\frac{11}{15} - \frac{7}{15}$	12. $\frac{9}{13} - \frac{4}{13}$
13. $\frac{11}{12} - \frac{5}{12}$	14. $\frac{7}{12} - \frac{5}{12}$
15. $\frac{19}{21} - \frac{4}{21}$	16. $\frac{17}{21} - \frac{8}{21}$
17. $\frac{5y}{8} - \frac{7}{8}$	18. $\frac{11z}{13} - \frac{8}{13}$
19. $-\frac{23}{u} - \frac{15}{u}$	20. $-\frac{29}{v} - \frac{26}{v}$
21. $-\frac{3}{5} - \left(-\frac{4}{5}\right)$	22. $-\frac{3}{7} - \left(-\frac{5}{7}\right)$
23. $-\frac{7}{9} - \left(-\frac{5}{9}\right)$	24. $-\frac{8}{11} - \left(-\frac{5}{11}\right)$

Mixed Practice

In the following exercises, simplify.

25. $-\frac{5}{18} \cdot \frac{9}{10}$	26. $-\frac{3}{14} \cdot \frac{7}{12}$
27. $\frac{n}{5} - \frac{4}{5}$	28. $\frac{6}{11} - \frac{s}{11}$
29. $-\frac{7}{24} + \frac{2}{24}$	30. $-\frac{5}{18} + \frac{1}{18}$
31. $\frac{8}{15} \div \frac{12}{5}$	32. $\frac{7}{12} \div \frac{9}{28}$

Add or Subtract Fractions with Different Denominators

In the following exercises, add or subtract.

33. $\frac{1}{2}+\frac{1}{7}$	34. $\frac{1}{3}+\frac{1}{8}$
35. $\frac{1}{3}-\left(-\frac{1}{9}\right)$	36. $\frac{1}{4}-\left(-\frac{1}{8}\right)$
37. $\frac{7}{12}+\frac{5}{8}$	38. $\frac{5}{12}+\frac{3}{8}$
39. $\frac{7}{12}-\frac{9}{16}$	40. $\frac{7}{16}-\frac{5}{12}$
41. $\frac{2}{3}-\frac{3}{8}$	42. $\frac{5}{6}-\frac{3}{4}$
43. $-\frac{11}{30}+\frac{27}{40}$	44. $-\frac{9}{20}+\frac{17}{30}$
45. $-\frac{13}{30}+\frac{25}{42}$	46. $-\frac{23}{30}+\frac{5}{48}$
47. $-\frac{39}{56}-\frac{22}{35}$	48. $-\frac{33}{49}-\frac{18}{35}$
49. $-\frac{2}{3}-\left(-\frac{3}{4}\right)$	50. $-\frac{3}{4}-\left(-\frac{4}{5}\right)$
51. $1+\frac{7}{8}$	52. $1-\frac{3}{10}$
53. $\frac{x}{3}+\frac{1}{4}$	54. $\frac{y}{2}+\frac{2}{3}$
55. $\frac{y}{4}-\frac{3}{5}$	56. $\frac{x}{5}-\frac{1}{4}$

Mixed Practice

In the following exercises, simplify.

57. a) $\frac{2}{3}+\frac{1}{6}$ b) $\frac{2}{3}\div\frac{1}{6}$	58. a) $-\frac{2}{5}-\frac{1}{8}$ b) $-\frac{2}{5}\cdot\frac{1}{8}$
59. a) $\frac{5n}{6}\div\frac{8}{15}$ b) $\frac{5n}{6}-\frac{8}{15}$	60. a) $\frac{3a}{8}\div\frac{7}{12}$ b) $\frac{3a}{8}-\frac{7}{12}$
61. $-\frac{3}{8}\div\left(-\frac{3}{10}\right)$	62. $-\frac{5}{12}\div\left(-\frac{5}{9}\right)$
63. $-\frac{3}{8}+\frac{5}{12}$	64. $-\frac{1}{8}+\frac{7}{12}$
65. $\frac{5}{6}-\frac{1}{9}$	66. $\frac{5}{9}-\frac{1}{6}$
67. $-\frac{7}{15}-\frac{y}{4}$	68. $-\frac{3}{8}-\frac{x}{11}$
69. $\frac{11}{12a}\cdot\frac{9a}{16}$	70. $\frac{10y}{13}\cdot\frac{8}{15y}$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

71. $\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$	72. $\frac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$
73. $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$	74. $\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$
75. $\frac{2}{\frac{1}{3}+\frac{1}{5}}$	76. $\frac{5}{\frac{1}{4}+\frac{1}{3}}$
77. $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$	78. $\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$
79. $\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}$	80. $\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4}$
81. $1-\frac{3}{5}\div\frac{1}{10}$	82. $1-\frac{5}{6}\div\frac{1}{12}$
83. $\frac{2}{3}+\frac{1}{6}+\frac{3}{4}$	84. $\frac{2}{3}+\frac{1}{4}+\frac{3}{5}$
85. $\frac{3}{8}-\frac{1}{6}+\frac{3}{4}$	86. $\frac{2}{5}+\frac{5}{8}-\frac{3}{4}$
87. $12\left(\frac{9}{20}-\frac{4}{15}\right)$	88. $8\left(\frac{15}{16}-\frac{5}{6}\right)$
89. $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$	90. $\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$
91. $\left(\frac{5}{9}+\frac{1}{6}\right)\div\left(\frac{2}{3}-\frac{1}{2}\right)$	92. $\left(\frac{3}{4}+\frac{1}{6}\right)\div\left(\frac{5}{8}-\frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

93. $x+\left(-\frac{5}{6}\right)$ when a) $x=\frac{1}{3}$ b) $x=-\frac{1}{6}$	94. $x+\left(-\frac{11}{12}\right)$ when a) $x=\frac{11}{12}$ b) $x=\frac{3}{4}$
95. $x-\frac{2}{5}$ when a) $x=\frac{3}{5}$ b) $x=-\frac{3}{5}$	96. $x-\frac{1}{3}$ when a) $x=\frac{2}{3}$ b) $x=-\frac{2}{3}$
97. $\frac{7}{10}-w$ when a) $w=\frac{1}{2}$ b) $w=-\frac{1}{2}$	98. $\frac{5}{12}-w$ when a) $w=\frac{1}{4}$ b) $w=-\frac{1}{4}$
99. $2x^2y^3$ when $x=-\frac{2}{3}$ and $y=-\frac{1}{2}$	100. $8u^2v^3$ when $u=-\frac{3}{4}$ and $v=-\frac{1}{2}$
101. $\frac{a+b}{a-b}$ when $a=-3, b=8$	102. $\frac{r-s}{r+s}$ when $r=10, s=-5$

Everyday Math

103. **Decorating** Laronda is making covers for the throw pillows on her sofa. For each pillow cover, she needs $\frac{1}{2}$ yard of print fabric and $\frac{3}{8}$ yard of solid fabric. What is the total amount of fabric Laronda needs for each pillow cover?	104. **Baking** Samuel is baking chocolate chip cookies and oatmeal cookies. He needs $\frac{1}{2}$ cup of sugar for the chocolate chip cookies and $\frac{1}{4}$ of sugar for the oatmeal cookies. How much sugar does he need altogether?

Writing Exercises

105. Why do you need a common denominator to add or subtract fractions? Explain.	106. How do you find the LCD of 2 fractions?

Answers

1. $\frac{11}{13}$	3. $\frac{x+3}{4}$	5. $-\frac{5}{8}$
7. $\frac{7}{17}$	9. $-\frac{16}{13}$	11. $\frac{4}{15}$
13. $\frac{1}{2}$	15. $\frac{5}{7}$	17. $\frac{5y-7}{8}$
19. $-\frac{38}{u}$	21. $\frac{1}{5}$	23. $-\frac{2}{9}$
25. $-\frac{1}{4}$	27. $\frac{n-4}{5}$	29. $-\frac{5}{24}$
31. $\frac{2}{9}$	33. $\frac{9}{14}$	35. $\frac{4}{9}$
37. $\frac{29}{24}$	39. $\frac{1}{48}$	41. $\frac{7}{24}$
43. $\frac{37}{120}$	45. $\frac{17}{105}$	47. $-\frac{53}{40}$
49. $\frac{1}{12}$	51. $\frac{15}{8}$	53. $\frac{4x+3}{12}$
55. $\frac{4y-12}{20}$	57. a)$\frac{5}{6}$b) 4	59. a)$\frac{25n}{16}$b)$\frac{25n-16}{30}$
61. $\frac{5}{4}$	63. $\frac{1}{24}$	65. $\frac{13}{18}$
67. $\frac{-28-15y}{60}$	69. $\frac{33}{64}$	71. 54
73. $\frac{49}{25}$	75. $\frac{15}{4}$	77. $\frac{5}{21}$
79. $\frac{7}{9}$	81. -5	83. $\frac{19}{12}$
85. $\frac{23}{24}$	87. $\frac{11}{5}$	89. 1
91. $\frac{13}{3}$	93. a) $-\frac{1}{2}$ b) -1	95. a) $\frac{1}{5}$ b) -1
97. a) $\frac{1}{5}$ b) $\frac{6}{5}$	99. $-\frac{1}{9}$	101. $-\frac{5}{11}$
103. $\frac{7}{8}$ yard	105. Answers may vary	

Attributions

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2.3 Decimals

Learning Objectives

By the end of this section, you will be able to:

- Name and write decimals
- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percent

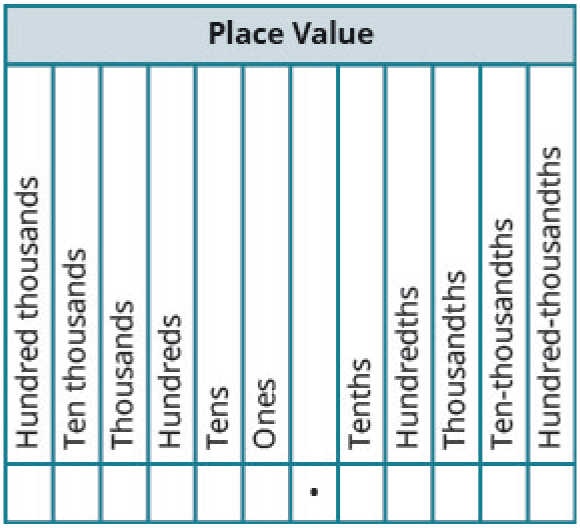
Name and Write Decimals

Decimals are another way of writing fractions whose denominators are powers of 10.

$$\begin{array}{rcll} 0.1 & = & \frac{1}{10} & \qquad\qquad 0.1 \text{ is "one tenth"} \\ 0.01 & = & \frac{1}{100} & \qquad\qquad 0.01 \text{ is "one hundredth"} \\ 0.001 & = & \frac{1}{1{,}000} & \qquad\qquad 0.001 \text{ is "one thousandth"} \\ 0.0001 & = & \frac{1}{10{,}000} & \qquad\qquad 0.0001 \text{ is "one ten-thousandth"} \end{array}$$

Notice that “ten thousand” is a number larger than one, but “one ten-thousand**th**” is a number smaller than one. The “th” at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as “ten thousand” and 10,000,000 as “ten million.” Likewise, the names of the decimal places correspond to their fraction values. Figure 1 shows the names of the place values to the left and right of the decimal point.
Place value of decimal numbers are shown to the left and right of the decimal point.



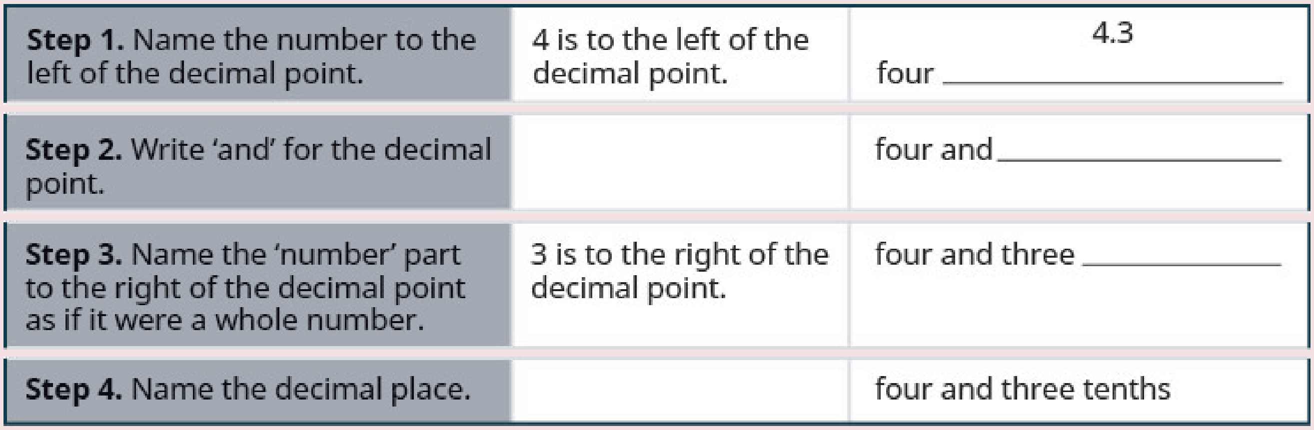
Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						•					

Figure 1

EXAMPLE 1

Name the decimal 4.3

Solution



Step 1. Name the number to the left of the decimal point.	4 is to the left of the decimal point.	4.3 four ________________
Step 2. Write 'and' for the decimal point.		four and ________________
Step 3. Name the 'number' part to the right of the decimal point as if it were a whole number.	3 is to the right of the decimal point.	four and three ____________
Step 4. Name the decimal place.		four and three tenths

TRY IT 1.1

Name the decimal: 6.7.

Show answer
six and seven tenths

TRY IT 1.2

Name the decimal: 5.8.

Show answer
five and eight tenths

We summarize the steps needed to name a decimal below.

HOW TO: Name a Decimal

1. Name the number to the left of the decimal point.
2. Write “and” for the decimal point.
3. Name the “number” part to the right of the decimal point as if it were a whole number.
4. Name the decimal place of the last digit.

EXAMPLE 2

Name the decimal: -15.571.

Solution

	-15.571
Name the number to the left of the decimal point.	negative fifteen ____
Write “and” for the decimal point.	negative fifteen and ___________________________________
Name the number to the right of the decimal point.	negative fifteen and five hundred seventy-one __________
The 1 is in the thousandths place.	negative fifteen and five hundred seventy-one thousandths

TRY IT 2.1

Name the decimal: -13.461.

Show answer
negative thirteen and four hundred sixty-one thousandths

TRY IT 2.2

Name the decimal: -2.053.

Show answer

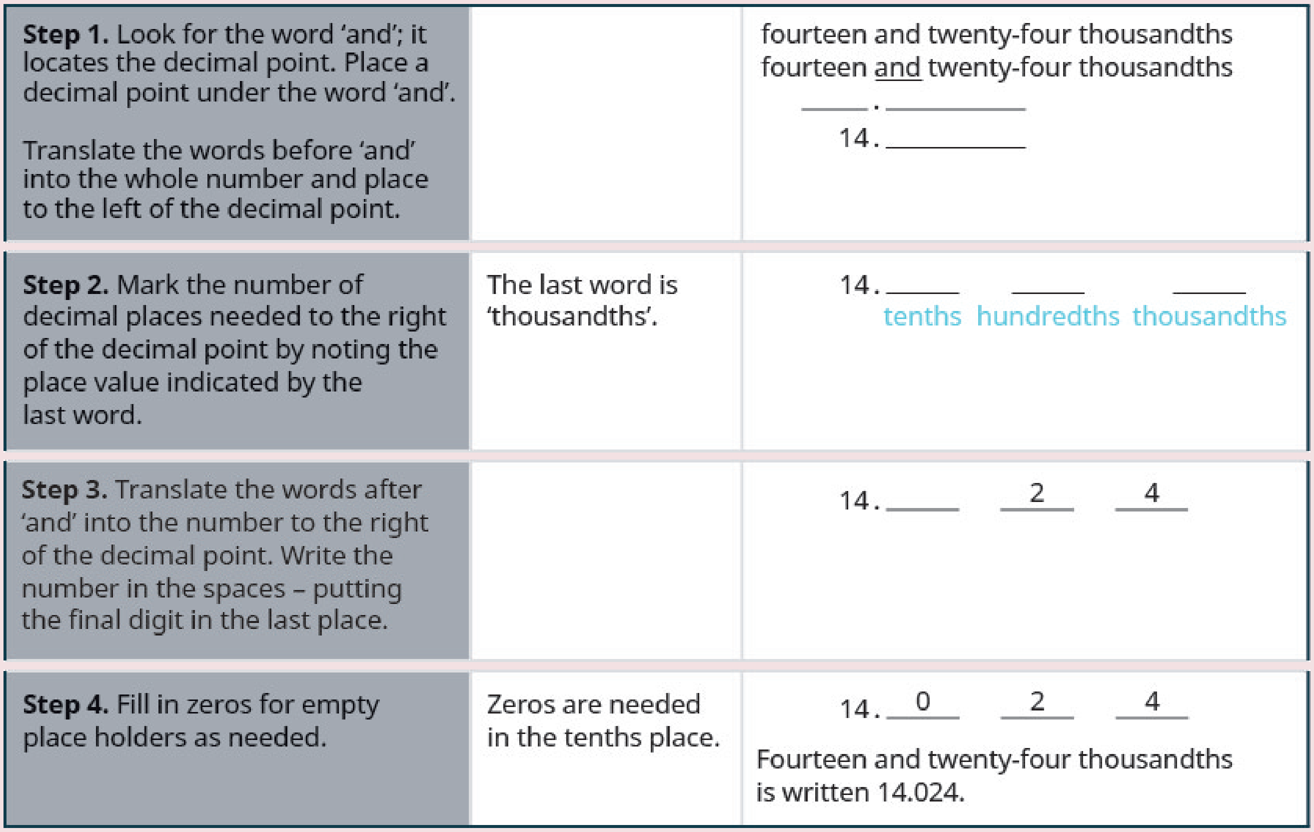
negative two and fifty-three thousandths

When we write a check we write both the numerals and the name of the number. Let’s see how to write the decimal from the name.

EXAMPLE 3

Write “fourteen and twenty-four thousandths” as a decimal.

Solution



Step 1. Look for the word ‘and’; it locates the decimal point. Place a decimal point under the word ‘and’. Translate the words before ‘and’ into the whole number and place to the left of the decimal point.		fourteen and twenty-four thousandths fourteen and twenty-four thousandths ___ . ______ 14 . ______
Step 2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.	The last word is ‘thousandths’.	14 . ___ ___ ___ tenths hundredths thousandths
Step 3. Translate the words after ‘and’ into the number to the right of the decimal point. Write the number in the spaces – putting the final digit in the last place.		14 . ___ 2 4
Step 4. Fill in zeros for empty place holders as needed.	Zeros are needed in the tenths place.	14 . 0 2 4 Fourteen and twenty-four thousandths is written 14.024.

TRY IT 3.1

Write as a decimal: thirteen and sixty-eight thousandths.

Show answer
13.68

TRY IT 3.2

Write as a decimal: five and ninety-four thousandths.

Show answer
5.94

We summarize the steps to writing a decimal.

HOW TO: Write a Decimal

1. Look for the word “and”—it locates the decimal point.
    - Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
    - If there is no “and,” write a “0” with a decimal point to its right.
2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
3. Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
4. Fill in zeros for place holders as needed.

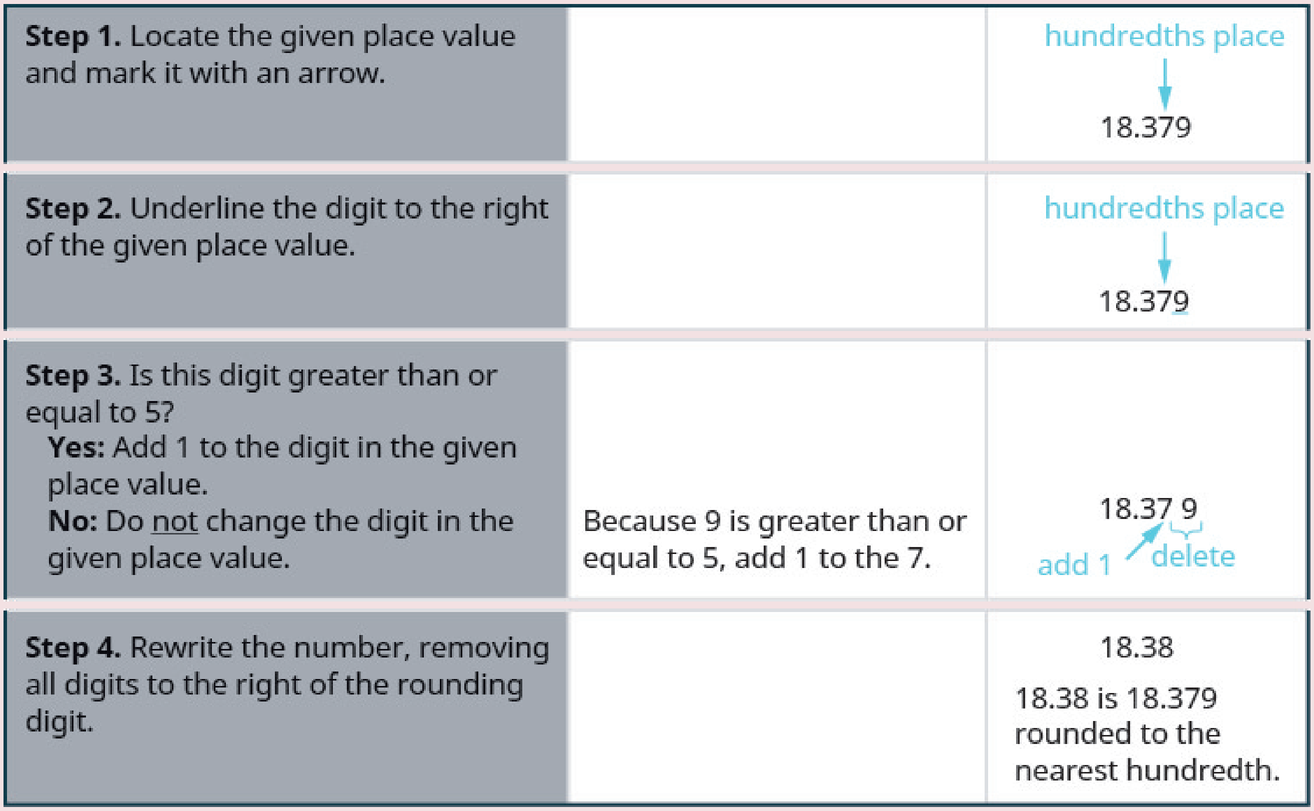
Round Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

EXAMPLE 4

Round 18.379 to the nearest hundredth.

Solution



Step 1. Locate the given place value and mark it with an arrow.		hundredths place 18.379
Step 2. Underline the digit to the right of the given place value.		hundredths place $18.37\underline{9}$
Step 3. Is this digit greater than or equal to 5? **Yes:** Add 1 to the digit in the given place value. **No:** Do <u>not</u> change the digit in the given place value.	Because 9 is greater than or equal to 5, add 1 to the 7.	18.37 9 add 1 delete
Step 4. Rewrite the number, removing all digits to the right of the rounding digit.		18.38 18.38 is 18.379 rounded to the nearest hundredth.

TRY IT 4.1

Round to the nearest hundredth: 1.047.

Show answer
1.05

TRY IT 4.2

Round to the nearest hundredth: 9.173.

Show answer
9.17

We summarize the steps for rounding a decimal here.

HOW TO: Round Decimals

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is this digit greater than or equal to 5?
    - Yes—add 1 to the digit in the given place value.
    - No—do not change the digit in the given place value.
4. Rewrite the number, deleting all digits to the right of the rounding digit.

EXAMPLE 5

Round 18.379 to the nearest a) tenth b) whole number.

Solution
Round 18.379

a) to the nearest tenth

Locate the tenths place with an arrow.	
Underline the digit to the right of the given place value.	
Because 7 is greater than or equal to 5, add 1 to the 3.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

b) to the nearest whole number

Locate the ones place with an arrow.	
Underline the digit to the right of the given place value.	
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

TRY IT 5.1

Round 6.582 to the nearest a) hundredth b) tenth c) whole number.

Show answer
a) 6.58 b) 6.6 c) 7

TRY IT 5.2

Round 15.2175 to the nearest a) thousandth b) hundredth c) tenth.

Show answer
a) 15.218 b) 15.22 c) 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

HOW TO: Add or Subtract Decimals

1. Write the numbers so the decimal points line up vertically.
2. Use zeros as place holders, as needed.
3. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

EXAMPLE 6

Add: $23.5 + 41.38$.

Solution

Write the numbers so the decimal points line up vertically.	$\begin{array}{r} 23.5 \\ +41.38 \\ \hline \end{array}$
Put 0 as a placeholder after the 5 in 23.5. Remember, $\frac{5}{10} = \frac{50}{100}$ so $0.5 = 0.50$.	$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$
Add the numbers as if they were whole numbers. Then place the decimal point in the sum.	$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$

TRY IT 6.1

Add: $4.8 + 11.69$.

Show answer
16.49

TRY IT 6.2

Add: $5.123 + 18.47$.

Show answer

23.593

EXAMPLE 7

Subtract: $20 - 14.65$.

Solution

	$20 - 14.65$
Write the numbers so the decimal points line up vertically. Remember, 20 is a whole number, so place the decimal point after the 0.	$\begin{array}{r} 20. \\ -14.65 \\ \hline \end{array}$
Put in zeros to the right as placeholders.	$\begin{array}{r} 20.00 \\ \underline{-14.65} \end{array}$
Subtract and place the decimal point in the answer.	$\begin{array}{r} \overset{1}{\overline{)2}}\overset{\overset{9}{\overline{)10}}}{\overline{)0}}\,.\,\overset{\overset{9}{\overline{)10}}}{\overline{)0}}\overset{)10}{\overline{)0}} \\ -\,1\,4\,.\,6\,5 \\ \hline 5\,.\,3\,5 \end{array}$

TRY IT 7.1

Subtract: $10 - 9.58$.

Show answer
0.42

TRY IT 7.2

Subtract: $50 - 37.42$.

Show answer
12.58

Multiply and Divide Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first convert them to fractions and then multiply.

So let's see what we would get as the product of decimals by converting them to fractions first. We will do two examples side-by-side. Look for a pattern!

	$(\underbrace{0.3}_{\text{1 place}})\quad(\underbrace{0.7}_{\text{1 place}})$	$(\underbrace{0.2}_{\text{1 place}})\quad(\underbrace{0.46}_{\text{2 places}})$
Convert to fractions.	$\frac{3}{10}\cdot\frac{7}{10}$	$\frac{2}{10}\cdot\frac{46}{100}$
Multiply.	$\frac{21}{100}$	$\frac{92}{1000}$
Convert to decimals.	$0.\underbrace{21}_{\text{2 places}}$	$0.\underbrace{092}_{\text{3 places}}$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point and the product had two decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places and the product had three decimal places.

We multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product.

The rules for multiplying positive and negative numbers apply to decimals, too, of course!

When *multiplying* two numbers,

- if their signs are the *same* the product is *positive*.
- if their signs are *different* the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

HOW TO: Multiply Decimals

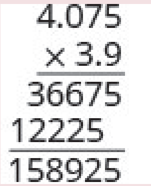
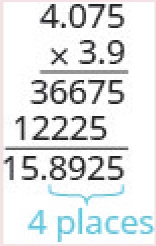
1. Determine the sign of the product.

2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

EXAMPLE 8

Multiply: $(-3.9)\,(4.075)$.

Solution

	(−3.9)(4.075)
The signs are different. The product will be negative.	
Write in vertical format, lining up the numbers on the right.	 $\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline \end{array}$
Multiply.	 $\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$
Add the number of decimal places in the factors (1 + 3).  $(-3.\underbrace{9}_{\text{1 place}})\quad(4.\underbrace{075}_{\text{3 places}})$ Place the decimal point 4 places from the right.	 $\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 15.\underbrace{8925}_{\text{4 places}} \end{array}$
The signs are different, so the product is negative.	(−3.9)(4.075) = −15.8925

TRY IT 8.1

Multiply: $-4.5\,(6.107)$.

Show answer
-27.4815

TRY IT 8.2

Multiply: $-10.79\,(8.12)$.

Show answer
-87.6148

In many of your other classes, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc.). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

HOW TO: Multiply a Decimal by a Power of Ten

1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
2. Add zeros at the end of the number as needed.

EXAMPLE 9

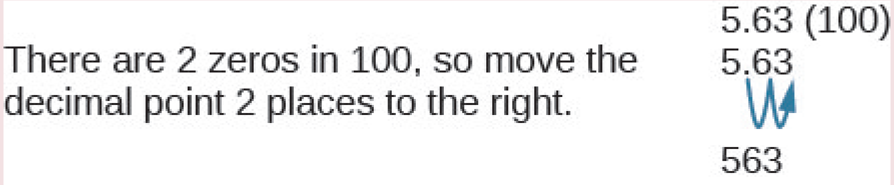
Multiply 5.63 a) by 10 b) by 100 c) by 1,000.

Solution
By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

a)

	5.63(10)
There is 1 zero in 10, so move the decimal point 1 place to the right.	

b)

	5.63(100)
There are 2 zeros in 100, so move the decimal point 2 places to the right.	 There are 2 zeros in 100, so move the decimal point 2 places to the right. 5.63 (100) 5.63 563

c)

	5.63(1,000)
There are 3 zeros in 1,000, so move the decimal point 3 places to the right.	 5.63
A zero must be added at the end.	 5,630

TRY IT 9.1

Multiply 2.58 a) by 10 b) by 100 c) by 1,000.

Show answer
a) 25.8 b) 258 c) 2,580

TRY IT 9.2

Multiply 14.2 a) by 10 b) by 100 c) by 1,000.

Show answer
a) 142 b) 1,420 c) 14,200

Just as with multiplication, division of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by to make it a whole number. Then multiply the numerator by that same power of 10. Because of the equivalent fractions property, we haven't changed the value of the fraction! The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example:

$$\frac{0.8}{0.4}$$
$$\frac{0.8(10)}{0.4(10)}$$
$$\frac{8}{4}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{\text{c}} \qquad\qquad \underset{\text{divisor}}{b}\ \overset{\overset{\text{c}}{\text{quotient}}}{\overline{\big)\ \underset{\text{dividend}}{a}}}$$

We'll write the steps to take when dividing decimals, for easy reference.

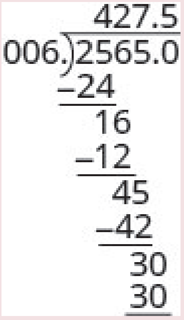
HOW TO: Divide Decimals

1. Determine the sign of the quotient.
2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

EXAMPLE 10

Divide: $-25.56 \div (-0.06)$.

Solution
Remember, you can "move" the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by “moving” the decimal point all the way to the right.	
“Move” the decimal point in the dividend the same number of places.	
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

TRY IT 10.1

Divide: $-23.492 \div (-0.04)$.

Show answer
687.3

TRY IT 10.2

Divide: $-4.11 \div (-0.12)$.

Show answer
34.25

A common application of dividing whole numbers into decimals is when we want to find the price of one item that is sold as part of a multi-pack. For example, suppose a case of 24 water bottles costs $3.99. To find the price of one water bottle, we would divide $3.99 by 24. We show this division in Example 11. In calculations with money, we will round the answer to the nearest cent (hundredth).

EXAMPLE 11

Divide: $\$3.99 \div 24$.

Solution

	$\$3.99 \div 24$
Place the decimal point in the quotient above the decimal point in the dividend.	
Divide as usual. When do we stop? Since this division involves money, we round it to the nearest cent (hundredth.) To do this, we must carry the division to the thousandths place.	$\begin{array}{r} 0.166 \\ 24\overline{)3.990} \\ \underline{24} \\ 159 \\ \underline{144} \\ 150 \\ \underline{144} \\ 6 \end{array}$
Round to the nearest cent.	$\$0.166 \approx \$0.17$ $\$3.99 \div 24 \approx \$0.17$

TRY IT 11.1

Divide: $\$6.99 \div 36$.

Show answer
$0.19

TRY IT 11.2

Divide: $\$4.99 \div 12$.

Show answer
$0.42

Convert Decimals and Fractions

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03 the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03

0 $0.03 = \frac{3}{100}$

Notice, when the number to the left of the decimal is zero, we get a fraction whose numerator is less than its denominator. Fractions like this are called proper fractions.

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

HOW TO: Covert a Decimal to a Proper Fraction

1. Determine the place value of the final digit.
2. Write the fraction.
    - numerator—the “numbers” to the right of the decimal point
    - denominator—the place value corresponding to the final digit

EXAMPLE 12

Write 0.374 as a fraction.

Solution

	0.374
Determine the place value of the final digit.	 0.3 7 4 tenths hundredths thousandths
Write the fraction for 0.374: • The numerator is 374. • The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$ so, $0.374 = \frac{187}{500}$

Did you notice that the number of zeros in the denominator of $\frac{374}{1,000}$ is the same as the number of decimal places in 0.374?

TRY IT 12.1

Write 0.234 as a fraction.

Show answer
$\frac{117}{500}$

TRY IT 12.2

Write 0.024 as a fraction.

Show answer
$\frac{3}{125}$

We've learned to convert decimals to fractions. Now we will do the reverse—convert fractions to decimals. Remember that the fraction bar means division. So $\frac{4}{5}$ can be written $4 \div 5$ or $5\overline{)4}$. This leads to the following method for converting a fraction to a decimal.

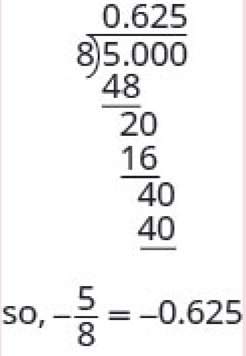
HOW TO: Covert a Fraction to a Decimal

To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

EXAMPLE 13

Write $-\frac{5}{8}$ as a decimal.

Solution
Since a fraction bar means division, we begin by writing $\frac{5}{8}$ as $8\overline{)5}$. Now divide.



so, $-\frac{5}{8} = -0.625$

TRY IT 13.1

Write $-\frac{7}{8}$ as a decimal.

Show answer
-0.875

TRY IT 13.2

Write $-\frac{3}{8}$ as a decimal.

Show answer
-0.375

When we divide, we will not always get a zero remainder. Sometimes the quotient ends up with a decimal that repeats. A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly. A bar is placed over the repeating block of digits to indicate it repeats.

Repeating Decimal

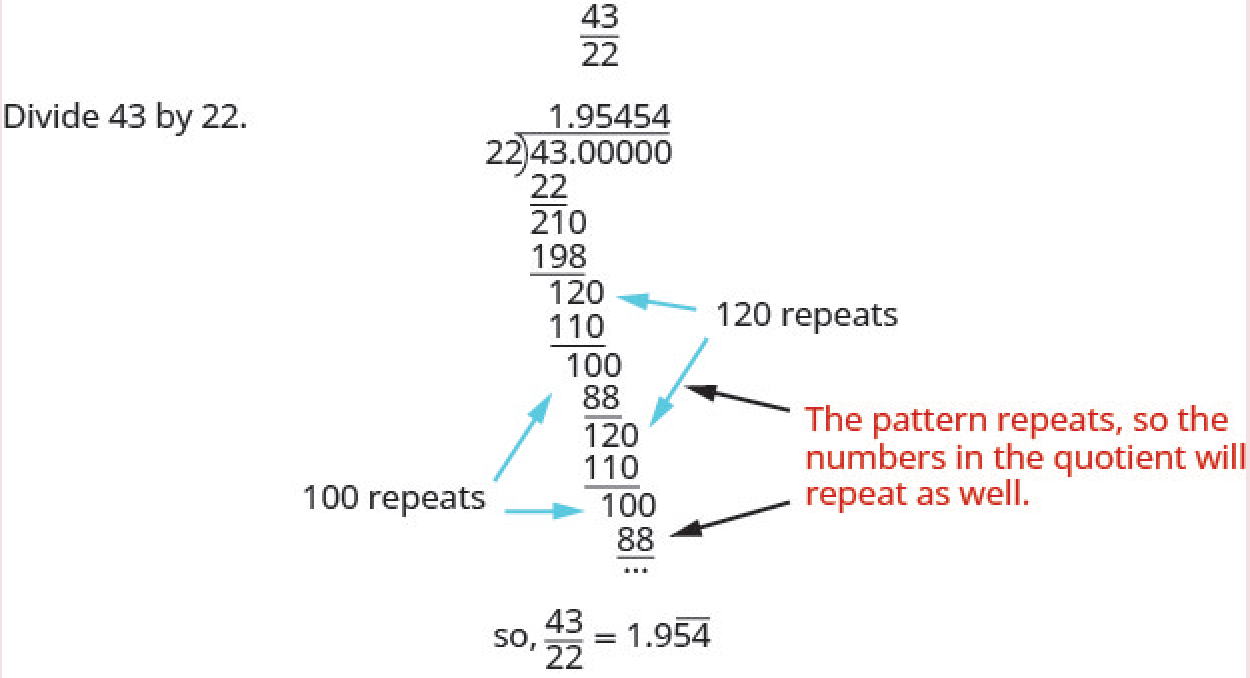
A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly.

A bar is placed over the repeating block of digits to indicate it repeats.

EXAMPLE 14

Write $\frac{43}{22}$ as a decimal.

Solution



TRY IT 14.1

Write $\frac{27}{11}$ as a decimal.

Show answer
2. 45

TRY IT 14.2

Write $\frac{51}{22}$ as a decimal.

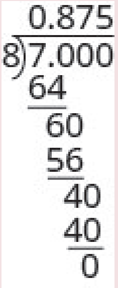
Show answer
2.3 18

Sometimes we may have to simplify expressions with fractions and decimals together.

EXAMPLE 15

Simplify: $\frac{7}{8}+6.4$.

Solution
First we must change one number so both numbers are in the same form. We can change the fraction to a decimal, or change the decimal to a fraction. Usually it is easier to change the fraction to a decimal.

		$\frac{7}{8}+6.4$
Change $\frac{7}{8}$ to a decimal.	0.875 8)7.000 64 60 56 40 40 0	
Add.		$0.875+6.4$
		7.275
		So, $\frac{7}{8}+6.4=7.275$

TRY IT 15.1

Simplify: $\frac{3}{8}+4.9$.

Show answer
5.275

TRY IT 15.2

Simplify: $5.7+\frac{13}{20}$.

Show answer
6.35

Key Concepts

- **Name a Decimal**
    1. Name the number to the left of the decimal point.
    2. Write ”and” for the decimal point.
    3. Name the “number” part to the right of the decimal point as if it were a whole number.
    4. Name the decimal place of the last digit.
- **Write a Decimal**
    1. Look for the word ‘and’—it locates the decimal point. Place a decimal point under the word ‘and.’ Translate the words before ‘and’ into the whole number and place it to the left of the decimal point. If there is no “and,” write a “0” with a decimal point to its right.
    2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
    3. Translate the words after ‘and’ into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
    4. Fill in zeros for place holders as needed.
- **Round a Decimal**
    1. Locate the given place value and mark it with an arrow.
    2. Underline the digit to the right of the place value.
    3. Is this digit greater than or equal to 5? Yes—add 1 to the digit in the given place value. No—do not change the digit in the given place value.
    4. Rewrite the number, deleting all digits to the right of the rounding digit.
- **Add or Subtract Decimals**
    1. Write the numbers so the decimal points line up vertically.
    2. Use zeros as place holders, as needed.
    3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.
- **Multiply Decimals**
    1. Determine the sign of the product.
    2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
    3. Place the decimal point. The number of decimal places in the product is the sum of the

decimal places in the factors.

4. Write the product with the appropriate sign.

- **Multiply a Decimal by a Power of Ten**
    1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
    2. Add zeros at the end of the number as needed.
- **Divide Decimals**
    1. Determine the sign of the quotient.
    2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places – adding zeros as needed.
    3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
    4. Write the quotient with the appropriate sign.
- **Convert a Decimal to a Proper Fraction**
    1. Determine the place value of the final digit.
    2. Write the fraction: numerator—the 'numbers' to the right of the decimal point; denominator—the place value corresponding to the final digit.
- **Convert a Fraction to a Decimal** Divide the numerator of the fraction by the denominator.

Practice Makes Perfect

Name and Write Decimals

In the following exercises, write as a decimal.

1. Twenty-nine and eighty-one hundredths	2. Sixty-one and seventy-four hundredths
3. Seven tenths	4. Six tenths
5. Twenty-nine thousandth	6. Thirty-five thousandths
7. Negative eleven and nine ten-thousandths	8. Negative fifty-nine and two ten-thousandths

In the following exercises, name each decimal.

9. 5.5	10. 14.02
11. 8.71	12. 2.64
13. 0.002	14. 0.479
15. -17.9	16. -31.4

Round Decimals

In the following exercises, round each number to the nearest tenth.

17. 0.67	18. 0.49
19. 2.84	20. 4.63

In the following exercises, round each number to the nearest hundredth.

21. 0.845	22. 0.761
23. 0.299	24. 0.697
25. 4.098	26. 7.096

In the following exercises, round each number to the nearest a) hundredth b) tenth c) whole number.

27. 5.781	28. 1.6381
29. 63.479	30. 84.281

Add and Subtract Decimals

In the following exercises, add or subtract.

31. $16.92+7.56$	32. $248.25-91.29$
33. $21.76-30.99$	34. $38.6+13.67$
35. $-16.53-24.38$	36. $-19.47-32.58$
37. $-38.69+31.47$	38. $29.83+19.76$
39. $72.5-100$	40. $86.2-100$
41. $15+0.73$	42. $27+0.87$
43. $91.95-(-10.462)$	44. $94.69-(-12.678)$
45. $55.01-3.7$	46. $59.08-4.6$
47. $2.51-7.4$	48. $3.84-6.1$

Multiply and Divide Decimals

In the following exercises, multiply.

49. $(0.24)(0.6)$	50. $(0.81)(0.3)$
51. $(5.9)(7.12)$	52. $(2.3)(9.41)$
53. $(-4.3)(2.71)$	54. $(-8.5)(1.69)$
55. $(-5.18)(-65.23)$	56. $(-9.16)(-68.34)$
57. $(0.06)(21.75)$	58. $(0.08)(52.45)$
59. $(9.24)(10)$	60. $(6.531)(10)$
61. $(55.2)(1000)$	62. $(99.4)(1000)$

In the following exercises, divide.

63. $4.75\div 25$	64. $12.04\div 43$
65. $\$117.25\div 48$	66. $\$109.24\div 36$
67. $0.6\div 0.2$	68. $0.8\div 0.4$
69. $1.44\div(-0.3)$	70. $1.25\div(-0.5)$
71. $-1.75\div(-0.05)$	72. $-1.15\div(-0.05)$
73. $5.2\div 2.5$	74. $6.5\div 3.25$
75. $11\div 0.55$	76. $14\div 0.35$

Convert Decimals and Fractions

In the following exercises, write each decimal as a fraction.

77. 0.04	78. 0.19
79. 0.52	80. 0.78
81. 1.25	82. 1.35
83. 0.375	84. 0.464
85. 0.095	86. 0.085

In the following exercises, convert each fraction to a decimal.

87. $\frac{17}{20}$	88. $\frac{13}{20}$
89. $\frac{11}{4}$	90. $\frac{17}{4}$
91. $-\frac{310}{25}$	92. $-\frac{284}{25}$
93. $\frac{15}{11}$	94. $\frac{18}{11}$
95. $\frac{15}{111}$	96. $\frac{25}{111}$
97. $2.4+\frac{5}{8}$	98. $3.9+\frac{9}{20}$

Everyday Math

99. **Salary Increase** Danny got a raise and now makes \$58,965.95 a year. Round this number to the nearest
a) dollar
b) thousand dollars
c) ten thousand dollars.

100. **New Car Purchase** Selena's new car cost \$23,795.95. Round this number to the nearest
a) dollar
b) thousand dollars
c) ten thousand dollars.

101. **Sales Tax** Hyo Jin lives in Vancouver. She bought a refrigerator for \$1,624.99 and when the clerk calculated the sales tax it came out to exactly \$142.186625. Round the sales tax to the nearest
a) penny and
b) dollar.

102. **Sales Tax** Jennifer bought a \$1,038.99 dining room set for her home in Burnaby. She calculated the sales tax to be exactly \$67.53435. Round the sales tax to the nearest
a) penny and
b) dollar.

103. **Paycheck** Annie has two jobs. She gets paid \$14.04 per hour for tutoring at Community College and \$8.75 per hour at a coffee shop. Last week she tutored for 8 hours and worked at the coffee shop for 15 hours.
a) How much did she earn?
b) If she had worked all 23 hours as a tutor instead of working both jobs, how much more would she have earned?

104. **Paycheck** Jake has two jobs. He gets paid \$7.95 per hour at the college cafeteria and \$20.25 at the art gallery. Last week he worked 12 hours at the cafeteria and 5 hours at the art gallery.
a) How much did he earn?
b) If he had worked all 17 hours at the art gallery instead of working both jobs, how much more would he have earned?

Writing Exercises

105. How does knowing about Canadian money help you learn about decimals?

106. Explain how you write "three and nine hundredths" as a decimal.

Glossary

decimal
A decimal is another way of writing a fraction whose denominator is a power of ten.

percent
A percent is a ratio whose denominator is 100.

repeating decimal
A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly.

Answers

1. 29.81	3. 0.7	5. 0.029
7. -11.0009	9. five and five tenths	11. eight and seventy-one hundredths
13. two thousandths	15. negative seventeen and nine tenths	17. 0.7
19. 2.8	21. 0.85	23. 0.30
25. 4.10	27. a) 5.78 b) 5.8 c) 6	29. a) 63.48 b) 63.5 c) 63
31. 24.48	33. -9.23	35. -40.91
37. -7.22	39. -27.5	41. 15.73
43. 102.212	45. 51.31	47. -4.89
49. 0.144	51. 42.008	53. -11.653
55. 337.8914	57. 1.305	59. 92.4
61. 55,200	63. 0.19	65. $2.44
67. 3	69. -4.8	71. 35
73. 2.08	75. 20	77. $\frac{1}{25}$
79. $\frac{13}{25}$	81. $\frac{5}{4}$	83. $\frac{3}{8}$
85. $\frac{19}{200}$	87. 0.85	89. 2.75
91. -12.4	93. $1.\ 36$	95. $0.\ 135$
97. 3.025	99. a) $58,966 b) $59,000 c) $60,000	101. a) $142.19; b) $142
103. a) $243.57 b) $79.35	105. Answers may vary.	107. Answers may vary.

Attributions

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2.4 Introduction to the Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions on the number line
- Locate decimals on the number line

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers?

Counting numbers	$1, 2, 3, 4,$
Whole numbers	$0, 1, 2, 3, 4,$
Integers	$-3, -2, -1, 0, 1, 2, 3,$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A rational number is a number that can be written as a ratio of two integers.

Rational Number

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

A rational number can be written as the ratio of two integers.

All signed fractions, such as $\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, -\frac{20}{3}$ are rational numbers. Each numerator and each denominator is an integer.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. Each integer can be written as a ratio of integers in many ways. For example, 3 is equivalent to $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}$

An easy way to write an integer as a ratio of integers is to write it as a fraction with denominator one.

$3 = \frac{3}{1}$ $\quad -8 = -\frac{8}{1}$ $\quad 0 = \frac{0}{1}$

Since any integer can be written as the ratio of two integers, *all integers are rational numbers*! Remember that the counting numbers and the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let's look at a few to see if we can write each of them as the ratio of two integers.

We've already seen that integers are rational numbers. The integer -8 could be written as the decimal -8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means $7\frac{3}{10}$, we can write it as an improper fraction, $\frac{73}{10}$. So 7.3 is the ratio of the integers 73 and 10. It is a rational number.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684 is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction.

EXAMPLE 1

Write as the ratio of two integers: a) -27 b) 7.31

Solution

a) Write it as a fraction with denominator 1.	-27 $\frac{-27}{1}$
b) Write it as a mixed number. Remember, 7 is the whole number and the decimal part, 0.31, indicates hundredths. Convert to an improper fraction.	7.31 $7\frac{31}{100}$ $\frac{731}{100}$

So we see that -27 and 7.31 are both rational numbers, since they can be written as the ratio of two integers.

TRY IT 1.1

Write as the ratio of two integers: a) -24 b) 3.57

Show answer
a) $\frac{-24}{1}$ b) $\frac{357}{100}$

TRY IT 1.2

Write as the ratio of two integers: a) -19 b) 8.41

Show answer
a) $\frac{-19}{1}$ b) $\frac{841}{100}$

Let’s look at the decimal form of the numbers we know are rational.

We have seen that *every integer is a rational number*, since $a = \frac{a}{1}$ for any integer, *a*. We can also change any integer to a decimal by adding a decimal point and a zero.

Integer	-2	-1	0	1	2	3
Decimal form	-2.0	-1.0	0.0	1.0	2.0	3.0
These decimal numbers stop.						

We have also seen that *every fraction is a rational number*. Look at the decimal form of the fractions we considered above.

Ratio of integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
The decimal form	0.8	-0.875	3.25	$-6.666...$
These decimals either stop or repeat.				

What do these examples tell us?

Every rational number can be written both as a ratio of integers, $\frac{p}{q}$,where p and q are integers and $q \neq 0$,and as a decimal that either stops or repeats.

Here are the numbers we looked at above expressed as a ratio of integers and as a decimal:

	Fractions				**Integers**					
Number	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	-2	-1	0	1	2	3
Ratio of Integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$
Decimal Form	0.8	-0.875	3.25	$-6.\overline{6}$	-2.0	-1.0	0.0	1.0	2.0	3.0

Rational Number

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes!

The number π (the Greek letter *pi*, pronounced “pie”), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654...$$

We can even create a decimal pattern that does not stop or repeat, such as

$$2.01001000100001\ldots$$

Numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call these numbers irrational. More on irrational numbers later on is this course.

Irrational Number

An irrational number is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let’s summarize a method we can use to determine whether a number is rational or irrational.

Rational or Irrational?

If the decimal form of a number

- *repeats or stops*, the number is **rational**.
- *does not repeat and does not stop*, the number is irrational

EXAMPLE 2

Given the numbers $0.58\overline{3}, 0.47, 3.605551275...$ list the a) rational numbers b) irrational numbers.

Solution

a) Look for decimals that repeat or stop.	The 3 repeats in $0.58\overline{3}$. The decimal 0.47 stops after the 7. So $0.58\overline{3}$ and 0.47 are rational.
b) Look for decimals that neither stop nor repeat.	3.605551275 has no repeating block of digits and it does not stop. So 3.605551275 is irrational.

TRY IT 2.1

For the given numbers list the a) rational numbers b) irrational numbers: $0.29, 0.81\overline{6}, 2.515115111\ldots$.

Show answer

a) $0.29, 0.81\overline{6}$ b) 2.515115111

TRY IT 2.2

For the given numbers list the a) rational numbers b) irrational numbers: $2.6\overline{3}, 0.125, 0.418302\ldots$

Show answer

a) $2.6\overline{3}, 0.125$ b) 0.418302

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of real numbers.

Real Number

A **real number** is a number that is either rational or irrational.

All the numbers we use in algebra are real numbers. Figure 1 illustrates how the number sets we've discussed in this section fit together.

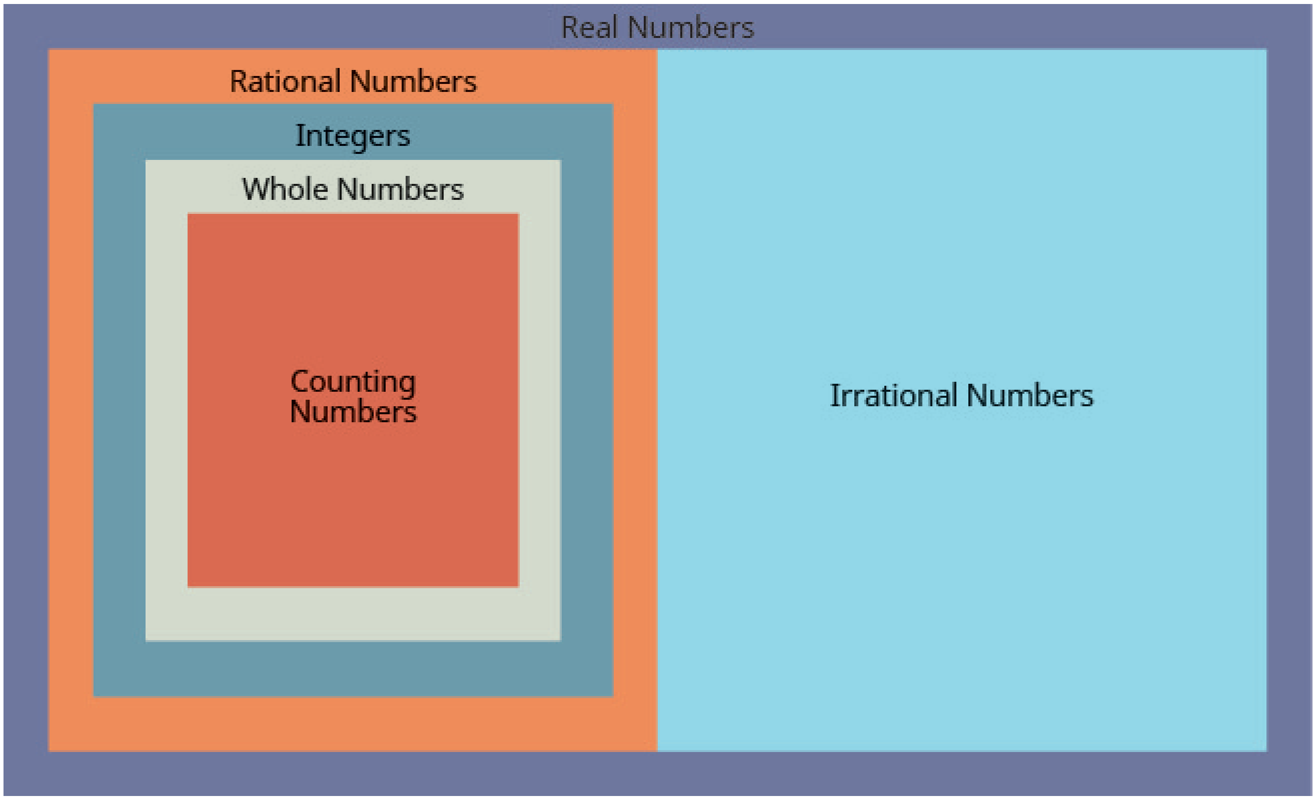


Figure 1 This chart shows the number sets that make up the set of real numbers. Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be?

EXAMPLE 3

Given the numbers $-7, \frac{14}{5}, 8, 0, 5.9,$ -6.457..., list the a) whole numbers b) integers c) rational numbers d) irrational numbers e) real numbers.

Solution

a) Remember, the whole numbers are 0, 1, 2, 3, … So 0 and 8 are the only whole numbers given.
b) The integers are the whole numbers, their opposites, and 0. So the whole numbers 0 and 8 are integers, and

-7 is the opposite of a whole number so it is an integer,too. So the integers are $-7, 0, and 8$.
c) Since all integers are rational, then $-7, 0, 8,$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ $\text{and } 5.9$ are rational. So the list of rational numbers is $-7, 0, \frac{14}{5}, 8, 5.9,$
d) Remember that 6.457… is a decimal that *does not repeat and does not stop* , so 6.457… is irrational.
e) All the numbers listed are real numbers.

TRY IT 3.1

For the given numbers, list the a) whole numbers b) integers c) rational numbers d) irrational numbers e) real numbers: $-3,$ -$\sqrt{2.97294...}$, $\frac{0}{5}$ $\bar{3}$, $\frac{9}{5}, 4, \sqrt{49}$.

Show answer

a) $4, \sqrt{49}$ b) $-3, 4, \sqrt{49}$ c) $-3, 0.\ \overline{3}, \frac{9}{5}, 4, \sqrt{49}$ d) $-\sqrt{2}$e)$-3, -\sqrt{2}, 0.\ \overline{3}, \frac{9}{5}, 4, \sqrt{49}$

TRY IT 3.2

For the given numbers, list the a) whole numbers b) integers c) rational numbers d) irrational numbers e) real numbers: -$0.25, -\frac{3}{8}, -1, 6, 2.041975...$

Show answer
a) $6, \sqrt{121}$ b) $-\sqrt{25}, -1, 6, \sqrt{121}$ c) $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$ d) 2.041975e) $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975...$

Locate Fractions on the Number Line

The last time we looked at the number line, it only had positive and negative integers on it. We now want to include fractions and decimals on it.

Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5, \text{ and } \frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and -5. because they are the easiest to plot. See Figure 2.

The proper fractions listed are $\frac{1}{5} \text{ and } -\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between $0 \text{ and } 1$. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$. See Figure 2.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$. See Figure 2.

Finally, look at the improper fractions $\frac{7}{4}, -\frac{9}{2}, \frac{8}{3}$. These are fractions in which the numerator is greater than the denominator. Locating these points may be easier if you change each of them to a mixed number. See Figure 2.

$$\frac{7}{4} = 1\frac{3}{4} \qquad -\frac{9}{2} = -4\frac{1}{2} \qquad \frac{8}{3} = 2\frac{2}{3}$$

Figure 2 shows the number line with all the points plotted.

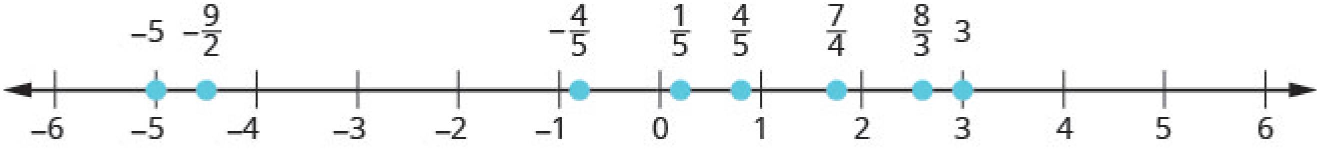


Figure 2

EXAMPLE 4

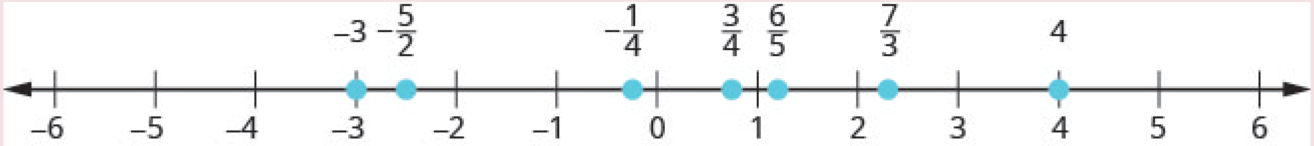
Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2}$, and $\frac{7}{3}$.

Solution

Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts then, we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

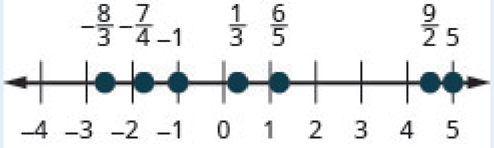
Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}, \frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



TRY IT 4.1

Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

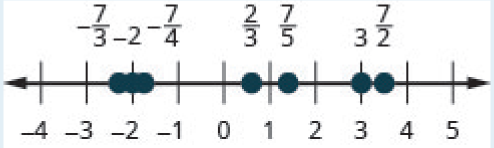
Show answer



TRY IT 4.2

Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

Show answer



In Example 5, we'll use the inequality symbols to order fractions. In previous chapters we used the number line to order numbers.

- $a < b$ "*a* is less than *b*" when *a* is to the left of *b* on the number line
- $a > b$ "*a* is greater than *b*" when *a* is to the right of *b* on the number line

As we move from left to right on a number line, the values increase.

EXAMPLE 5

Order each of the following pairs of numbers, using < or >. It may be helpful to refer Figure 3.

a) $-\frac{2}{3}$____-1 b) $-3\frac{1}{2}$____-3 c) $\frac{3}{4}$____$-\frac{1}{4}$ d) -2____$-\frac{8}{3}$

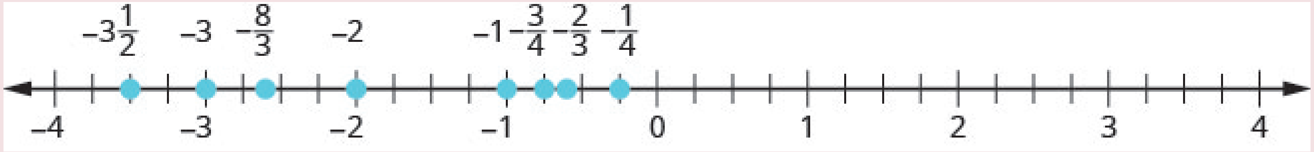


Figure 3

$-3\frac{1}{2}$____ -3

$-3\frac{1}{2} < -3$

Solution

a) $-\frac{2}{3}$ is to the right of -1 on the number line.	$-\frac{2}{3}$___-1 $-\frac{2}{3}$ > -1
b) $-3\frac{1}{2}$ is to the right of -3 on the number line.	$-3\frac{1}{2}$____ -3 $-3\frac{1}{2} < -3$
c) $-\frac{3}{4}$ is to the right of $-\frac{1}{4}$ on the number line.	$-\frac{3}{4}$____ $-\frac{1}{4}$ $-\frac{3}{4} < -\frac{1}{4}$
d) -2 is to the right of $-\frac{8}{3}$ on the number line.	-2____ $-\frac{8}{3}$ $-2 > -\frac{8}{3}$

TRY IT 5.1

Order each of the following pairs of numbers, using < or >:

a) $-\frac{1}{3}$____ -1 b) $-1\frac{1}{2}$____ -2 c) $-\frac{2}{3}$____ $-\frac{1}{3}$ d) -3____ $-\frac{7}{3}$.

Show answer

a) > b) > c) < d) <

TRY IT 5.2

Order each of the following pairs of numbers, using < or >:

a) -1____ $-\frac{2}{3}$ b) $-2\frac{1}{4}$____ -2 c) $-\frac{3}{5}$____ $-\frac{4}{5}$ d) -4____ $-\frac{10}{3}$.

Show answer

a) < b) < c) > d) <

Locate Decimals on the Number Line

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

EXAMPLE 6

Locate 0.4 on the number line.

Solution

A proper fraction has value less than one. The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 and 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line. See Figure 4.

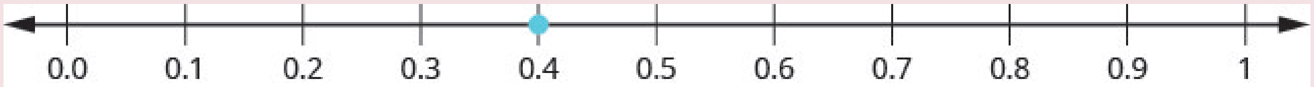
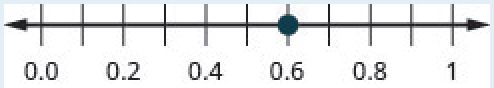


Figure 4

TRY IT 6.1

Locate on the number line: 0.6

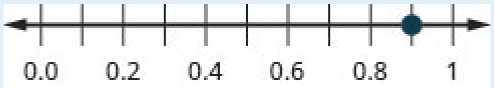
Show answer



TRY IT 6.2

Locate on the number line: 0.9

Show answer



EXAMPLE 7

Locate -0.74 on the number line.

Solution

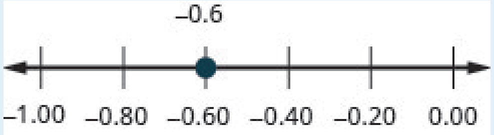
The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and -1. On a number line, mark off and label the hundredths in the interval between 0 and -1. See Figure 5.

Figure 5

TRY IT 7.1

Locate on the number line: -0.6.

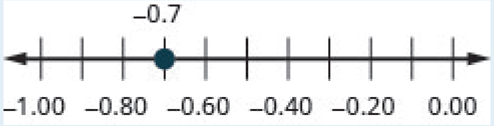
Show answer



TRY IT 7.2

Locate on the number line: -0.7.

Show answer



Which is larger, 0.04 or 0.40? If you think of this as money, you know that ?0.40 (forty cents) is greater than ?0.04 (four cents). So,

$$0.40 > 0.04$$

Again, we can use the number line to order numbers.

- $a < b$ “*a* is less than *b*” when *a* is to the left of *b* on the number line
- $a > b$ “*a* is greater than *b*” when *a* is to the right of *b* on the number line

Where are 0.04 and 0.40 located on the number line? See Figure 6.

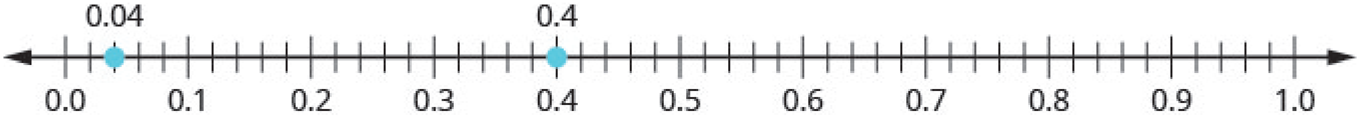


Figure 6

We see that 0.40 is to the right of 0.04 on the number line. This is another way to demonstrate that 0.40 > 0.04

How does 0.31 compare to 0.308? This doesn't translate into money to make it easy to compare. But if we convert 0.31 and 0.308 into fractions, we can tell which is larger.

	0.31	0.308
Convert to fractions.	$\frac{31}{100}$	$\frac{308}{1000}$
We need a common denominator to compare them.	$\frac{31 \cdot 10}{100 \cdot 10}$	$\frac{308}{1000}$
	$\frac{310}{1000}$	$\frac{308}{1000}$

Because 310 > 308, we know that $\frac{310}{1000} > \frac{308}{1000}$. Therefore, 0.31 > 0.308

Notice what we did in converting 0.31 to a fraction—we started with the fraction $\frac{31}{100}$ and ended with the equivalent fraction $\frac{310}{1000}$. Converting $\frac{310}{1000}$ back to a decimal gives 0.310. So 0.31 is equivalent to 0.310. Writing zeros at the end of a decimal does not change its value!

$$\frac{31}{100} = \frac{310}{1000} \quad \text{and} \quad 0.31 = 0.310$$

We say 0.31 and 0.310 are equivalent decimals.

Equivalent Decimals

Two decimals are equivalent if they convert to equivalent fractions.

We use equivalent decimals when we order decimals.

The steps we take to order decimals are summarized here.

HOW TO: Order Decimals.

1. Write the numbers one under the other, lining up the decimal points.
2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
3. Compare the numbers as if they were whole numbers.
4. Order the numbers using the appropriate inequality sign.

EXAMPLE 8

Order $0.64\_\_\_\_0.6$ using < or >.

Solution

Write the numbers one under the other, lining up the decimal points.	0.64 $\ 0.6$
Add a zero to 0.6 to make it a decimal with 2 decimal places. Now they are both hundredths.	0.64 0.60
64 is greater than 60.	$64 > 60$
64 hundredths is greater than 60 hundredths.	$0.64 > 0.60$
	$0.64 > 0.6$

TRY IT 8.1

Order each of the following pairs of numbers, using $<$ or > : $0.42\_\_\_\_0.4$.

Show answer
>

TRY IT 8.2

Order each of the following pairs of numbers, using $<$ or > : $0.18\_\_\_\_0.1$.

Show answer
>

EXAMPLE 9

Order $0.83\_\_\_\_0.803$ using < or >.

Solution

	$0.83\_\_\_\_0.803$
Write the numbers one under the other, lining up the decimals.	0.83 0.803
They do not have the same number of digits. Write one zero at the end of 0.83.	0.830 0.803
Since $830 > 803$, 830 thousandths is greater than 803 thousandths.	$0.830 > 0.803$
	$0.83 > 0.803$

TRY IT 9.1

Order the following pair of numbers, using $<$ or $>$: $0.76\_\_\_\_0.706.$

Show answer

>

TRY IT 9.2

Order the following pair of numbers, using $<$ or $>$: $0.305\_\_\_\_0.35.$

Show answer

<

When we order negative decimals, it is important to remember how to order negative integers. Recall that larger numbers are to the right on the number line. For example, because -2 lies to the right of -3 on the number line, we know that $-2 > -3$. Similarly, smaller numbers lie to the left on the number line. For example, because -9 lies to the left of -6 on the number line, we know that $-9 < -6$. See Figure 7.

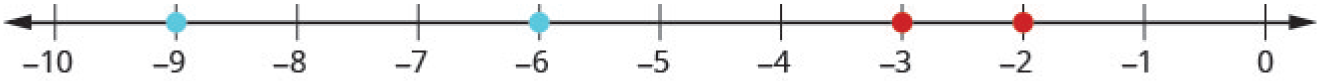


Figure 7

If we zoomed in on the interval between 0 and -1, as shown in Example 10, we would see in the same way that $-0.2 > -0.3$ and $-0.9 < -0.6$.

EXAMPLE 10

Use < or > to order $-0.1\_\_\_\_ -0.8$.

Solution

	$-0.1\_\_\_\_ -0.8$
Write the numbers one under the other, lining up the decimal points. They have the same number of digits.	-0.1 -0.8
Since $-1 > -8$, −1 tenth is greater than −8 tenths.	$-0.1 > -0.8$

TRY IT 10.1

Order the following pair of numbers, using < or >: $-0.3\_\_\_\_ -0.5$.

Show answer
>

TRY IT 10.2

Order the following pair of numbers, using < or >: $-0.6\_\_\_\_ -0.7$.

Show answer
>

Key Concepts

- **Order Decimals**
  1. Write the numbers one under the other, lining up the decimal points.
  2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
  3. Compare the numbers as if they were whole numbers.
  4. Order the numbers using the appropriate inequality sign.

Glossary

equivalent decimals

Two decimals are equivalent if they convert to equivalent fractions.

irrational number

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

rational number

A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. A rational number can be written as the ratio of two integers. Its decimal form stops or repeats.

real number

A real number is a number that is either rational or irrational.

Practice Makes Perfect

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, write as the ratio of two integers.

1. a) 5 b) 3.19	2. a) 8 b) 1.61
3. a) -12 b) 9.279	4. a) -16 b) 4.399

In the following exercises, list the a) rational numbers, b) irrational numbers

5. $0.75, 0.22\overline{3}, 1.39174$	6. $0.36, 0.94729\ldots, 2.52\overline{8}$
7. $0.4\overline{5}, 1.919293\ldots, 3.59$	8. $0.1\overline{3}, 0.42982\ldots, 1.875$

In the following exercises, list the a) whole numbers, b) integers, c) rational numbers, d) irrational numbers, e) real numbers for each set of numbers.

9. $-8, 0, 1.95286\ldots, \frac{12}{5}, 9$	10. $-9, -3\frac{4}{9}, 0.40\overline{9}, \frac{11}{6}, 7$
11. $-7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$	12. $-6, -\frac{5}{2}, 0, 0.\overline{714285}, 2\frac{1}{5}$

Locate Fractions on the Number Line

In the following exercises, locate the numbers on a number line.

13. $\frac{3}{4}, \frac{8}{5}, \frac{10}{3}$	14. $\frac{1}{4}, \frac{9}{5}, \frac{11}{3}$
15. $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$	16. $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$
17. $\frac{2}{5}, -\frac{2}{5}$	18. $\frac{3}{4}, -\frac{3}{4}$
19. $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$	20. $\frac{1}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

In the following exercises, order each of the pairs of numbers, using < or >.

21. -1____ $-\frac{1}{4}$	22. -1____ $-\frac{1}{3}$
23. $-2\frac{1}{2}$____ -3	24. $-1\frac{3}{4}$____ -2
25. $-\frac{5}{12}$____ $-\frac{7}{12}$	26. $-\frac{9}{10}$____ $-\frac{3}{10}$
27. -3____ $-\frac{13}{5}$	28. -4____ $-\frac{23}{6}$

Locate Decimals on the Number Line In the following exercises, locate the number on the number line.

29. 0.8	30. -0.9
31. -1.6	32. 3.1

In the following exercises, order each pair of numbers, using < or >.

33. 0.37____0.63	34. 0.86____0.69
35. 0.91____0.901	36. 0.415____0.41
37. -0.5____ -0.3	38. -0.1____ -0.4
39. -0.62____ -0.619	40. -7.31____ -7.3

Everyday Math

41. **Field trip.** All the 5th graders at Lord Selkirk Elementary School will go on a field trip to the science museum. Counting all the children, teachers, and chaperones, there will be 147 people. Each bus holds 44 people.

a) How many buses will be needed?
b) Why must the answer be a whole number?
c) Why shouldn't you round the answer the usual way, by choosing the whole number closest to the exact answer?

42. **Child care.** Serena wants to open a licensed child care center. Her state requires there be no more than 12 children for each teacher. She would like her child care centre to serve 40 children.

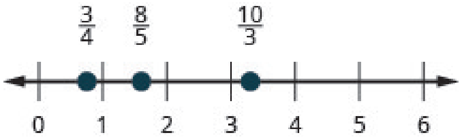
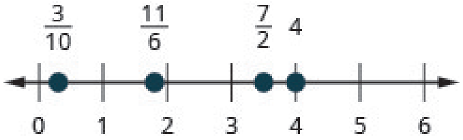
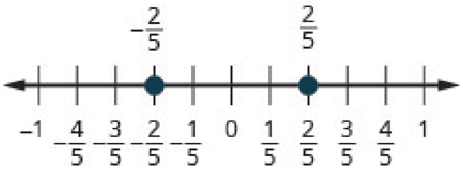
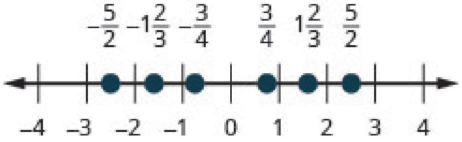
a) How many teachers will be needed?
b) Why must the answer be a whole number?
c) Why shouldn't you round the answer the usual way, by choosing the whole number closest to the exact answer?

Writing Exercises

43. In your own words, explain the difference between a rational number and an irrational number.

44. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Answers

1. a) $\frac{5}{1}$ b) $\frac{319}{100}$	3. a) $\frac{-12}{1}$ b) $\frac{9297}{1000}$	5. a) $0.75, 0.22\overline{3}$ b) 1.39174
7. a) $0.4\overline{5}, 3.59$ b) 1.919293	9. a) $0, 9$ b) $-8, 9$ c) $-8, 0, \frac{12}{5}, 9$ d)1.95286 e) $-8, 0, 1.95286..., \frac{12}{5}, 9$	11. a) none b) $-7, -1$ c) -7, $-\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$ d) none e) $-7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$
13. 	15. 	17. 
19. 	21. <	23. >
25. >	27. <	29. 
31. 	33. <	35. >
37. <	39. <	41. a) 4 buses b) answers may vary c) answers may vary
43. Answers may vary.		

Attributions

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2.5 Properties of Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the identity and inverse properties of addition and multiplication
- Use the properties of zero
- Simplify expressions using the distributive property

Use the Commutative and Associative Properties

Think about adding two numbers, say 5 and 3. The order we add them doesn't affect the result, does it?

$$5+3 \qquad 3+5$$

$$8 \qquad 8$$

$$5+3=3+5$$

The results are the same.

As we can see, the order in which we add does not matter!

What about multiplying $5 \text{ and } 3?$

$$5\cdot 3 \qquad 3\cdot 5$$

$$15 \qquad 15$$

$$5\cdot 3=3\cdot 5$$

Again, the results are the same!

The order in which we multiply does not matter!

These examples illustrate the commutative property. When adding or multiplying, changing the *order* gives the same result.

Commutative Property

of Addition	If a, b are real numbers, then	$a+b = b+a$
of Multiplication	If a, b are real numbers, then	$a \cdot b = b \cdot a$

When adding or multiplying, changing the *order* gives the same result.

The commutative property has to do with order. If you change the order of the numbers when adding or multiplying, the result is the same.

What about subtraction? Does order matter when we subtract numbers? Does $7-3$ give the same result as $3-7$?

$$\begin{array}{ll} 7-3 & 3-7 \\ 4 & -4 \end{array}$$

$$\begin{aligned} 4 &\neq -4 \\ 7-3 &\neq 3-7 \end{aligned}$$

The results are not the same.

Since changing the order of the subtraction did not give the same result, we know that *subtraction is not commutative*.

Let's see what happens when we divide two numbers. Is division commutative?

$$\begin{array}{ll} 12 \div 4 & 4 \div 12 \\ \frac{12}{4} & \frac{4}{12} \\ 3 & \frac{1}{3} \end{array}$$

$$\begin{aligned} 3 &\neq \tfrac{1}{3} \\ 12 \div 4 &\neq 4 \div 12 \end{aligned}$$

The results are not the same.

Since changing the order of the division did not give the same result, *division is not commutative*. The commutative properties only apply to addition and multiplication!

- Addition and multiplication *are* commutative.
- Subtraction and Division *are not* commutative.

If you were asked to simplify this expression, how would you do it and what would your answer be?

$7+8+2$

Some people would think $7+8$ is 15 and then $15+2$ is 17. Others might start with $8+2 \text{ makes } 10$ and then $7+10 \text{ makes } 17$.

Either way gives the same result. Remember, we use parentheses as grouping symbols to indicate which operation should be done first.

Add $7+8$. Add.	$(7+8)+2$ $15+2$ 17
Add $8+2$. Add.	$7+(8+2)$ $7+10$ 17
	$(7+8)+2=7+(8+2)$

When adding three numbers, changing the grouping of the numbers gives the same result.

This is true for multiplication, too.

Multiply. $5\cdot\frac{1}{3}$ Multiply.	$\left(5\cdot\frac{1}{3}\right)\cdot 3$ $\frac{5}{3}\cdot 3$ 5
Multiply. $\frac{1}{3}\cdot 3$. Multiply.	$5\cdot\left(\frac{1}{3}\cdot 3\right)$ $5\cdot 1$ 5
	$\left(5\cdot\frac{1}{3}\right)\cdot 3=5\cdot\left(\frac{1}{3}\cdot 3\right)$

When multiplying three numbers, changing the grouping of the numbers gives the same result.

You probably know this, but the terminology may be new to you. These examples illustrate the associative property.

Associative Property

of Addition	If a, b, c are real numbers, then $(a+b)+c=a+(b+c)$
of Multiplication	If a, b, c are real numbers, then $(a\cdot b)\cdot c=a\cdot(b\cdot c)$

When adding or multiplying, changing the *grouping* gives the same result.

Let's think again about multiplying $5\cdot\frac{1}{3}\cdot 3$. We got the same result both ways, but which way was easier? Multiplying $\frac{1}{3}$ and 3 first, as shown above on the right side, eliminates the fraction in the first step. Using the associative property can make the math easier!

The associative property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the commutative property of addition to write the like terms together.

EXAMPLE 1

Simplify: $18p+6q+15p+5q$.

Solution

	$18p+6q+15p+5q$
Use the commutative property of addition to re-order so that like terms are together.	$18p+15p+6q+5q$
Add like terms.	$33p+11q$

TRY IT 1.1

Simplify: $23r+14s+9r+15s$.

Show answer
$32r+29s$

TRY IT 1.2

Simplify: $37m+21n+4m-15n$.

Show answer
$41m+6n$

When we have to simplify algebraic expressions, we can often make the work easier by applying the commutative or associative property first, instead of automatically following the order of operations. When adding or subtracting fractions, combine those with a common denominator first.

EXAMPLE 2

Simplify: $\left(\frac{5}{13}+\frac{3}{4}\right)+\frac{1}{4}$.

Solution

	$\left(\frac{5}{13}+\frac{3}{4}\right)+\frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13}+\left(\frac{3}{4}+\frac{1}{4}\right)$
Add in parentheses first.	$\frac{5}{13}+\left(\frac{4}{4}\right)$
Simplify the fraction.	$\frac{5}{13}+1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

TRY IT 2.1

Simplify: $\left(\frac{7}{15}+\frac{5}{8}\right)+\frac{3}{8}$.

Show answer

$1\frac{7}{15}$

TRY IT 2.2

Simplify: $\left(\frac{2}{9}+\frac{7}{12}\right)+\frac{5}{12}$.

Show answer

$1\frac{2}{9}$

EXAMPLE 3

Use the associative property to simplify $6\,(3x)$.

Solution

	$6\,(3x)$
Change the grouping.	$(6\cdot 3)\,x$
Multiply in the parentheses.	$18x$

Notice that we can multiply $6\cdot 3$ but we could not multiply 3*x* without having a value for *x*.

TRY 3.1

Use the associative property to simplify 8(4*x*).

Show answer
32*x*

TRY IT 3.2

Use the associative property to simplify $-9\,(7y)$.

Show answer
$-63y$

Use the Identity and Inverse Properties of Addition and Multiplication

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the additive identity.

For example,

$$\begin{array}{lll} 13+0 & -14+0 & 0+(-8) \\ \;\;13 & \;\;-14 & \;\;-8 \end{array}$$

These examples illustrate the Identity Property of Addition that states that for any real number a, $a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the multiplicative identity.

For example,

$$\begin{array}{ccc} 43 \cdot 1 & -27 \cdot 1 & 1 \cdot \frac{3}{5} \\ 43 & -27 & \frac{3}{5} \end{array}$$

These examples illustrate the Identity Property of Multiplication that states that for any real number a, $a \cdot 1 = a$ and $1 \cdot a = a$.

We summarize the Identity Properties below.

Identity Property

of addition For any real number a: 0 is the additive identity	$a + 0 = a$	$0 + a = a$
of multiplication For any real number a: 1 is the multiplicative identity	$a \cdot 1 = a$	$1 \cdot a = a$

What number added to 5 gives the additive identity, 0?

$5 + \underline{\qquad} = 0$ We know $5 + (-5) = 0$

What number added to –6 gives the additive identity, 0?

$-6 + \underline{\qquad} = 0$ We know $-6 + 6 = 0$

Notice that in each case, the missing number was the opposite of the number!

We call $-a$. the additive inverse of *a*. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the Inverse Property of Addition that states for any real number a, $a + (-a) = 0$. Remember, a number and its opposite add to zero.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1?

$\frac{2}{3} \cdot \underline{\qquad} = 1$ We know $\frac{2}{3} \cdot \frac{3}{2} = 1$

What number multiplied by 2 gives the multiplicative identity, 1? In other words 2 times what results in 1?

$2 \cdot \underline{\qquad} = 1$ We know $2 \cdot \frac{1}{2} = 1$

Notice that in each case, the missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the multiplicative inverse of *a*. *The reciprocal of a number is its multiplicative inverse.* A number and its reciprocal multiply to one, which is the multiplicative identity. This leads to the Inverse Property of Multiplication that states that for any real number a, $a \neq 0$, $a \cdot \frac{1}{a} = 1$.

We'll formally state the inverse properties here:

Inverse Property

of addition	For any real number a, $-a$ is the **additive inverse** of a. A number and its opposite add to zero.	$a + (-a) = 0$
of multiplication	For any real number a, $\frac{1}{a}$ is the **multiplicative inverse** of a. A number and its reciprocal multiply to one.	$a \cdot \frac{1}{a} = 1$

EXAMPLE 4

Find the additive inverse of a) $\frac{5}{8}$ b) 0.6 c) -8 d) $-\frac{4}{3}$.

Solution
To find the additive inverse, we find the opposite.

a. The additive inverse of $\frac{5}{8}$ is the opposite of $\frac{5}{8}$. The additive inverse of $\frac{5}{8}$ is $-\frac{5}{8}$.
b. The additive inverse of 0.6 is the opposite of 0.6. The additive inverse of 0.6 is -0.6.
c. The additive inverse of -8 is the opposite of -8. We write the opposite of -8 as $-(-8)$, and then simplify it to 8. Therefore, the additive inverse of -8 is 8.
d. The additive inverse of $-\frac{4}{3}$ is the opposite of $-\frac{4}{3}$. We write this as $-\left(-\frac{4}{3}\right)$, and then simplify to $\frac{4}{3}$. Thus, the additive inverse of $-\frac{4}{3}$ is $\frac{4}{3}$.

TRY IT 4.1

Find the additive inverse of: a) $\frac{7}{9}$ b) 1.2 c) -14 d) $-\frac{9}{4}$.

Show answer
a) $-\frac{7}{9}$ b) -1.2 c)14 d) $\frac{9}{4}$

Exercises

Find the additive inverse of: a) $\frac{7}{13}$ b) 8.4 c) -46 d) $-\frac{5}{2}$.

Show answer
a) $-\frac{7}{13}$ b) -8.4 c) 46 d) $\frac{5}{2}$

EXAMPLE 5

Find the multiplicative inverse of a) 9 b) $-\frac{1}{9}$ c) 0.9.

Solution
To find the multiplicative inverse, we find the reciprocal.

a. The multiplicative inverse of 9 is the reciprocal of 9, which is $\frac{1}{9}$. Therefore, the multiplicative inverse of 9 is $\frac{1}{9}$.
b. The multiplicative inverse of $-\frac{1}{9}$ is the reciprocal of $-\frac{1}{9}$, which is -9. Thus, the multiplicative inverse of $-\frac{1}{9}$ is -9.
c. To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal of the fraction. The reciprocal of $\frac{9}{10}$ is $\frac{10}{9}$. So the multiplicative inverse of 0.9 is $\frac{10}{9}$.

TRY IT 5.1

Find the multiplicative inverse of a) 4 b) $-\frac{1}{7}$ c) 0.3

Show answer
a) $\frac{1}{4}$ b) -7 c) $\frac{10}{3}$

TRY IT 5.2

Find the multiplicative inverse of a) 18 b) $-\frac{4}{5}$ c) 0.6.

Show answer
a) $\frac{1}{18}$ b) $-\frac{5}{4}$ c) $\frac{5}{3}$

Use the Properties of Zero

The identity property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

Multiplication by Zero

For any real number *a*.

$a \cdot 0 = 0 \qquad 0 \cdot a = 0$

The product of any real number and 0 is 0.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So,
$0 \div 3 = 0$

We can check division with the related multiplication fact.
$12 \div 6 = 2$ because $2 \cdot 6 = 12$.

So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

Division of Zero

For any real number *a*, except $0, \frac{0}{a} = 0$ and $0 \div a = 0$.

Zero divided by any real number except zero is zero.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact: $4 \div 0 = ?$ means $? \cdot 0 = 4$. Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4

We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is undefined.

Division by Zero

For any real number *a*, except 0, $\frac{a}{0}$ and $a \div 0$ are undefined.

Division by zero is undefined.

We summarize the properties of zero below.

Properties of Zero

Multiplication by Zero: For any real number *a*,

$a \cdot 0 = 0 \quad 0 \cdot a = 0$	The product of any number and 0 is 0.

Division of Zero, Division by Zero: For any real number $a, a \neq 0$

$\frac{0}{a} = 0$	Zero divided by any real number except itself is zero.
$\frac{a}{0}$ is undefined	Division by zero is undefined.

EXAMPLE 6

Simplify: a) $-8 \cdot 0$ b) $\frac{0}{-2}$ c) $\frac{-32}{0}$.

Solution

a) The product of any real number and 0 is 0.	$-8 \cdot 0$ 0
b) The product of any real number and 0 is 0.	$\frac{0}{-2}$ 0
c) Division by 0 is undefined.	$\frac{-32}{0}$ Undefined

TRY IT 6.1

Simplify: a) $-14 \cdot 0$ b) $\frac{0}{-6}$ c) $\frac{-2}{0}$.

Show answer

a) 0 b) 0 c) undefined

TRY IT 6.2

Simplify: a) $0\left(-17\right)$ b) $\frac{0}{-10}$ c) $\frac{-5}{0}$.

Show answer
a) 0 b) 0 c) undefined

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

EXAMPLE 7

Simplify: a) $\frac{0}{n+5}$, where $n\ne -5$ b) $\frac{10-3p}{0}$, where $10-3p\ne 0$.

Solution

a) Zero divided by any real number except itself is 0.	$\frac{0}{n+5}$ 0
b) Division by 0 is undefined.	$\frac{10-3p}{0}$ Undefined

TRY IT 7.1

Simplify: a) $\frac{0}{m+7}$, where $m\ne -7$ b) $\frac{18-6c}{0}$, where $18-6c\ne 0$.

Show answer
a) 0 b) undefined

TRY IT 7.2

Simplify: a) $\frac{0}{d-4}$, where $d\ne 4$ b) $\frac{15-4q}{0}$, where $15-4q\ne 0$.

Show answer
a) 0 b) undefined

EXAMPLE 8

Simplify: $-84n+(-73n)+84n$.

Solution

	$-84n+(-73n)+84n$
Notice that the first and third terms are opposites; use the commutative property of addition to re-order the terms.	$-84n+84n+(-73n)$
Add left to right.	$0+(-73)$
Add.	$-73n$

TRY IT 8.1

Simplify: $-27a+(-48a)+27a$.

Show answer
$-48a$

TRY IT 8.2

Simplify: $39x+(-92x)+(-39x)$.

Show answer
$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1

EXAMPLE 9

Simplify: $\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$.

Solution

	$\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$
Notice that the first and third terms are reciprocals, so use the commutative property of multiplication to re-order the factors.	$\frac{7}{15} \cdot \frac{15}{7} \cdot \frac{8}{23}$
Multiply left to right.	$1 \cdot \frac{8}{23}$
Multiply.	$\frac{8}{23}$

TRY IT 9.1

Simplify: $\frac{9}{16} \cdot \frac{5}{49} \cdot \frac{16}{9}$.

Show answer

$\frac{5}{49}$

TRY IT 9.2

Simplify: $\frac{6}{17} \cdot \frac{11}{25} \cdot \frac{17}{6}$.

Show answer

$\frac{11}{25}$

EXAMPLE 10

Simplify: $\frac{3}{4} \cdot \frac{4}{3}\left(6x + 12\right)$.

Solution

	$\frac{3}{4} \cdot \frac{4}{3}\left(6x + 12\right)$
There is nothing to do in the parentheses, so multiply the two fractions first—notice, they are reciprocals.	$1\left(6x + 12\right)$
Simplify by recognizing the multiplicative identity.	$6x + 12$

TRY IT 10.1

Simplify: $\frac{2}{5} \cdot \frac{5}{2}\left(20y+50\right)$.

Show answer

$20y+50$

TRY IT 10.2

Simplify: $\frac{3}{8} \cdot \frac{8}{3}\left(12z+16\right)$.

Show answer

$12z+16$

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27, and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the **distributive property**.

Distributive Property

$$\begin{array}{rl} \text{If } a, b, c \text{ are real numbers, then} & a\left(b+c\right)=ab+ac \\ \\ \text{Also,} & \left(b+c\right)a=ba+ca \\ & a\left(b-c\right)=ab-ac \\ & \left(b-c\right)a=ba-ca \end{array}$$

Back to our friends at the movies, we could find the total amount of money they need like this:

3(9.25)
3(9 + 0.25)
3(9) + 3(0.25)
27 + 0.75
27.75

In algebra, we use the distributive property to remove parentheses as we simplify expressions.

For example, if we are asked to simplify the expression $3\left(x+4\right)$, the order of operations says to work in the parentheses first. But we cannot add *x* and 4, since they are not like terms. So we use the distributive property, as shown in (Example 11).

EXAMPLE 11

Simplify: $3\left(x+4\right)$.

Solution

	$3\left(x+4\right)$
Distribute.	$3\cdot x+3\cdot 4$
Multiply.	$3x+12$

TRY IT 11.1

Simplify: $4\left(x+2\right)$.

Show answer

$4x+8$

TRY IT 11.2

Simplify: $6\left(x+7\right)$.

Show answer

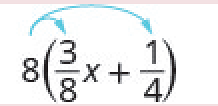
$6x+42$

Some students find it helpful to draw in arrows to remind them how to use the distributive property. Then the first step in (Example 11) would look like this:

EXAMPLE 12

Simplify: $8\left(\frac{3}{8}x+\frac{1}{4}\right)$.

Solution

	
Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

TRY IT 12.1

Simplify: $6\left(\frac{5}{6}y+\frac{1}{2}\right)$.

Show answer
$5y+3$

TRY IT 12.2

Simplify: $12\left(\frac{1}{3}n+\frac{3}{4}\right)$.

Show answer
$4n+9$

Using the distributive property as shown in (Example 13) will be very useful when we solve money applications in later chapters.

EXAMPLE 13

Simplify: $100\left(0.3+0.25q\right)$.

Solution

	$100(0.3+0.25q)$
Distribute.	$100(0.3)+100(0.25q)$
Multiply.	$30+25q$

TRY IT 13.1

Simplify: $100\left(0.7+0.15p\right)$.

Show answer
$70+15p$

TRY IT 13.2

Simplify: $100\left(0.04+0.35d\right)$.

Show answer
$4+35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

EXAMPLE 14

Simplify: $-2\left(4y+1\right)$.

Solution

	$-2(4y+1)$
Distribute.	$-2\cdot 4y+(-2)\cdot 1$
Multiply.	$-8y-2$

TRY IT 14.1

Simplify: $-3\,(6m+5)$.

Show answer
$-18m-15$

TRY IT 14.2

Simplify: $-6\,(8n+11)$.

Show answer
$-48n-66$

EXAMPLE 15

Simplify: $-11\,(4-3a)$.

Solution

Distribute.	$-11(4-3a)$
Multiply.	$-11\cdot 4-(-11)\cdot 3a$ $-44-(-33a)$
Simplify.	$-44+33a$

Notice that you could also write the result as $33a-44$. Do you know why?

TRY IT 15.1

Simplify: $-5\left(2-3a\right)$.

Show answer
$-10+15a$

TRY IT 15.2

Simplify: $-7\left(8-15y\right)$.

Show answer
$-56+105y$

(Example 16) will show how to use the distributive property to find the opposite of an expression.

EXAMPLE 16

Simplify: $-\left(y+5\right)$.

Solution

	$\left(y+5\right)$
Multiplying by −1 results in the opposite.	$-1\left(y+5\right)$
Distribute.	$-1\cdot y+\left(-1\right)\cdot 5$
Simplify.	$-y+\left(-5\right)$
	$-y-5$

TRY IT 16.1

Simplify: $-\left(z-11\right)$.

Show answer
$-z+11$

TRY IT 16.2

Simplify: $-(x-4)$.

Show answer
$-x+4$

There will be times when we'll need to use the distributive property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the distributive property, which removes the parentheses. The next two examples will illustrate this.

EXAMPLE 17

Simplify: $8-2(x+3)$.

Be sure to follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

Solution

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

TRY IT 17.1

Simplify: $9-3(x+2)$.

Show answer
$3-3x$

TRY IT 17.2

Simplify: $7x - 5(x + 4)$.

Show answer
$2x - 20$

EXAMPLE 18

Simplify: $4(x - 8) - (x + 3)$.

Solution

	$4(x - 8) - (x + 3)$
Distribute.	$4x - 32 - x - 3$
Combine like terms.	$3x - 35$

TRY IT 18.1

Simplify: $6(x - 9) - (x + 12)$.

Show answer
$5x - 66$

TRY IT 18.2

Simplify: $8(x - 1) - (x + 5)$.

Show answer
$7x - 13$

All the properties of real numbers we have used in this chapter are summarized in the table below.

Commutative Property	**of addition** If a, b are real numbers, then	$a+b=b+a$
	of multiplication If a, b are real numbers, then	$a \cdot b=b \cdot a$
Associative Property	**of addition** If a, b, c are real numbers, then	$(a+b)+c=a+(b+c)$
	of multiplication If a, b, c are real numbers, then	$(a \cdot b) \cdot c=a \cdot (b \cdot c)$
Distributive Property	If a, b, c are real numbers, then	$a\,(b+c)=ab+ac$
Identity Property	**of addition** For any real number a: 0 is the **additive identity**	$a+0=a$ $0+a=a$
	of multiplication For any real number a: 1 is the **multiplicative identity**	$a \cdot 1=a$ $1 \cdot a=a$
Inverse Property	**of addition** For any real number a , $-a$ is the **additive inverse** of a	$a+(-a)=0$
	of multiplication For any real number $a, a \neq 0$ $\frac{1}{a}$ is the **multiplicative inverse** of a.	$a \cdot \frac{1}{a}=1$
Properties of Zero	For any real number *a*, For any real number $a, a \neq 0$ For any real number $a, a \neq 0$	$a \cdot 0=0$ $0 \cdot a=0$ $\frac{0}{a}=0$ $\frac{a}{0}$ is undefined

Key Concepts

- **Commutative Property of**
    - **Addition:** If a, b are real numbers, then $a+b=b+a$.
    - **Multiplication:** If a, b are real numbers, then $a \cdot b=b \cdot a$. When adding or multiplying, changing the *order* gives the same result.
- **Associative Property of**
    - **Addition:** If a, b, c are real numbers, then $(a+b)+c=a+(b+c)$.
    - **Multiplication:** If a, b, c are real numbers, then $(a \cdot b) \cdot c=a \cdot (b \cdot c)$. When adding or multiplying, changing the *grouping* gives the same result.
- **Distributive Property:** If a, b, c are real numbers, then

  - $a\left(b+c\right)=ab+ac$
  - $\left(b+c\right)a=ba+ca$
  - $a\left(b-c\right)=ab-ac$
  - $\left(b-c\right)a=ba-ca$
- **Identity Property**
  - **of Addition:** For any real number a: $a+0=a\quad 0+a=a$
    0 is the **additive identity**
  - **of Multiplication:** For any real number a: $a\cdot 1=a\quad 1\cdot a=a$
    1 is the **multiplicative identity**
- **Inverse Property**
  - **of Addition:** For any real number a, $a+\left(-a\right)=0$. A number and its *opposite* add to zero. $-a$ is the **additive inverse** of a.
  - **of Multiplication:** For any real number a, $\left(a\neq 0\right)\ a\cdot\frac{1}{a}=1$. A number and its *reciprocal* multiply to one. $\frac{1}{a}$ is the **multiplicative inverse** of a.
- **Properties of Zero**
  - For any real number a,
    $a\cdot 0=0\quad 0\cdot a=0$ – The product of any real number and 0 is 0.
  - $\frac{0}{a}=0$ for $a\neq 0$ – Zero divided by any real number except zero is zero.
  - $\frac{a}{0}$ is undefined – Division by zero is undefined.

Glossary

additive identity
The additive identity is the number 0; adding 0 to any number does not change its value.

additive inverse
The opposite of a number is its additive inverse. A number and it additive inverse add to 0.

multiplicative identity
The multiplicative identity is the number 1; multiplying 1 by any number does not change the value of the number.

multiplicative inverse
The reciprocal of a number is its multiplicative inverse. A number and its multiplicative inverse multiply to one.

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, use the associative property to simplify.

1. $3(4x)$	2. $4(7m)$
3. $(y+12)+28$	4. $(n+17)+33$

In the following exercises, simplify.

5. $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$	6. $\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$
7. $\frac{3}{20}\cdot\frac{49}{11}\cdot\frac{20}{3}$	8. $\frac{13}{18}\cdot\frac{25}{7}\cdot\frac{18}{13}$
9. $-24\cdot 7\cdot\frac{3}{8}$	10. $-36\cdot 11\cdot\frac{4}{9}$
11. $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$	12. $\left(\frac{11}{12}+\frac{4}{9}\right)+\frac{5}{9}$
13. $17(0.25)(4)$	14. $36(0.2)(5)$
15. $[2.48(12)](0.5)$	16. $[9.731(4)](0.75)$
17. $7(4a)$	18. $9(8w)$
19. $-15\,(5m)$	20. $-23\,(2n)$
21. $12\left(\frac{5}{6}p\right)$	22. $20\left(\frac{3}{5}q\right)$
23. $43m+(-12n)+(-16m)+(-9n)$	24. $-22p+17q+(-35p)+(-27q)$
25. $\frac{3}{8}g+\frac{1}{12}h+\frac{7}{8}g+\frac{5}{12}h$	26. $\frac{5}{6}a+\frac{3}{10}b+\frac{1}{6}a+\frac{9}{10}b$
27. $6.8p+9.14q+(-4.37p)+(-0.88q)$	28. $9.6m+7.22n+(-2.19m)+(-0.65n)$

Use the Identity and Inverse Properties of Addition and Multiplication

In the following exercises, find the additive inverse of each number.

29. a) $\frac{2}{5}$ b) 4.3 c) -8 d) $-\frac{10}{3}$	30. a) $\frac{5}{9}$ b) 2.1 c) -3 d) $-\frac{9}{5}$
31. a) $-\frac{7}{6}$ b) -0.075 c) 23 d) $\frac{1}{4}$	32. a) $-\frac{8}{3}$ b) -0.019 c) 52 d) $\frac{5}{6}$

In the following exercises, find the multiplicative inverse of each number.

33. a) 6 b) $-\frac{3}{4}$ c) 0.7	34. a) 12 b) $-\frac{9}{2}$ c) 0.13
35. a) $\frac{11}{12}$ b) -1.1 c) -4	36. a) $\frac{17}{20}$ b) -1.5 c) -3

Use the Properties of Zero

In the following exercises, simplify.

37. $\frac{0}{6}$	38. $\frac{3}{0}$
39. $0 \div \frac{11}{12}$	40. $\frac{6}{0}$
41. $\frac{0}{3}$	42. $0 \cdot \frac{8}{15}$
43. $(-3.14)\,(0)$	44. $\frac{\frac{1}{10}}{0}$

Mixed Practice

In the following exercises, simplify.

45. $19a + 44 - 19a$	46. $27c + 16 - 27c$
47. $10\,(0.1d)$	48. $100\,(0.01p)$
49. $\frac{0}{u-4.99}$, where $u \neq 4.99$	50. $\frac{0}{v-65.1}$, where $v \neq 65.1$
51. $0 \div \left(x - \frac{1}{2}\right)$, where $x \neq \frac{1}{2}$	52. $0 \div \left(y - \frac{1}{6}\right)$, where $x \neq \frac{1}{6}$
53. $\frac{32-5a}{0}$, where $32 - 5a \neq 0$	54. $\frac{28-9b}{0}$, where $28 - 9b \neq 0$
55. $\left(\frac{3}{4} + \frac{9}{10}m\right) \div 0$ where $\frac{3}{4} + \frac{9}{10}m \neq 0$	56. $\left(\frac{5}{16}n - \frac{3}{7}\right) \div 0$ where $\frac{5}{16}n - \frac{3}{7} \neq 0$
57. $15 \cdot \frac{3}{5}\,(4d + 10)$	58. $18 \cdot \frac{5}{6}\,(15h + 24)$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the distributive property.

59. $8\left(4y+9\right)$	60. $9\left(3w+7\right)$
61. $6\left(c-13\right)$	62. $7\left(y-13\right)$
63. $\frac{1}{4}\left(3q+12\right)$	64. $\frac{1}{5}\left(4m+20\right)$
65. $9\left(\frac{5}{9}y-\ \frac{1}{3}\right)$	66. $10\left(\frac{3}{10}x-\ \frac{2}{5}\right)$
67. $12\left(\frac{1}{4}+\frac{2}{3}r\right)$	68. $12\left(\frac{1}{6}+\frac{3}{4}s\right)$
69. $r\left(s-18\right)$	70. $u\left(v-10\right)$
71. $\left(y+4\right)p$	72. $\left(a+7\right)x$
73. $-7\left(4p+1\right)$	74. $-9\left(9a+4\right)$
75. $-3\left(x-6\right)$	76. $-4\left(q-7\right)$
77. $-\left(3x-7\right)$	78. $-\left(5p-4\right)$
79. $16-3\left(y+8\right)$	80. $18-4\left(x+2\right)$
81. $4-11\left(3c-2\right)$	82. $9-6\left(7n-5\right)$
83. $22-\left(a+3\right)$	84. $8-\left(r-7\right)$
85. $\left(5m-3\right)-\left(m+7\right)$	86. $\left(4y-1\right)-\left(y-2\right)$
87. $5\left(2n+9\right)+12\left(n-3\right)$	88. $9\left(5u+8\right)+2\left(u-6\right)$
89. $9\left(8x-3\right)-\left(-2\right)$	90. $4\left(6x-1\right)-\left(-8\right)$
91. $14\left(c-1\right)-8\left(c-6\right)$	92. $11\left(n-7\right)-5\left(n-1\right)$
93. $6\left(7y+8\right)-\left(30y-15\right)$	94. $7\left(3n+9\right)-\left(4n-13\right)$

Everyday Math

95. **Insurance copayment** Carrie had to have 5 fillings done. Each filling cost \$80. Her dental insurance required her to pay 20% of the cost as a copay. Calculate Carrie's copay:

a) First, by multiplying 0.20 by 80 to find her copay for each filling and then multiplying your answer by 5 to find her total copay for 5 fillings.

b) Next, by multiplying [5(0.20)](80)

c) Which of the properties of real numbers says that your answers to parts (a), where you multiplied 5[(0.20)(80)] and (b), where you multiplied [5(0.20)](80), should be equal?

96. **Cooking time** Matt bought a 24-pound turkey for his family's Thanksgiving dinner and wants to know what time to put the turkey in to the oven. He wants to allow 20 minutes per pound cooking time. Calculate the length of time needed to roast the turkey:

a) First, by multiplying $24 \cdot 20$ to find the total number of minutes and then multiplying the answer by $\frac{1}{60}$ to convert minutes into hours.

b) Next, by multiplying $24\left(20 \cdot \frac{1}{60}\right)$.

c) Which of the properties of real numbers says that your answers to parts (a), where you multiplied $\left(24 \cdot 20\right)\frac{1}{60}$, and (b), where you multiplied $24\left(20 \cdot \frac{1}{60}\right)$, should be equal?

97. **Buying by the case.** Trader Joe's grocery stores sold a can of Coke Zero for \$1.99. They sold a case of 12 cans for \$23.88. To find the cost of 12 cans at \$1.99, notice that 1.99 is $2 - 0.01$.

a) Multiply 12(1.99) by using the distributive property to multiply $12\left(2 - 0.01\right)$.

b) Was it a bargain to buy Coke Zero by the case?

98. **Multi-pack purchase.** Adele's shampoo sells for \$3.99 per bottle at the grocery store. At the warehouse store, the same shampoo is sold as a 3 pack for \$10.49. To find the cost of 3 bottles at \$3.99, notice that 3.99 is $4 - 0.01$.

a) Multiply 3(3.99) by using the distributive property to multiply $3\left(4 - 0.01\right)$.

b) How much would Adele save by buying 3 bottles at the warehouse store instead of at the grocery store?

Writing Exercises

99. In your own words, state the commutative property of addition.

100. What is the difference between the additive inverse and the multiplicative inverse of a number?

101. Simplify $8\left(x - \frac{1}{4}\right)$ using the distributive property and explain each step.

102. Explain how you can multiply 4(\$5.97) without paper or calculator by thinking of \$5.97 as $6 - 0.03$ and then using the distributive property.

Answers

1. 12x	3. $y+40$	5. $\frac{7}{8}$
7. $\frac{49}{11}$	9. -63	11. $1\frac{5}{6}$
13. 17	15. 14.88	17. 28a
19. $-75m$	21. 10p	23. $27m+(-21n)$
25. $\frac{5}{4}g+\frac{1}{2}h$	27. $2.43p+8.26q$	29. a) $-\frac{2}{5}$ b) -4.3 c) 8 d) $\frac{10}{3}$
31. a) $\frac{7}{6}$ b) 0.075 c) -23 d) $-\frac{1}{4}$	33. a) $\frac{1}{6}$ b) $-\frac{4}{3}$ c) $\frac{10}{7}$	35. a) $\frac{12}{11}$ b) $-\frac{10}{11}$ c) $-\frac{1}{4}$
37. 0	39. 0	41. 0
43. 0	45. 44	47. d
49. 0	51. 0	53. undefined
55. undefined	57. $36d+90$	59. $32y+72$
61. $6c-78$	63. $\frac{3}{4}q+3$	65. $5y-3$
67. $3+8r$	69. $rs-18r$	71. $yp+4p$
73. $-28p-7$	75. $-3x+18$	77. $-3x+7$
79. $-3y-8$	81. $-33c+26$	83. $-a+19$
85. $4m-10$	87. $22n+9$	89. $72x-25$
91. $6c+34$	93. $12y+63$	95. a) $80 b) $80 c) answers will vary
97. a) $23.88 b) no, the price is the same	99. Answers may vary	101. Answers may vary

Attributions

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2.6 Chapter Review

Review Exercises

Find Equivalent Fractions

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

1. $\frac{1}{4}$	2. $\frac{1}{3}$
3. $\frac{5}{6}$	4. $\frac{2}{7}$

Simplify Fractions

In the following exercises, simplify.

5. $\frac{7}{21}$	6. $\frac{8}{24}$
7. $\frac{15}{20}$	8. $\frac{12}{18}$
9. $-\frac{168}{192}$	10. $-\frac{140}{224}$
11. $\frac{11x}{11y}$	12. $\frac{15a}{15b}$

Multiply Fractions

In the following exercises, multiply.

13. $\frac{2}{5} \cdot \frac{1}{3}$	14. $\frac{1}{2} \cdot \frac{3}{8}$
15. $\frac{7}{12}(-\frac{8}{21})$	16. $\frac{5}{12}(-\frac{8}{15})$
17. $-28p(-\frac{1}{4})$	18. $-51q(-\frac{1}{3})$
19. $\frac{14}{5}(-15)$	20. $-1(-\frac{3}{8})$

Divide Fractions

In the following exercises, divide.

21. $\frac{1}{2} \div \frac{1}{4}$	22. $\frac{1}{2} \div \frac{1}{8}$
23. $-\frac{4}{5} \div \frac{4}{7}$	24. $-\frac{3}{4} \div \frac{3}{5}$
25. $\frac{5}{8} \div \frac{a}{10}$	26. $\frac{5}{6} \div \frac{c}{15}$
27. $\frac{7p}{12} \div \frac{21p}{8}$	28. $\frac{5q}{12} \div \frac{15q}{8}$
29. $\frac{2}{5} \div (-10)$	30. $-18 \div -\left(\frac{9}{2}\right)$

In the following exercises, simplify.

31. $\dfrac{\frac{2}{3}}{\frac{8}{9}}$	32. $\dfrac{\frac{4}{5}}{\frac{8}{15}}$
33. $\dfrac{-\frac{9}{10}}{3}$	34. $\dfrac{2}{\frac{5}{8}}$
35. $\dfrac{\frac{r}{5}}{\frac{s}{3}}$	36. $\dfrac{-\frac{x}{6}}{-\frac{8}{9}}$

Simplify Expressions Written with a Fraction Bar

In the following exercises, simplify.

37. $\dfrac{4+11}{8}$	38. $\dfrac{9+3}{7}$
39. $\dfrac{30}{7-12}$	40. $\dfrac{15}{4-9}$
41. $\dfrac{22-14}{19-13}$	42. $\dfrac{15+9}{18+12}$
43. $\dfrac{5\cdot 8}{-10}$	44. $\dfrac{3\cdot 4}{-24}$
45. $\dfrac{15\cdot 5-5^2}{2\cdot 10}$	46. $\dfrac{12\cdot 9-3^2}{3\cdot 18}$
47. $\dfrac{2+4\left(3\right)}{-3-2^2}$	48. $\dfrac{7+3\left(5\right)}{-2-3^2}$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

49. the quotient of *c* and the sum of *d* and 9.	50. the quotient of the difference of *h* and *k*, and -5.

Add and Subtract Fractions with a Common Denominator

In the following exercises, add.

51. $\dfrac{4}{9}+\dfrac{1}{9}$	52. $\dfrac{2}{9}+\dfrac{5}{9}$
53. $\dfrac{y}{3}+\dfrac{2}{3}$	54. $\dfrac{7}{p}+\dfrac{9}{p}$
55. $-\dfrac{1}{8}+\left(-\dfrac{3}{8}\right)$	56. $-\dfrac{1}{8}+\left(-\dfrac{5}{8}\right)$

In the following exercises, subtract.

57. $\frac{4}{5}-\frac{1}{5}$	58. $\frac{4}{5}-\frac{3}{5}$
59. $\frac{y}{17}-\frac{9}{17}$	60. $\frac{x}{19}-\frac{8}{19}$
61. $-\frac{8}{d}-\frac{3}{d}$	62. $-\frac{7}{c}-\frac{7}{c}$

Add or Subtract Fractions with Different Denominators

In the following exercises, add or subtract.

63. $\frac{1}{3}+\frac{1}{5}$	64. $\frac{1}{4}+\frac{1}{5}$
65. $\frac{1}{5}-\left(-\frac{1}{10}\right)$	66. $\frac{1}{2}-\left(-\frac{1}{6}\right)$
67. $\frac{2}{3}+\frac{3}{4}$	68. $\frac{3}{4}+\frac{2}{5}$
69. $\frac{11}{12}-\frac{3}{8}$	70. $\frac{5}{8}-\frac{7}{12}$
71. $-\frac{9}{16}-\left(-\frac{4}{5}\right)$	72. $-\frac{7}{20}-\left(-\frac{5}{8}\right)$
73. $1+\frac{5}{6}$	74. $1-\frac{5}{9}$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

75. $\frac{\left(\frac{1}{5}\right)^2}{2+3^2}$	76. $\frac{\left(\frac{1}{3}\right)^2}{5+2^2}$
77. $\frac{\frac{2}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{2}{3}}$	78. $\frac{\frac{3}{4}+\frac{1}{2}}{\frac{5}{6}-\frac{2}{3}}$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

79. $x+\frac{1}{2}$ when a) $x=-\frac{1}{8}$ b) $x=-\frac{1}{2}$	80. $x+\frac{2}{3}$ when a) $x=-\frac{1}{6}$ b) $x=-\frac{5}{3}$
81. $4p^2q$ when $p=-\frac{1}{2}$ and $q=\frac{5}{9}$	82. $5m^2n$ when $m=-\frac{2}{5}$ and $n=\frac{1}{3}$
83. $\frac{u+v}{w}$ when $u=-4, v=-8, w=2$	84. $\frac{m+n}{p}$ when $m=-6, n=-2, p=4$

Name and Write Decimals

In the following exercises, write as a decimal.

85. Eight and three hundredths	86. Nine and seven hundredths
87. One thousandth	88. Nine thousandths

In the following exercises, name each decimal.

89. 7.8	90. 5.01
91. 0.005	92. 0.381

Round Decimals

In the following exercises, round each number to the nearest a) hundredth b) tenth c) whole number.

93. 5.7932	94. 3.6284
95. 12.4768	96. 25.8449

Add and Subtract Decimals

In the following exercises, add or subtract.

97. $18.37+9.36$	98. $256.37-85.49$
99. $15.35-20.88$	100. $37.5+12.23$
101. $-4.2+(-9.3)$	102. $-8.6+(-8.6)$
103. $100-64.2$	104. $100-65.83$
105. $2.51+40$	106. $9.38+60$

Multiply and Divide Decimals

In the following exercises, multiply.

107. $(0.3)\,(0.4)$	108. $(0.6)\,(0.7)$
109. $(8.52)\,(3.14)$	110. $(5.32)\,(4.86)$
111. $(0.09)\,(24.78)$	112. $(0.04)\,(36.89)$

In the following exercises, divide.

113. 0.15 ÷ 5	114. 0.27 ÷ 3
115. $8.49 ÷ 12	116. $16.99 ÷ 9
117. 12 ÷ 0.08	118. 5 ÷ 0.04

Convert Decimals and Fractions

In the following exercises, write each decimal as a fraction.

119. 0.08	120. 0.17
121. 0.425	122. 0.184
123. 1.75	124. 0.035

In the following exercises, convert each fraction to a decimal.

125. $\frac{2}{5}$	126. $\frac{4}{5}$
127. $-\frac{3}{8}$	128. $-\frac{5}{8}$
129. $\frac{5}{9}$	130. $\frac{2}{9}$
131. $\frac{1}{2} + 6.5$	132. $\frac{1}{4} + 10.75$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, write as the ratio of two integers.

133. a) 9 b) 8.47	134. a) -15b) 3.591

In the following exercises, list the a) rational numbers, b) irrational numbers.

135. $0.84, 0.79132\ldots, 1.\overline{3}$	136. $2.3\overline{8}, 0.572, 4.93814\ldots$

In the following exercises, list the a) whole numbers, b) integers, c) rational numbers, d) irrational numbers, e) real numbers for each set of numbers.

137. $-4, 0, \frac{5}{6}, 17, 5.2537\ldots$	138. $-2, 0.\overline{36}, \frac{13}{3}, 6.9152\ldots, 10\frac{1}{2}$

Locate Fractions on the Number Line

In the following exercises, locate the numbers on a number line.

139. $\frac{2}{3}, \frac{5}{4}, \frac{12}{5}$	140. $\frac{1}{3}, \frac{7}{4}, \frac{13}{5}$
141. $2\frac{1}{3}, -2\frac{1}{3}$	142. $1\frac{3}{5}, -1\frac{3}{5}$

In the following exercises, order each of the following pairs of numbers, using < or >.

143. -1 ___ $-\frac{1}{8}$	144. $-3\frac{1}{4}$___ -4
145. $-\frac{7}{9}$___ $-\frac{4}{9}$	146. -2 ___ $-\frac{19}{8}$

Locate Decimals on the Number Line

In the following exercises, locate on the number line.

147. 0.3	148. -0.2
149. -2.5	150. 2.7

In the following exercises, order each of the following pairs of numbers, using < or >.

151. 0.9 ___ 0.6	152. 0.7 ___ 0.8
153. -0.6 ___ -0.59	154. -0.27 ___ -0.3

Use the Commutative and Associative Properties

In the following exercises, use the Associative Property to simplify.

155. $-12\,(4m)$	156. $30\left(\frac{5}{6}q\right)$
157. $(a+16)+31$	158. $(c+0.2)+0.7$

In the following exercises, simplify.

159. $6y+37+(-6y)$	160. $\frac{1}{4}+\frac{11}{15}+\left(-\frac{1}{4}\right)$
161. $\frac{14}{11}\cdot\frac{35}{9}\cdot\frac{14}{11}$	162. $-18\cdot 15\cdot\frac{2}{9}$
163. $\left(\frac{7}{12}+\frac{4}{5}\right)+\frac{1}{5}$	164. $(3.98d+0.75d)+1.25d$
165. $11x+8y+16x+15y$	166. $52m+(-20n)+(-18m)+(-5n)$

Use the Identity and Inverse Properties of Addition and Multiplication

In the following exercises, find the additive inverse of each number.

167. a) $\frac{1}{3}$ b) 5.1 c) -14 d) $-\frac{8}{5}$	168. a) $-\frac{7}{8}$ b) -0.03 c) 17 d) $\frac{12}{5}$

In the following exercises, find the multiplicative inverse of each number.

169. a) 10 b) $-\frac{4}{9}$ c) 0.6	170. a) $-\frac{9}{2}$ b) -7 c) 2.1

Use the Properties of Zero

In the following exercises, simplify.

171. $83 \cdot 0$	172. $\frac{0}{9}$
173. $\frac{5}{0}$	174. 0\div $\frac{2}{3}$

In the following exercises, simplify.

175. $43 + 39 + (-43)$	176. $(n + 6.75) + 0.25$
177. $\frac{5}{13} \cdot 57 \cdot \frac{13}{5}$	178. $\frac{1}{6} \cdot 17 \cdot 12$
179. $\frac{2}{3} \cdot 28 \cdot \frac{3}{7}$	180. $9\,(6x - 11) + 15$

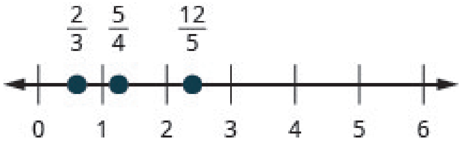
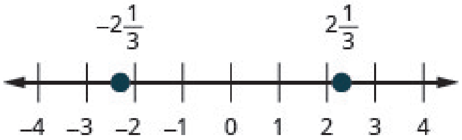
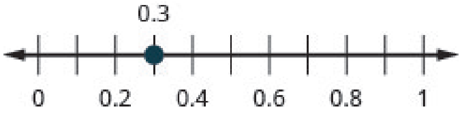
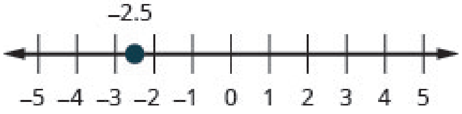
Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

181. $7\left(x+9\right)$	182. $9\left(u-4\right)$
183. $-3\left(6m-1\right)$	184. $-8\left(-7a-12\right)$
185. $\dfrac{1}{3}\left(15n-6\right)$	186. $\left(y+10\right)\cdot p$
187. $\left(a-4\right)-\left(6a+9\right)$	188. $4\left(x+3\right)-8\left(x-7\right)$

Review Exercise Answers

1. $\frac{2}{8}, \frac{3}{12}, \frac{4}{16}$ answers may vary	3. $\frac{10}{12}, \frac{15}{18}, \frac{20}{24}$ answers may vary	5. $\frac{1}{3}$
7. $\frac{3}{4}$	9. $-\frac{7}{8}$	11. $\frac{x}{y}$
13. $\frac{2}{15}$	15. $-\frac{2}{9}$	17. $7p$
19. -42	21. 2	23. $-\frac{7}{5}$
25. $\frac{25}{4a}$	27. $\frac{2}{9}$	29. $-\frac{1}{25}$
31. $\frac{3}{4}$	33. $-\frac{3}{10}$	35. $\frac{3r}{5s}$
37. $\frac{15}{8}$	39. -6	41. $\frac{4}{3}$
43. -4	45. $\frac{5}{21}$	47. -2
49. $\frac{c}{d+9}$	51. $\frac{5}{9}$	53. $\frac{y+2}{3}$
55. $-\frac{1}{2}$	57. $\frac{3}{5}$	59. $\frac{y-9}{17}$
61. $-\frac{11}{d}$	63. $\frac{8}{15}$	65. $\frac{3}{10}$
67. $\frac{17}{12}$	69. $\frac{13}{24}$	71. $\frac{19}{80}$
73. $\frac{11}{6}$	75. $\frac{1}{275}$	77. 14
79. a) $\frac{3}{8}$ b) 0	81. $\frac{5}{9}$	83. -6
85. 8.03	87. 0.001	89. seven and eight tenths
91. five thousandths	93. a) 5.79 b) 5.8 c) 6	95. a) 12.48 b) 12.5 c) 12
97. 27.73	99. −5.53	101. −13.5
103. 35.8	105. 42.51	107. 0.12
109. 26.7528	111. 2.2302	113. 0.03
115. $0.71	117. 150	119. $\frac{2}{25}$

121. $\frac{17}{40}$	123. $\frac{7}{4}$	125. 0.4
127. -0.375	129. $0.\overline{5}$	131. 7
133. a) $\frac{9}{1}$ b) $\frac{847}{100}$	135. a) $0.84, 1.\overline{3}$ b) $0.79132\ldots,$	137. a) 0, 17 b) -4,0,17 c) $-4, 0, \frac{5}{6}, 17$ d) $5.2537\ldots$ e) $-4, 0, 17, \frac{5}{6}, 5.2537\ldots$
139. 	141. 	143. <
145. >	147. 	149. 
151. >	153. >	155. $-48m$
157. $a+47$	159. 37	161. $\frac{35}{9}$
163. $1\frac{7}{12}$	165. $27x+23y$	167. a) $-\frac{1}{3}$ b) -5.1 c) 14 d) $\frac{8}{5}$
169. a) $\frac{1}{10}$ b) $-\frac{9}{4}$ c) $\frac{5}{3}$	171. 0	173. undefined
175. 39	177. 57	179. 8
181. $7x+63$	183. $-18m+3$	185. $5n-2$
187. $-5a-13$		

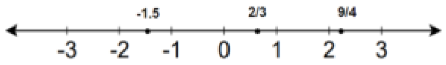
Practice Test

1. Convert 1.85 to a fraction and simplify.	2. Locate $\frac{2}{3}$, -1.5, and $\frac{9}{4}$ on a number line.

In the following exercises, simplify each expression.

3. $4+10(3+9)-5^2$	4. $-85+42$
5. $-19-25$	6. $(-2)^4$
7. $-5(-9)\div15$	8. $\frac{3}{8}\cdot\frac{11}{12}$
9. $\frac{4}{5}\div\frac{9}{20}$	10. $\frac{12+3\cdot5}{15-6}$
11. $\frac{m}{7}+\frac{10}{7}$	12. $\frac{7}{12}-\frac{3}{8}$
13. $-5.8+(-4.7)$	14. $100-64.25$
15. $(0.07)(31.95)$	16. 9 ÷ 0.05
17. $-14\left(\frac{5}{7}p\right)$	18. $(u+8)-9$
19. $6x+(-4y)+9x+8y$	20. $\frac{0}{23}$
21. $\frac{75}{0}$	22. $-2(13q-5)$

Practice Test Answers

1. $\frac{37}{20}$		3. 99
4. -43	5. -44	6. 16
7. 3	8. $\frac{11}{32}$	9. $\frac{16}{9}$
10 3	11. $\frac{m+10}{7}$	12. $\frac{5}{24}$
13. -10.5	14. 35.75	15. 2.2365
16. 180	17. $-10p$	18. $u-1$
19. $15x+4y$	20. 0	21. undefined
22. $-26q+10$		

CHAPTER 3 Measurement, Perimeter, Area, and Volume

Note the many individual shapes in this building.

We are surrounded by all sorts of geometry. Architects use geometry to design buildings. Artists create vivid images out of colorful geometric shapes. Street signs, automobiles, and product packaging all take advantage of geometric properties. In this chapter, we will begin with learning about two measurement systems used in Canada and then we will explore geometry and solve problems related to everyday situations.

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3.1 Systems of Measurement

Learning Objectives

By the end of this section, you will be able to:

- Make unit conversions in the imperial system
- Use mixed units of measurement in the imperial system
- Make unit conversions in the metric system
- Use mixed units of measurement in the metric system
- Convert between the imperial and the metric systems of measurement
- Convert between Fahrenheit and Celsius temperatures

Make Unit Conversions in the Imperial System

There are two systems of measurement commonly used around the world. Most countries use the metric system. Canada uses the metric system, and the United States use the imperial system of measurement. However, people in Canada often use imperial measurements as well. We will look at the imperial system first.

The imperial system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart, and gallons. Both the imperial system and the metric system measure time in seconds, minutes, and hours.

The equivalencies of measurements are shown in the table below. The table also shows, in parentheses, the common abbreviations for each measurement.

Imperial System of Measurement

Length	1 foot (ft.)	=	12 inches (in.)
	1 yard (yd.)	=	3 feet (ft.)
	1 mile (mi.)	=	5,280 feet (ft.)
Volume	3 teaspoons (t)	=	1 tablespoon (T)
	16 tablespoons (T)	=	1 cup (C)
	1 cup (C)	=	8 fluid ounces (fl. oz.)
	1 pint (pt.)	=	2 cups (C)
	1 quart (qt.)	=	2 pints (pt.)
	1 gallon (gal)	=	4 quarts (qt.)
Weight	1 pound (lb.)	=	16 ounces (oz.)
	1 ton	=	2000 pounds (lb.)
Time	1 minute (min)	=	60 seconds (sec)
	1 hour (hr)	=	60 minutes (min)
	1 day	=	24 hours (hr)
	1 week (wk)	=	7 days
	1 year (yr)	=	365 days

In many real-life applications, we need to convert between units of measurement, such as feet and yards, minutes and seconds, quarts and gallons, etc. We will use the identity property of multiplication to do these conversions. We'll restate the identity property of multiplication here for easy reference.

Identity Property of Multiplication

For any real number a:

$$a \cdot 1 = a$$

$$1 \cdot a = a$$

1 is the **multiplicative identity**.

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we want to change inches to feet. We know that 1 foot is equal to 12 inches, so we will write 1 as the fraction $\frac{1 \text{ foot}}{12 \text{ inches}}$. When we multiply by this fraction we do not change the value, but just change the units.

But $\frac{12 \text{ inches}}{1 \text{ foot}}$ also equals 1. How do we decide whether to multiply by $\frac{1 \text{ foot}}{12 \text{ inches}}$ or $\frac{12 \text{ inches}}{1 \text{ foot}}$? We choose the fraction that will make the units we want to convert *from* divide out. Treat the unit words like factors and "divide out" common units like we do common factors. If we want to convert 66 inches to feet, which multiplication will eliminate the inches?

$$66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \quad \text{or} \quad \xcancel{66 \text{ inches} \cdot \frac{12 \text{ inches}}{1 \text{ foot}}}$$

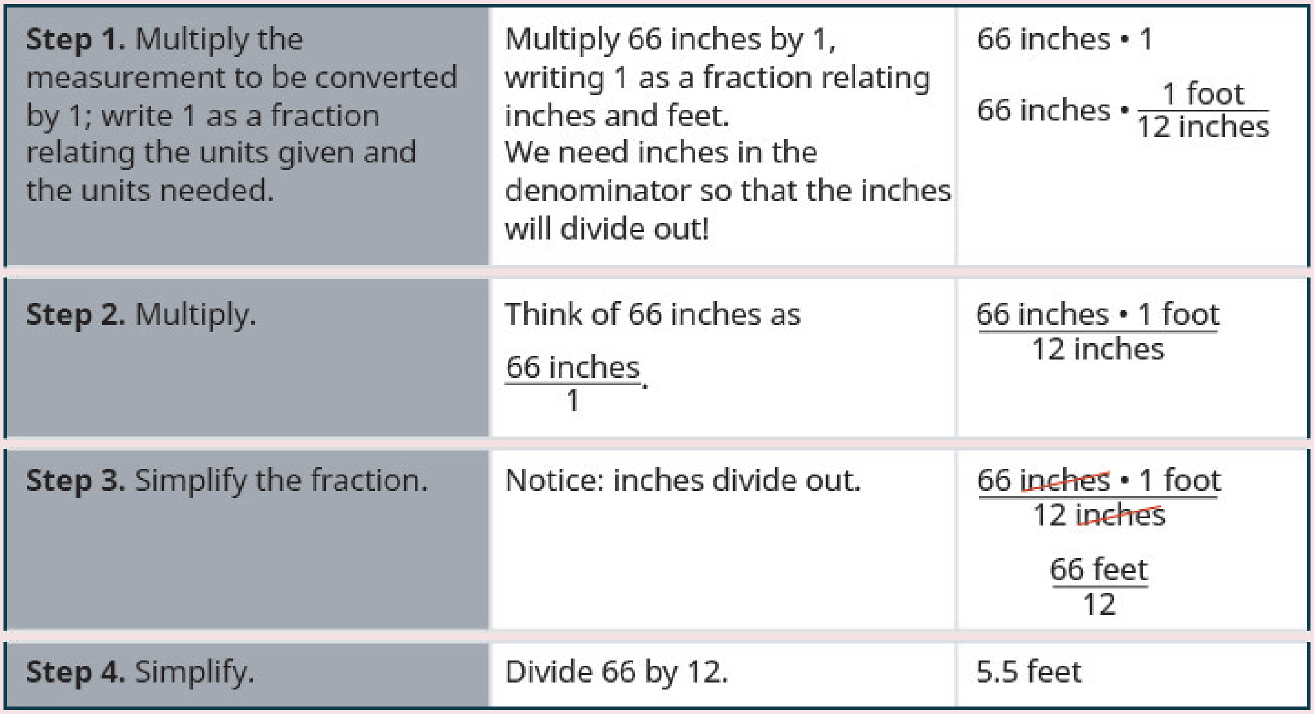
The first form works since $66 \cancel{\text{inches}} \cdot \frac{1 \text{ foot}}{12 \cancel{\text{inches}}}$.

The inches divide out and leave only feet. The second form does not have any units that will divide out and so will not help us.

EXAMPLE 1

MaryAnne is 66 inches tall. Convert her height into feet.

Solution



Step 1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.	Multiply 66 inches by 1, writing 1 as a fraction relating inches and feet. We need inches in the denominator so that the inches will divide out!	$66\text{ inches}\cdot 1$ $66\text{ inches}\cdot\frac{1\text{ foot}}{12\text{ inches}}$
Step 2. Multiply.	Think of 66 inches as $\frac{66\text{ inches}}{1}$.	$\frac{66\text{ inches}\cdot 1\text{ foot}}{12\text{ inches}}$
Step 3. Simplify the fraction.	Notice: inches divide out.	$\frac{66\cancel{\text{ inches}}\cdot 1\text{ foot}}{12\cancel{\text{ inches}}}$ $\frac{66\text{ feet}}{12}$
Step 4. Simplify.	Divide 66 by 12.	5.5 feet

TRY IT 1.1

Lexie is 30 inches tall. Convert her height to feet.

Show answer
2.5 feet

TRY IT 1.2

Rene bought a hose that is 18 yards long. Convert the length to feet.

Show answer

54 feet

HOW TO: Make unit conversions

1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
2. Multiply.
3. Simplify the fraction.
4. Simplify.

When we use the identity property of multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator.

EXAMPLE 2

A female orca in the Salish Sea weighs almost 3.2 tons. Convert her weight to pounds.

Solution
We will convert 3.2 tons into pounds. We will use the identity property of multiplication, writing 1 as the fraction $\frac{2000\text{ pounds}}{1\text{ ton}}$.

	3.2 tons
Multiply the measurement to be converted, by 1.	$3.2\text{ tons}\cdot 1$
Write 1 as a fraction relating tons and pounds.	$3.2\text{ tons}\cdot\frac{2{,}000\text{ pounds}}{1\text{ ton}}$
Simplify.	$\frac{3.2\,\cancel{\text{tons}}\cdot 2{,}000\text{ pounds}}{1\,\cancel{\text{ton}}}$
Multiply.	$6{,}400\text{ pounds}$
	The female orca weighs almost 6,400 pounds.

TRY IT 2.1

Arnold’s SUV weighs about 4.3 tons. Convert the weight to pounds.

Show answer
8,600 pounds

TRY IT 2.2

The Carnival *Destiny* cruise ship weighs 51,000 tons. Convert the weight to pounds.

Show answer
102,000,000 pounds

Sometimes, to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

EXAMPLE 3

Juliet is going with her family to their summer home. She will be away from her boyfriend for 9 weeks. Convert the time to minutes.

Solution
To convert weeks into minutes we will convert weeks into days, days into hours, and then hours into minutes. To do this we will multiply by conversion factors of 1.

	9 weeks
Write 1 as $\frac{7\text{ days}}{1\text{ week}}$, $\frac{24\text{ hours}}{1\text{ day}}$, and $\frac{60\text{ minutes}}{1\text{ hour}}$.	$\frac{9\text{ wk}}{1}\cdot\frac{7\text{ days}}{1\text{ wk}}\cdot\frac{24\text{ hr}}{1\text{ day}}\cdot\frac{60\text{ min}}{1\text{ hr}}$
Divide out the common units.	$\frac{9\ \ \text{wk}}{1}\cdot\frac{7\text{ days}}{1\text{ wk}}\cdot\frac{24\text{ hr}}{1\text{ day}}\cdot\frac{60\text{ min}}{1\text{ hr}}$
Multiply.	$\frac{9\cdot7\cdot24\cdot60\text{ min}}{1\cdot1\cdot1\cdot1}$
Multiply.	$90{,}720\text{ min}$

Juliet and her boyfriend will be apart for 90,720 minutes (although it may seem like an eternity!).

TRY IT 3.1

The distance between the earth and the moon is about 250,000 miles. Convert this length to yards.

Show answer
440,000,000 yards

TRY IT 3.2

The astronauts of Expedition 28 on the International Space Station spend 15 weeks in space. Convert the time to minutes.

Show answer
151,200 minutes

EXAMPLE 4

How many ounces are in 1 gallon?

Solution
We will convert gallons to ounces by multiplying by several conversion factors. Refer to the table on Imperial Systems of Measurement.

	1 gallon
Multiply the measurement to be converted by 1.	$\frac{1\text{ gallon}}{1} \cdot \frac{4\text{ quarts}}{1\text{ gallon}} \cdot \frac{2\text{ pints}}{1\text{ quart}} \cdot \frac{2\text{ cups}}{1\text{ pint}} \cdot \frac{8\text{ ounces}}{1\text{ cup}}$
Use conversion factors to get to the right unit. Simplify.	$\frac{1\text{ gallon}}{1} \cdot \frac{4\text{ quarts}}{1\text{ gallon}} \cdot \frac{2\text{ pints}}{1\text{ quart}} \cdot \frac{2\text{ cups}}{1\text{ pint}} \cdot \frac{8\text{ ounces}}{1\text{ cup}}$
Multiply.	$\frac{1 \cdot 4 \cdot 2 \cdot 2 \cdot 8\text{ ounces}}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}$
Simplify.	128 ounces There are 128 ounces in a gallon.

TRY IT 4.1

How many cups are in 1 gallon?

Show answer
16 cups

TRY IT 4.2

How many teaspoons are in 1 cup?

Show answer
48 teaspoons

Use Mixed Units of Measurement in the Imperial System

We often use mixed units of measurement in everyday situations. Suppose Joe is 5 feet 10 inches tall, stays at work for 7 hours and 45 minutes, and then eats a 1 pound 2 ounce steak for dinner—all these measurements have mixed units.

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units!

EXAMPLE 5

Seymour bought three steaks for a barbecue. Their weights were 14 ounces; 1 pound, 2 ounces; and 1 pound, 6 ounces. How many total pounds of steak did he buy?

Solution
We will add the weights of the steaks to find the total weight of the steaks.

Add the ounces. Then add the pounds.	$\begin{array}{rr} & 14\text{ ounces} \\ 1\text{ pound} & 2\text{ ounces} \\ +\ 1\text{ pound} & 6\text{ ounces} \\ \hline 2\text{ pounds} & 22\text{ ounces} \end{array}$
Convert 22 ounces to pounds and ounces.	1 pound, 6 ounces
Add the pounds and ounces.	2 pounds + 1 pound + 6 ounces
Answer	Seymour bought 3 pounds 6 ounces of steak.

TRY IT 5.1

Laura gave birth to triplets weighing 3 pounds 3 ounces, 3 pounds 3 ounces, and 2 pounds 9 ounces. What was the total birth weight of the three babies?

Show answer
9 lbs. 8 oz

TRY IT 5.2

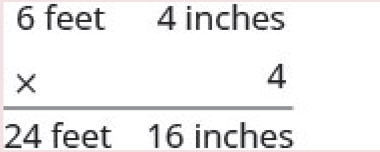
Stan cut two pieces of crown molding for his family room that were 8 feet 7 inches and 12 feet 11 inches. What was the total length of the molding?

Show answer
21 ft. 6 in.

EXAMPLE 6

Anthony bought four planks of wood that were each 6 feet 4 inches long. What is the total length of the wood he purchased?

Solution
We will multiply the length of one plank to find the total length.

Multiply the inches and then the feet.	 6 feet 4 inches × 4 24 feet 16 inches
Convert the 16 inches to feet. Add the feet.	 24 feet + 1 foot 4 inches 25 feet 4 inches
	Anthony bought 25 feet and 4 inches of wood.

TRY IT 6.1

Henri wants to triple his vegan spaghetti sauce recipe that uses 1 pound 8 ounces of black beans. How many pounds of black beans will he need?

Show answer

4 lbs. 8 oz.

TRY IT 6.2

Joellen wants to double a solution of 5 gallons 3 quarts. How many gallons of solution will she have in all?

Show answer

11 gallons 2 qt.

Make Unit Conversions in the Metric System

In the metric system, units are related by powers of 10. The roots words of their names reflect this relation. For example, the basic unit for measuring length is a metre. One kilometre is 1,000 metres; the prefix *kilo* means *thousand*. One centimetre is $\frac{1}{100}$ of a metre, just like one cent is $\frac{1}{100}$ of one dollar.

The equivalencies of measurements in the metric system are shown in the table below. The common abbreviations for each measurement are given in parentheses.

Metric System of Measurement

Length	**Mass**	**Capacity**
1 kilometre (km) = 1,000 m	1 kilogram (kg) = 1,000 g	1 kilolitre (kL) = 1,000 L
1 hectometre (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectolitre (hL) = 100 L
1 dekametre (dam) = 10 m	1 dekagram (dag) = 10 g	1 dekalitre (daL) = 10 L
1 metre (m) = 1 m	1 gram (g) = 1 g	1 litre (L) = 1 L
1 decimetre (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 decilitre (dL) = 0.1 L
1 centimetre (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centilitre (cL) = 0.01 L
1 millimetre (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 millilitre (mL) = 0.001 L
1 metre = 100 centimetres	1 gram = 100 centigrams	1 litre = 100 centilitre s
1 metre = 1,000 millimetres	1 gram = 1,000 milligrams	1 litre = 1,000 millilitre s

To make conversions in the metric system, we will use the same technique we did in the Imperial system. Using the identity property of multiplication, we will multiply by a conversion factor of one to get to the correct units.

Have you ever run a 5K or 10K race? The length of those races are measured in kilometres. The metric system is commonly used in Canada when talking about the length of a race.

EXAMPLE 7

Nick ran a 10K race. How many metres did he run?

Solution
We will convert kilometres to metres using the identity property of multiplication.

	10 kilometres
Multiply the measurement to be converted by 1.	10 kilometres × 1
Write 1 as a fraction relating kilometres and metres.	$10 \text{ kilometres} \times \frac{1,000 \text{ metres}}{1 \text{ kilometres}}$
Simplify.	$10 \text{ kilometres} \times \frac{1,000 \text{ metres}}{1 \text{ kilometres}}$
Multiply.	10,000 metres
	Nick ran 10,000 metres.

TRY IT 7.1

Sandy completed her first 5K race! How many metres did she run?

Show answer
5,000 metres

TRY IT 7.2

Herman bought a rug 2.5 metres in length. How many centimetres is the length?

Show answer
250 centimetres

EXAMPLE 8

Eleanor’s newborn baby weighed 3,200 grams. How many kilograms did the baby weigh?

Solution
We will convert grams into kilograms.

	$3{,}200 \text{ grams}$
Multiply the measurement to be converted by 1.	$3{,}200 \text{ grams} \cdot 1$
Write 1 as a function relating kilograms and grams.	$3{,}200 \text{ grams} \cdot \frac{1 \text{ kg}}{1{,}000 \text{ grams}}$
Simplify.	$3{,}200 \cancel{\text{grams}} \cdot \frac{1 \text{ kg}}{1{,}000 \cancel{\text{grams}}}$
Multiply.	$\frac{3{,}200 \text{ kilograms}}{1{,}000}$
Divide.	3.2 kilograms The baby weighed 3.2 kilograms.

TRY IT 8.1

Kari’s newborn baby weighed 2,800 grams. How many kilograms did the baby weigh?

Show answer
2.8 kilograms

TRY IT 8.2

Anderson received a package that was marked 4,500 grams. How many kilograms did this package weigh?

Show answer
4.5 kilograms

As you become familiar with the metric system you may see a pattern. Since the system is based on multiples of ten, the calculations involve multiplying by multiples of ten. We have learned how to simplify these calculations by just moving the decimal.

To multiply by 10, 100, or 1,000, we move the decimal to the right one, two, or three places, respectively. To multiply by 0.1, 0.01, or 0.001, we move the decimal to the left one, two, or three places, respectively.

We can apply this pattern when we make measurement conversions in the metric system. In Example 8, we changed 3,200 grams to kilograms by multiplying by $\frac{1}{1000}$ (or 0.001). This is the same as moving the decimal three places to the left.

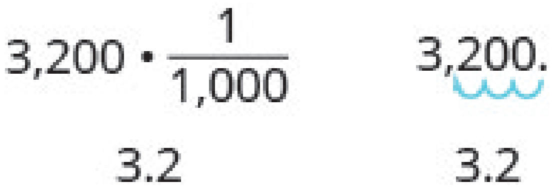


Figure.1

EXAMPLE 9

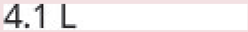
Convert a) 350 L to kilolitres b) 4.1 L to millilitre s.

Solution

a. We will convert litres to kilolitres. In the Metric System of Measurement table, we see that $1 \text{ kilolitre} = 1{,}000 \text{ litres}.$

	350 L
Multiply by 1, writing 1 as a fraction relating litres to kilolitres.	$350 \text{ L} \cdot \dfrac{1 \text{ kL}}{1{,}000 \text{ L}}$
Simplify.	$350\text{L} \cdot \dfrac{1 \text{ kL}}{1,000\text{L}}$
Move the decimal 3 units to the left.	0.35 kL

b. We will convert litres to millilitre s. From Metric System of Measurement table we see that $1 \text{ litre} = 1{,}000 \text{ millilitre s.}$

	 4.1 L
Multiply by 1, writing 1 as a fraction relating litres to millilitre s.	 $4.1 \text{ L} \cdot \dfrac{1{,}000 \text{ mL}}{1 \text{ L}}$
Simplify.	 $4.1 \cancel{\text{L}} \cdot \dfrac{1{,}000 \text{ mL}}{1 \cancel{\text{L}}}$
Move the decimal 3 units to the right.	 4.100 mL
	 $4{,}100 \text{ mL}$

TRY IT 9.1

Convert: a) 725 L to kilolitres b) 6.3 L to millilitre s

Show answer
a) 7,250 kilolitres b) 6,300 millilitre s

TRY IT 9.2

Convert: a) 350 hL to litres b) 4.1 L to centilitre s

Show answer
a) 35,000 litres b) 410 centilitre s

Use Mixed Units of Measurement in the Imperial System

Performing arithmetic operations on measurements with mixed units of measures in the imperial system requires the same care we used in the Canadian system. Make sure to add or subtract like units.

EXAMPLE 10

Ryland is 1.6 metres tall. His younger brother is 85 centimetres tall. How much taller is Ryland than his younger brother?

Solution
We can convert both measurements to either centimetres or metres. Since metres is the larger unit, we will subtract the lengths in metres. We convert 85 centimetres to metres by moving the decimal 2 places to the left.

Write the 85 centimetres as metres.	$\begin{array}{r} 1.60\text{m} \\ \underline{-0.85\text{m}} \\ 0.75\text{m} \end{array}$

Ryland is 0.75 m taller than his brother.

TRY IT 10.1

Mariella is 1.58 metres tall. Her daughter is 75 centimetres tall. How much taller is Mariella than her daughter? Write the answer in centimetres.

Show answer
83 centimetres

TRY IT 10.2

The fence around Hank’s yard is 2 metres high. Hank is 96 centimetres tall. How much shorter than the fence is Hank? Write the answer in metres.

Show answer

1.04 metres

EXAMPLE 11

Dena’s recipe for lentil soup calls for 150 millilitre s of olive oil. Dena wants to triple the recipe. How many litres of olive oil will she need?

Solution
We will find the amount of olive oil in millileters then convert to litres.

	Triple 150 mL
Translate to algebra.	$3\cdot 150\text{ mL}$
Multiply.	450 mL
Convert to litres.	$450\cdot\dfrac{0.001\text{ L}}{1\text{ mL}}$
Simplify.	0.45 L
	Dena needs 0.45 litres of olive oil.

TRY IT 11.1

A recipe for Alfredo sauce calls for 250 millilitre s of milk. Renata is making pasta with Alfredo sauce for a big party and needs to multiply the recipe amounts by 8. How many litres of milk will she need?

Show answer
2 litres

TRY IT 11.2

To make one pan of baklava, Dorothea needs 400 grams of filo pastry. If Dorothea plans to make 6 pans of baklava, how many kilograms of filo pastry will she need?

Show answer
2.4 kilograms

Convert Between the Imperial and the Metric Systems of Measurement

Many measurements in Canada are made in metric units. Our soda may come in 2-litre bottles, our calcium may come in 500-mg capsules, and we may run a 5K race. To work easily in both systems, we need to be able to convert between the two systems.

The table below shows some of the most common conversions.

Conversion Factors Between Imperial and Metric Systems

Length	**Mass**	**Capacity**
1 in. = 2.54 cm 1 ft. = 0.305 m 1 yd. = 0.914 m 1 mi. = 1.61 km 1 m = 3.28 ft.	1 lb. = 0.45 kg 1 oz. = 28 g 1 kg = 2.2 lb.	1 qt. = 0.95 L 1 fl. oz. = 30 mL 1 L = 1.06 qt.

(Figure.2) shows how inches and centimetres are related on a ruler.

Figure.2

(Figure.3) shows the ounce and millilitre markings on a measuring cup.

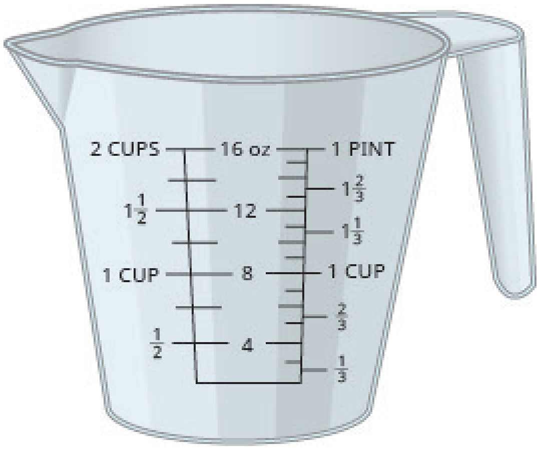


Figure.3

(Figure.4) shows how pounds and kilograms marked on a bathroom scale.

Figure.4

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

EXAMPLE 12

Lee’s water bottle holds 500 mL of water. How many ounces are in the bottle? Round to the nearest tenth of an ounce.

Solution

	500 mL
Multiply by a unit conversion factor relating mL and ounces.	$500 \text{ millilitre s} \cdot \dfrac{1 \text{ ounce}}{30 \text{ millilitre s}}$
Simplify.	$\dfrac{50 \text{ ounce}}{30}$
Divide.	16.7 ounces.
	The water bottle has 16.7 ounces.

TRY IT 12.1

How many quarts of soda are in a 2-L bottle?

Show answer
2.12 quarts

TRY IT 12.2

How many litres are in 4 quarts of milk?

Show answer
3.8 litres

EXAMPLE 13

Soleil was on a road trip and saw a sign that said the next rest stop was in 100 kilometres. How many miles until the next rest stop?

Solution

	100 kilometres
Multiply by a unit conversion factor relating km and mi.	$100 \text{ kilometres} \cdot \dfrac{1 \text{ mile}}{1.61 \text{ kilometre}}$
Simplify.	$\dfrac{100 \text{ miles}}{1.61}$
Divide.	62 ounces.
	Soleil will travel 62 miles.

TRY IT 13.1

The height of Mount Kilimanjaro is 5,895 metres. Convert the height to feet.

Show answer
19,335.6 feet

TRY IT 13.2

The flight distance from Toronto to Vancouver is 3,364 kilometres. Convert the distance to miles.

Show answer
2,090 miles

Convert between Fahrenheit and Celsius Temperatures

Have you ever been in a foreign country and heard the weather forecast? If the forecast is for $71°F$ what does that mean?

The Canadian and imperial systems use different scales to measure temperature. The Canadian system uses degrees Celsius, written °C. The imperial system uses degrees Fahrenheit, written°F. (Figure.5) shows the relationship between the two systems.
The diagram shows normal body temperature, along with the freezing and boiling temperatures of water in degrees Fahrenheit and degrees Celsius.

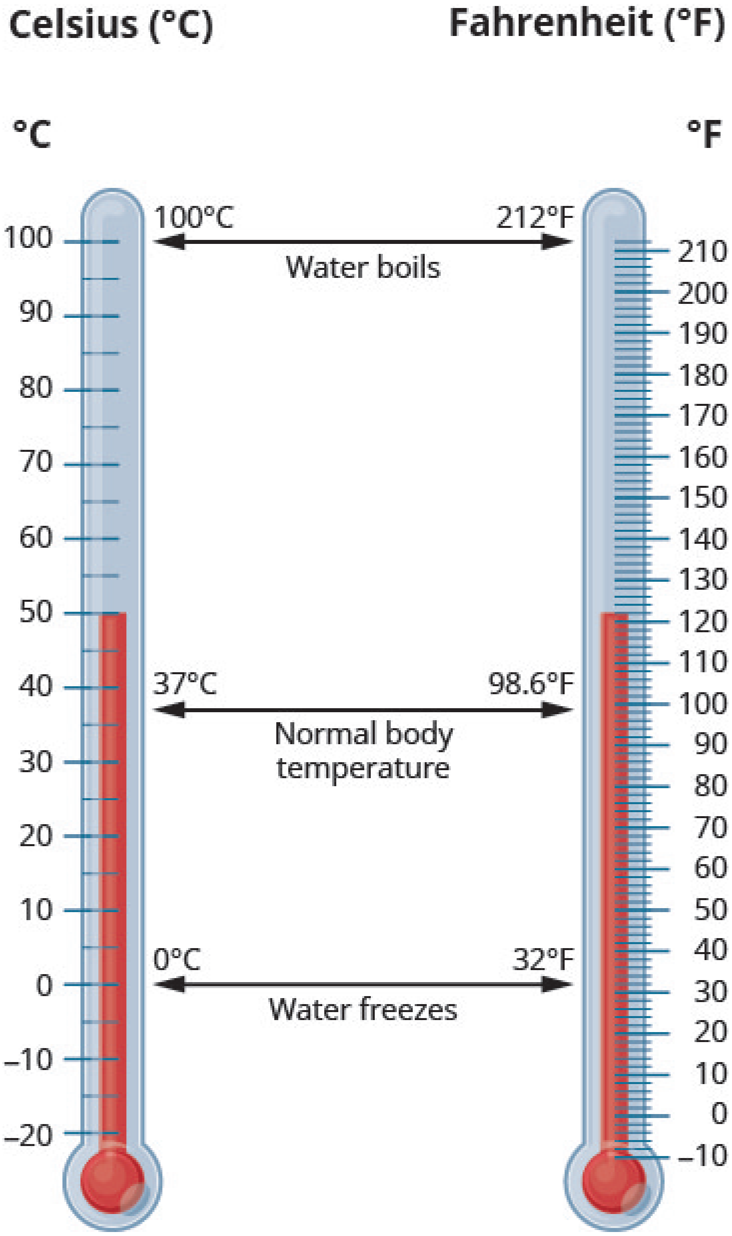


Figure.5

Temperature Conversion

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula

$$C = \frac{5}{9}(F - 32).$$

To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula

$$F = \frac{9}{5}C + 32.$$

EXAMPLE 14

Convert 50° Fahrenheit into degrees Celsius.

Solution

We will substitute 50°F into the formula to find C.

	$C = \frac{5}{9}(F - 32)$
Substitute 50 for F.	$C = \frac{5}{9}(50 - 32)$
Simplify in parentheses.	$C = \frac{5}{9}(18)$
Multiply.	$C = 10$
	So we found that 50°F is equivalent to 10°C.

TRY IT 14.1

Convert the Fahrenheit temperature to degrees Celsius: 59° Fahrenheit.

Show answer
15°C

TRY IT 14.2

Convert the Fahrenheit temperature to degrees Celsius: 41° Fahrenheit.

Show answer
5°C

EXAMPLE 15

While visiting Paris, Woody saw the temperature was 20° Celsius. Convert the temperature into degrees Fahrenheit.

Solution

We will substitute 20°C into the formula to find F.

	$F=\frac{9}{5}C+32$
Substitute 20 for C.	$F=\frac{9}{5}(20)+32$
Multiply.	$F=36+32$
Add.	$F=68$
	So we found that 20°C is equivalent to 68°F.

TRY IT 15.1

Convert the Celsius temperature to degrees Fahrenheit: the temperature in Helsinki, Finland, was 15° Celsius.

Show answer
59°F

TRY IT 15.2

Convert the Celsius temperature to degrees Fahrenheit: the temperature in Sydney, Australia, was 10° Celsius.

Show answer

50° F

Key Concepts

- **Metric System of Measurement**
    - **Length**
      1 kilometre (km) = 1,000 m
      1 hectometre (hm) = 100 m
      1 dekametre (dam) = 10 m
      1 metre (m) = 1 m
      1 decimetre (dm) = 0.1 m
      1 centimetre (cm) = 0.01 m
      1 millimetre (mm) = 0.001 m
      1 metre = 100 centimetres
      1 metre = 1,000 millimetres
    - **Mass**
      1 kilogram (kg) = 1,000 g
      1 hectogram (hg) = 100 g
      1 dekagram (dag) = 10 g
      1 gram (g) = 1 g
      1 decigram (dg) = 0.1 g
      1 centigram (cg) = 0.01 g
      1 milligram (mg) = 0.001 g
      1 gram = 100 centigrams
      1 gram = 1,000 milligrams
    - **Capacity**
      1 kilolitre (kL) = 1,000 L
      1 hectolitre (hL) = 100 L
      1 dekalitre (daL) = 10 L
      1 litre (L) = 1 L
      1 decilitre (dL) = 0.1 L
      1 centilitre (cL) = 0.01 L
      1 millilitre (mL) = 0.001 L
      1 litre = 100 centilitre s
      1 litre = 1,000 millilitre s
- **Temperature Conversion**
    - To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula
      $\mathrm{C} = \frac{5}{9}(\mathrm{F} - 32)$

- To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula $\text{F} = \frac{9}{5}\text{C} + 32$

Practice Makes Perfect

Make Unit Conversions in the Imperial System

In the following exercises, convert the units.

1. A park bench is 6 feet long. Convert the length to inches.	2. A floor tile is 2 feet wide. Convert the width to inches.
3. A ribbon is 18 inches long. Convert the length to feet.	4. Carson is 45 inches tall. Convert his height to feet.
5. A football field is 160 feet wide. Convert the width to yards.	6. On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.
7. Ulises lives 1.5 miles from school. Convert the distance to feet.	8. Denver, Colorado, is 5,183 feet above sea level. Convert the height to miles.
9. A killer whale weighs 4.6 tons. Convert the weight to pounds.	10. Blue whales can weigh as much as 150 tons. Convert the weight to pounds.
11. An empty bus weighs 35,000 pounds. Convert the weight to tons.	12. At take-off, an airplane weighs 220,000 pounds. Convert the weight to tons.
13. Rocco waited $1\frac{1}{2}$ hours for his appointment. Convert the time to seconds.	14. Misty’s surgery lasted $2\frac{1}{4}$ hours. Convert the time to seconds.
15. How many teaspoons are in a pint?	16. How many tablespoons are in a gallon?
17. JJ’s cat, Posy, weighs 14 pounds. Convert her weight to ounces.	18. April’s dog, Beans, weighs 8 pounds. Convert his weight to ounces.
19. Crista will serve 20 cups of juice at her son’s party. Convert the volume to gallons.	20. Lance needs 50 cups of water for the runners in a race. Convert the volume to gallons.
21. Jon is 6 feet 4 inches tall. Convert his height to inches.	22. Faye is 4 feet 10 inches tall. Convert her height to inches.
23. The voyage of the *Mayflower* took 2 months and 5 days. Convert the time to days.	24. Lynn’s cruise lasted 6 days and 18 hours. Convert the time to hours.
25. Baby Preston weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.	26. Baby Audrey weighted 6 pounds 15 ounces at birth. Convert her weight to ounces.

Use Mixed Units of Measurement in the Imperial System

In the following exercises, solve.

27. Eli caught three fish. The weights of the fish were 2 pounds 4 ounces, 1 pound 11 ounces, and 4 pounds 14 ounces. What was the total weight of the three fish?	28. Judy bought 1 pound 6 ounces of almonds, 2 pounds 3 ounces of walnuts, and 8 ounces of cashews. How many pounds of nuts did Judy buy?
29. One day Anya kept track of the number of minutes she spent driving. She recorded 45, 10, 8, 65, 20, and 35. How many hours did Anya spend driving?	30. Last year Eric went on 6 business trips. The number of days of each was 5, 2, 8, 12, 6, and 3. How many weeks did Eric spend on business trips last year?
31. Renee attached a 6 feet 6 inch extension cord to her computer's 3 feet 8 inch power cord. What was the total length of the cords?	32. Fawzi's SUV is 6 feet 4 inches tall. If he puts a 2 feet 10 inch box on top of his SUV, what is the total height of the SUV and the box?
33. Leilani wants to make 8 placemats. For each placemat she needs 18 inches of fabric. How many yards of fabric will she need for the 8 placemats?	34. Mireille needs to cut 24 inches of ribbon for each of the 12 girls in her dance class. How many yards of ribbon will she need altogether?

Make Unit Conversions in the Metric System

In the following exercises, convert the units.

35. Ghalib ran 5 kilometres. Convert the length to metres.	36. Kitaka hiked 8 kilometres. Convert the length to metres.
37. Estrella is 1.55 metres tall. Convert her height to centimetres.	38. The width of the wading pool is 2.45 metres. Convert the width to centimetres.
39. Mount Whitney is 3,072 metres tall. Convert the height to kilometres.	40. The depth of the Mariana Trench is 10,911 metres. Convert the depth to kilometres.
41. June's multivitamin contains 1,500 milligrams of calcium. Convert this to grams.	42. A typical ruby-throated hummingbird weights 3 grams. Convert this to milligrams.
43. One stick of butter contains 91.6 grams of fat. Convert this to milligrams.	44. One serving of gourmet ice cream has 25 grams of fat. Convert this to milligrams.
45. The maximum mass of an airmail letter is 2 kilograms. Convert this to grams.	46. Dimitri's daughter weighed 3.8 kilograms at birth. Convert this to grams.
47. A bottle of wine contained 750 millilitre s. Convert this to litres.	48. A bottle of medicine contained 300 millilitre s. Convert this to litres.

Use Mixed Units of Measurement in the Metric System

In the following exercises, solve.

49. Matthias is 1.8 metres tall. His son is 89 centimetres tall. How much taller is Matthias than his son?	50. Stavros is 1.6 metres tall. His sister is 95 centimetres tall. How much taller is Stavros than his sister?
51. A typical dove weighs 345 grams. A typical duck weighs 1.2 kilograms. What is the difference, in grams, of the weights of a duck and a dove?	52. Concetta had a 2-kilogram bag of flour. She used 180 grams of flour to make biscotti. How many kilograms of flour are left in the bag?
53. Harry mailed 5 packages that weighed 420 grams each. What was the total weight of the packages in kilograms?	54. One glass of orange juice provides 560 milligrams of potassium. Linda drinks one glass of orange juice every morning. How many grams of potassium does Linda get from her orange juice in 30 days?
55. Jonas drinks 200 millilitre s of water 8 times a day. How many litres of water does Jonas drink in a day?	56. One serving of whole grain sandwich bread provides 6 grams of protein. How many milligrams of protein are provided by 7 servings of whole grain sandwich bread?

Convert Between the Imperial and the Metric Systems of Measurement

In the following exercises, make the unit conversions. Round to the nearest tenth.

57. Bill is 75 inches tall. Convert his height to centimetres.	58. Frankie is 42 inches tall. Convert his height to centimetres.
59. Marcus passed a football 24 yards. Convert the pass length to metres	60. Connie bought 9 yards of fabric to make drapes. Convert the fabric length to metres.
61. According to research conducted by the CRC, Canadians regrettably produce more garbage per capita than any other country on earth, at 2,172.6 pounds per person annually. Convert the waste to kilograms.	62. An average Canadian will throw away 163,000 pounds of trash over his or her lifetime. Convert this weight to kilograms.
63. A 5K run is 5 kilometres long. Convert this length to miles.	64. Kathryn is 1.6 metres tall. Convert her height to feet.
65. Dawn’s suitcase weighed 20 kilograms. Convert the weight to pounds.	66. Jackson’s backpack weighed 15 kilograms. Convert the weight to pounds.
67. Ozzie put 14 gallons of gas in his truck. Convert the volume to litres.	68. Bernard bought 8 gallons of paint. Convert the volume to litres.

Convert between Fahrenheit and Celsius Temperatures

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

69. 86° Fahrenheit	70. 77° Fahrenheit
71. 104° Fahrenheit	72. 14° Fahrenheit
73. 72° Fahrenheit	74. 4° Fahrenheit
75. 0° Fahrenheit	76. 120° Fahrenheit

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

77. 5° Celsius	78. 25° Celsius
79. -10° Celsius	80. -15° Celsius
81. 22° Celsius	82. 8° Celsius
83. 43° Celsius	84. 16° Celsius

Everyday Math

85. **Nutrition** Julian drinks one can of soda every day. Each can of soda contains 40 grams of sugar. How many kilograms of sugar does Julian get from soda in 1 year?	86. **Reflectors** The reflectors in each lane-marking stripe on a highway are spaced 16 yards apart. How many reflectors are needed for a one mile long lane-marking stripe?

Writing Exercises

87. Some people think that 65° to 75° Fahrenheit is the ideal temperature range. a) What is your ideal temperature range? Why do you think so? b) Convert your ideal temperatures from Fahrenheit to Celsius.	88. a) Did you grow up using the Canadian. or the Imperial system of measurement? b) Describe two examples in your life when you had to convert between the two systems of measurement.

Answers

1. 72 inches	3. 1.5 feet	5. $53\frac{1}{3}$ yards
7. 7,920 feet	9. 9,200 pounds	11. $17\frac{1}{2}$ tons
13. 5,400 s	15. 96 teaspoons	17. 224 ounces
19. $1\frac{1}{4}$ gallons	21. 76 in.	23. 65 days
25. 115 ounces	27. 8 lbs. 13 oz.	29. 3.05 hours
31. 10 ft. 2 in.	33. 4 yards	35. 5,000 metres
37. 155 centimetres	39. 3.072 kilometres	41. 1.5 grams
43. 91,600 milligrams	45. 2,000 grams	47. 0.75 litres
49. 91 centimetres	49. 91 centimetres	49. 91 centimetres
53. 2.1 kilograms	55. 1.6 litres	57. 190.5 centimetres
59. 21.9 metres	61. 985.5 kilograms	63. 3.1 miles
65. 44 pounds	67. 53.2 litres	69. 30°C
71. 40°C	73. 22.2°C	75. -17.8°C
77. 41°F	79. 14°F	81. 71.6°F
83. 109.4°F	85. 14.6 kilograms	87. Answers may vary.

Attributions

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3.2 Use Properties of Rectangles, Triangles, and Trapezoids

Learning Objectives

By the end of this section, you will be able to:

- Understand linear, square, and cubic measure
- Use properties of rectangles
- Use properties of triangles
- Use properties of trapezoids

Understand Linear, Square, and Cubic Measure

When you measure your height or the length of a garden hose, you use a ruler or tape measure (Figure.1). A tape measure might remind you of a line—you use it for linear measure, which measures length. Inch, foot, yard, mile, centimetre and metre are units of linear measure.
This tape measure measures inches along the top and centimetres along the bottom.

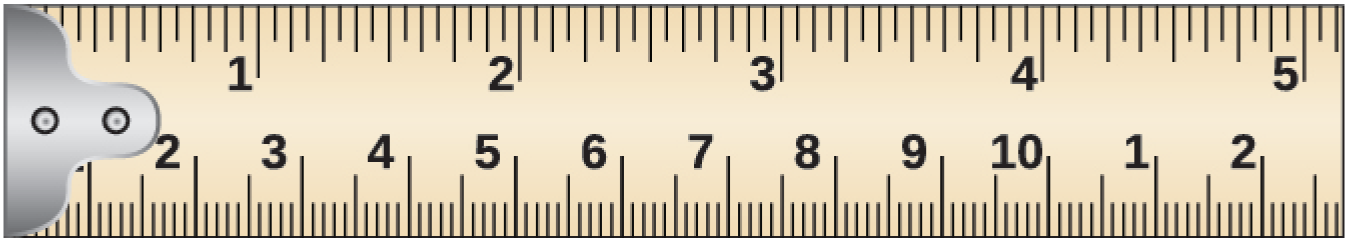


Figure.1

When you want to know how much tile is needed to cover a floor, or the size of a wall to be painted, you need to know the area, a measure of the region needed to cover a surface. Area is measured is square units. We often use square inches, square feet, square centimetres, or square miles to measure area. A square centimetre is a square that is one centimetre (cm) on each side. A square inch is a square that is one inch on each side (Figure.2).

Square measures have sides that are each 1 unit in length.

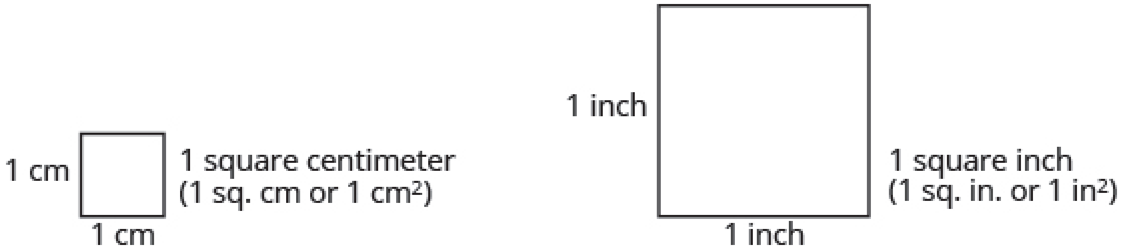


Figure.2

(Figure.3) shows a rectangular rug that is 2 feet long by 3 feet wide. Each square is 1 foot wide by 1 foot long, or 1 square foot. The rug is made of 6 squares. The area of the rug is6 square feet.

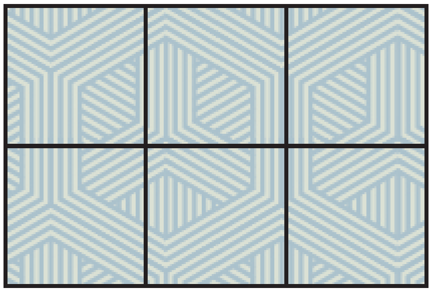


Figure 3 The rug contains six squares of 1 square foot each, so the total area of the rug is 6 square feet.

When you measure how much it takes to fill a container, such as the amount of gasoline that can fit in a tank, or the amount of medicine in a syringe, you are measuring volume. Volume is measured in cubic units such as cubic inches or cubic centimetres. When measuring the volume of a rectangular solid, you measure how many cubes fill the container. We often use cubic centimetres, cubic inches, and cubic feet. A cubic centimetre is a cube that measures one centimetre on each side, while a cubic inch is a cube that measures one inch on each side (Figure.4).

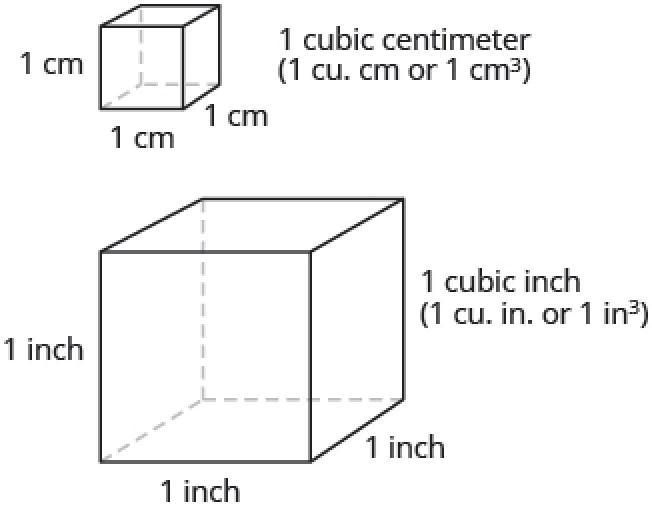


Figure 4 Cubic measures have sides that are 1 unit in length.

Suppose the cube in (Figure.5) measures 3 inches on each side and is cut on the lines shown. How many little cubes does it contain? If we were to take the big cube apart, we would find 27 little cubes, with each one measuring one inch on all sides. So each little cube has a volume of 1 cubic inch, and the volume of the big cube is 27 cubic inches.

A cube that measures 3 inches on each side is made up of 27 one-inch cubes, or 27 cubic inches.

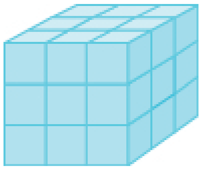


Figure.5

EXAMPLE 1

For each item, state whether you would use linear, square, or cubic measure:

a) amount of carpeting needed in a room

b) extension cord length

c) amount of sand in a sandbox

d) length of a curtain rod

e) amount of flour in a canister

f) size of the roof of a doghouse.

Solution

a) You are measuring how much surface the carpet covers, which is the area.	square measure
b) You are measuring how long the extension cord is, which is the length.	linear measure
c) You are measuring the volume of the sand.	cubic measure
d) You are measuring the length of the curtain rod.	linear measure
e) You are measuring the volume of the flour.	cubic measure
f) You are measuring the area of the roof.	square measure

TRY IT 1.1

Determine whether you would use linear, square, or cubic measure for each item.

a) amount of paint in a can b) height of a tree c) floor of your bedroom d) diametre of bike wheel e) size of a piece of sod f) amount of water in a swimming pool

Show answer

a. cubic
b. linear
c. square
d. linear
e. square
f. cubic

TRY IT 1.2

Determine whether you would use linear, square, or cubic measure for each item.

a) volume of a packing box b) size of patio c) amount of medicine in a syringe d) length of a piece of yarn e) size of housing lot f) height of a flagpole

Show answer

a. cubic
b. square
c. cubic
d. linear
e. square
f. linear

Many geometry applications will involve finding the perimeter or the area of a figure. There are also many applications of perimeter and area in everyday life, so it is important to make sure you understand what they each mean.

Picture a room that needs new floor tiles. The tiles come in squares that are a foot on each side—one square foot. How many of those squares are needed to cover the floor? This is the area of the floor.

Next, think about putting new baseboard around the room, once the tiles have been laid. To figure out how many strips are needed, you must know the distance around the room. You would use a tape measure to measure the number of feet around the room. This distance is the perimeter.

Perimeter and Area

The perimeter is a measure of the distance around a figure.

The area is a measure of the surface covered by a figure.

(Figure. 6) shows a square tile that is 1 inch on each side. If an ant walked around the edge of the tile, it would walk 4 inches. This distance is the perimeter of the tile.

Since the tile is a square that is 1 inch on each side, its area is one square inch. The area of a shape is measured by determining how many square units cover the shape.

$$\text{Perimeter} = 4\,\text{inches}$$
$$\text{Area} = 1\,\text{square inch}$$

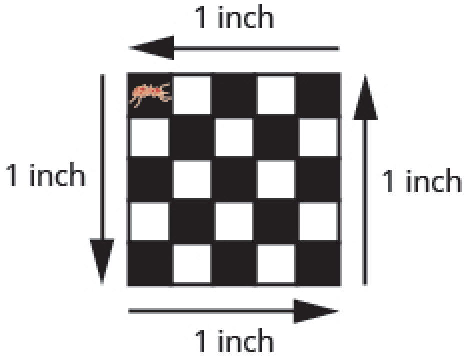


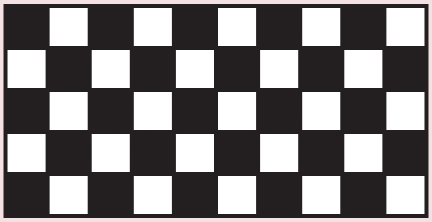
Figure 6 When the ant walks completely around the tile on its edge, it is tracing the perimeter of the tile. The area of the tile is 1 square inch.

EXAMPLE 2

Each of two square tiles is 1 square inch. Two tiles are shown together.

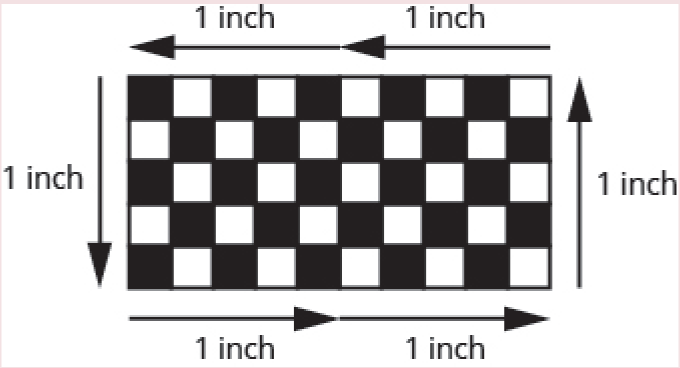
a) What is the perimeter of the figure?

b) What is the area?



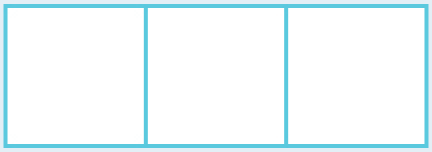
Solution
a) The perimeter is the distance around the figure. The perimeter is 6 inches.

b) The area is the surface covered by the figure. There are 2 square inch tiles so the area is 2 square inches.



TRY IT 2.1

Find the a) perimeter and b) area of the figure:

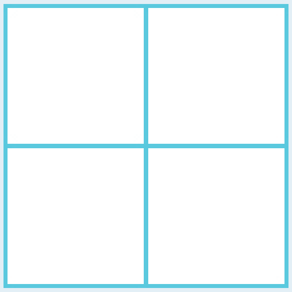


Show answer

a. 8 inches
b. 3 sq. inches

TRY IT 2.2

Find the a) perimeter and b) area of the figure:



Show answer

a. 8 centimetres

b. 4 sq. centimetres

Use the Properties of Rectangles

A rectangle has four sides and four right angles. The opposite sides of a rectangle are the same length. We refer to one side of the rectangle as the length, L, and the adjacent side as the width, W. See (Figure.7).

A rectangle has four sides, and four right angles. The sides are labeled L for length and W for width.

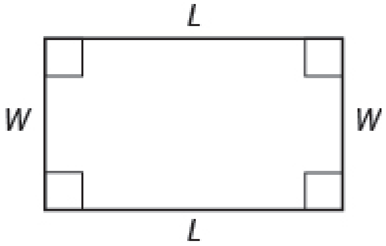


Figure.7

The perimeter, P, of the rectangle is the distance around the rectangle. If you started at one corner and walked around the rectangle, you would walk $L + W + L + W$ units, or two lengths and two widths. The perimeter then is

$$P = L + W + L + W$$
$$\text{or}$$
$$P = 2L + 2W$$

What about the area of a rectangle? Remember the rectangular rug from the beginning of this section. It was 2 feet long by 3 feet wide, and its area was 6 square feet. See (Figure.8). Since $A = 2 \cdot 3$, we see that the area, A, is the length, L, times the width, W, so the area of a rectangle is $A = L \cdot W$.

The area of this rectangular rug is 6 square feet, its length times its width.

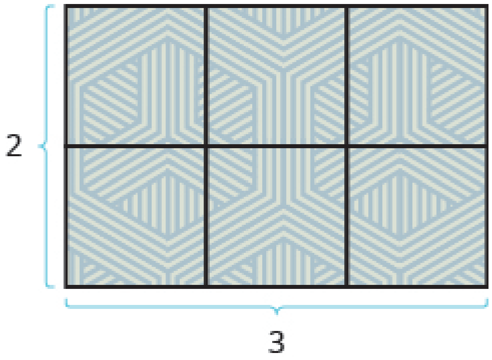


Figure.8

Properties of Rectangles

- Rectangles have four sides and four right $(90)°$ angles.
- The lengths of opposite sides are equal.
- The perimeter, P, of a rectangle is the sum of twice the length and twice the width. See (Figure 8). $P = 2L + 2W$
- The area, A, of a rectangle is the length times the width. $A = L \cdot W$

For easy reference as we work the examples in this section, we will state the Problem Solving Strategy for Geometry Applications here.

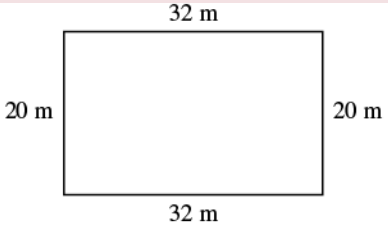
HOW TO: Use a Problem Solving Strategy for Geometry Applications

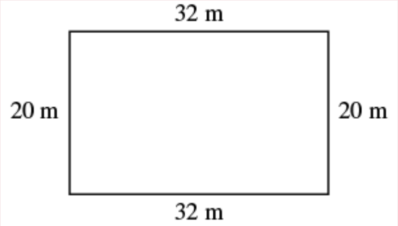
1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

EXAMPLE 3

The length of a rectangle is 32 metres and the width is 20 metres. Find a) the perimeter, and b) the area.

Solution

a)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the perimeter of a rectangle
Step 3. **Name.** Choose a variable to represent it.	Let *P* = the perimeter
Step 4. **Translate.** Write the appropriate formula. Substitute.	$\underbrace{P}_{P} \; \underbrace{=}_{=} \; \underbrace{2L}_{2(32)} \; \underbrace{+}_{+} \; \underbrace{2W}_{2(20)}$
Step 5. **Solve** the equation.	$P = 64 + 40$ $P = 104$
Step 6. **Check:**	$P \stackrel{?}{=} 104$ $20 + 32 + 20 + 32 \stackrel{?}{=} 104$ $104 = 104$ ✓
Step 7. **Answer** the question.	The perimeter of the rectangle is 104 metres.

b)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of a rectangle
Step 3. **Name.** Choose a variable to represent it.	Let *A* = the area
Step 4. **Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}_{A} \underbrace{=}_{=} \underbrace{L}_{32\text{ m}} \underbrace{\cdot}_{\cdot} \underbrace{W}_{20\text{ m}}$
Step 5. **Solve** the equation.	$A = 640$
Step 6. **Check:**	$A \overset{?}{=} 640$ $32 \cdot 20 \overset{?}{=} 640$ $640 = 640$ ✓
Step 7. **Answer** the question.	The area of the rectangle is 60 square metres.

TRY IT 3.1

The length of a rectangle is 120 yards and the width is 50 yards. Find a) the perimeter and b) the area.

Show answer

a. 340 yd
b. 6000 sq. yd

TRY IT 3.2

The length of a rectangle is 62 feet and the width is 48 feet. Find a) the perimeter and b) the area.

Show answer

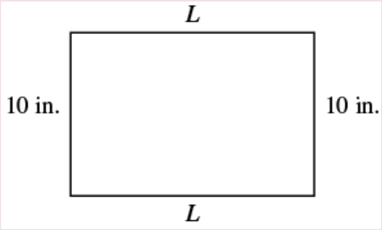
a. 220 ft

b. 2976 sq. ft

EXAMPLE 4

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the length of the rectangle
Step 3. **Name.** Choose a variable to represent it.	Let *L* = the length
Step 4. **Translate.** Write the appropriate formula. Substitute.	$\underbrace{P}_{50} \underbrace{=}_{=} \underbrace{2L}_{2L} \underbrace{+}_{+} \underbrace{2W}_{2(10)}$
Step 5. **Solve** the equation.	$50 - 20 = 2L + 20 - 20$ $30 = 2L$ $\frac{30}{2} = \frac{2L}{2}$ $15 = L$
Step 6. **Check:**	$P = 50$ $15 + 10 + 15 + 10 \overset{?}{=} 50$ $50 = 50 \checkmark$
Step 7. **Answer** the question.	The length is 15 inches.

TRY IT 4.1

Find the length of a rectangle with a perimeter of 80 inches and width of 25 inches.

Show answer
15 in.

TRY IT 4.2

Find the length of a rectangle with a perimeter of 30 yards and width of 6 yards.

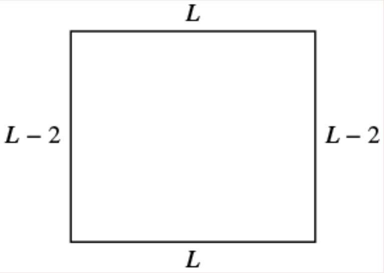
Show answer
9 yd

In the next example, the width is defined in terms of the length. We’ll wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

EXAMPLE 5

The width of a rectangle is two inches less than the length. The perimeter is 52 inches. Find the length and width.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length and width of the rectangle
Step 3. **Name.** Choose a variable to represent it. Now we can draw a figure using these expressions for the length and width.	Since the width is defined in terms of the length, we let L = length. The width is two feet less that the length, so we let $L - 2$ = width 
Step 4.**Translate.** Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information. Substitute in the given information.	$\underbrace{P}_{52} \; \underbrace{=}_{=} \; \underbrace{2L}_{2L} \; \underbrace{+}_{+} \; \underbrace{2W}_{2(L-2)}$
Step 5. **Solve** the equation.	$52 = 2L + 2L - 4$
Combine like terms.	$52 = 4L - 4$
Add 4 to each side.	$56 = 4L$
Divide by 4.	$\frac{56}{4} = \frac{4L}{4}$
	$14 = L$
	The length is 14 inches.
Now we need to find the width.	
The width is $L - 2$.	$L - 2$ $14 - 2$ 12 The width is 12 inches.
Step 6. **Check:** Since $14 + 12 + 14 + 12 = 52$, this works!	
Step 7. **Answer** the question.	The length is 14 feet and the width is 12 feet.

TRY IT 5.1

The width of a rectangle is seven metres less than the length. The perimeter is 58 metres. Find the length and width.

Show answer

18 m, 11 m

TRY IT 5.2

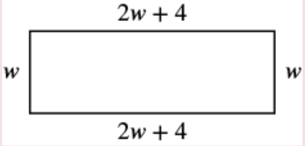
The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

Show answer
11 ft , 19 ft

EXAMPLE 6

The length of a rectangle is four centimetres more than twice the width. The perimeter is 32 centimetres. Find the length and width.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length and width
Step 3. **Name.** Choose a variable to represent it.	let W = width The length is four more than twice the width. $2w + 4$ = length 
Step 4.**Translate.** Write the appropriate formula and substitute in the given information.	$\underbrace{P}_{32}\ \underbrace{=}_{=}\ \underbrace{2L}_{2(2w+4)}\ \underbrace{+}_{+}\ \underbrace{2W}_{2w}$
Step 5. **Solve** the equation.	$32 = 4w + 8 + 2w$ $32 = 6w + 8$ $24 = 6w$ $4 = w$ width $2w + 4$ length $2(4) + 4$ 12 The length is 12 cm.
Step 6. **Check:**	$p = 2L + 2W$ $32 \stackrel{?}{=} 2 \cdot 12 + 2 \cdot 4$ $32 = 32$
Step 7. **Answer** the question.	The length is 12 cm and the width is 4 cm.

TRY IT 6.1

The length of a rectangle is eight more than twice the width. The perimeter is 64 feet. Find the length and width.

Show answer
8 ft, 24 ft

TRY IT 6.2

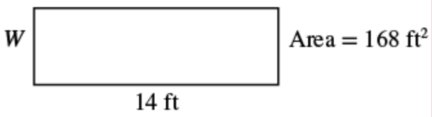
The width of a rectangle is six less than twice the length. The perimeter is 18 centimetres. Find the length and width.

Show answer
5 cm, 4 cm

EXAMPLE 7

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the width of a rectangular room
Step 3. **Name.** Choose a variable to represent it.	Let W = width
Step 4.**Translate.** Write the appropriate formula and substitute in the given information.	$A = LW$ $168 = 14W$
Step 5. **Solve** the equation.	$\frac{168}{14} = \frac{14W}{14}$ $12 = W$
Step 6. **Check:**	$A = LW$ $168 \overset{?}{=} 14 \cdot 12$ $168 = 168 \checkmark$
Step 7. **Answer** the question.	The width of the room is 12 feet.

TRY IT 7.1

The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Show answer
26 ft

TRY IT 7.2

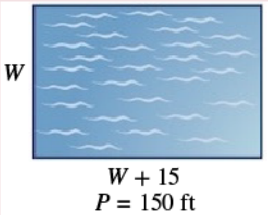
The width of a rectangle is 21 metres. The area is 609 square metres. What is the length?

Show answer
29 m

EXAMPLE 8

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the length and width of the pool
Step 3. **Name.** Choose a variable to represent it. The length is 15 feet more than the width.	Let $W = \text{width}$ $W + 15 = \text{length}$
Step 4.**Translate.** Write the appropriate formula and substitute.	$\underbrace{P}_{150} \quad \underbrace{=}_{=} \quad \underbrace{2L}_{2(w+15)} \quad \underbrace{+}_{+} \quad \underbrace{2W}_{2w}$
Step 5. **Solve** the equation.	$150 = 2w + 30 + 2w$ $150 = 4w + 30$ $120 = 4w$ $30 = w$ the width of the pool $w + 15$ the length of the pool $30 + 15$ 45
Step 6. **Check:**	$p = 2L + 2W$ $150 \overset{?}{=} 2(45) + 2(30)$ $150 = 150$
Step 7. **Answer** the question.	The length of the pool is 45 feet and the width is 30 feet.

TRY IT 8.1

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Show answer
30 ft, 70 ft

TRY IT 8.2

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

Show answer
60 yd, 90 yd

Use the Properties of Triangles

We now know how to find the area of a rectangle. We can use this fact to help us visualize the formula for the area of a triangle. In the rectangle in (Figure.9), we've labeled the length b and the width h, so it's area is bh.

The area of a rectangle is the base, b, times the height, h.

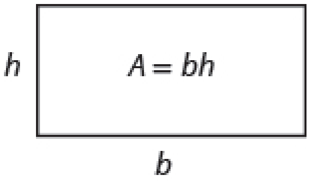


Figure.9

We can divide this rectangle into two congruent triangles (Figure.10). Triangles that are congruent have identical side lengths and angles, and so their areas are equal. The area of each triangle is one-half the area of the rectangle, or $\frac{1}{2}bh$. This example helps us see why the formula for the area of a triangle is $A = \frac{1}{2}bh$.

A rectangle can be divided into two triangles of equal area. The area of each triangle is one-half the area of the rectangle.

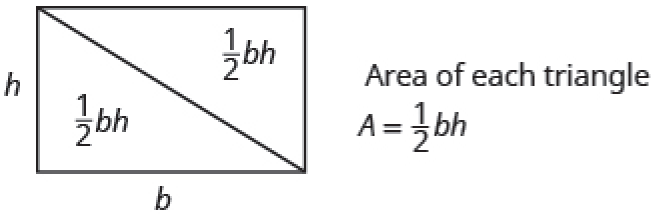


Figure.10

The formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height.

To find the area of the triangle, you need to know its base and height. The base is the length of one side of the triangle, usually the side at the bottom. The height is the length of the line that connects the base to the opposite vertex, and makes a $90°$ angle with the base. (Figure.11) shows three triangles with the base and height of each marked.

The height h of a triangle is the length of a line segment that connects the the base to the opposite vertex and makes a $90°$ angle with the base.

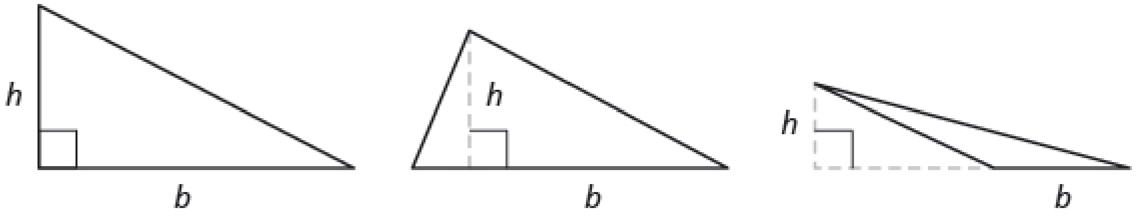


Figure.11

Triangle Properties

For any triangle ΔABC, the sum of the measures of the angles is $180°$.

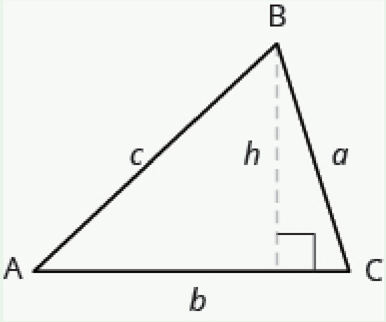
$m\angle A + m\angle B + m\angle C = 180°$

The perimeter of a triangle is the sum of the lengths of the sides.

$P = a + b + c$

The area of a triangle is one-half the base, b, times the height, h.

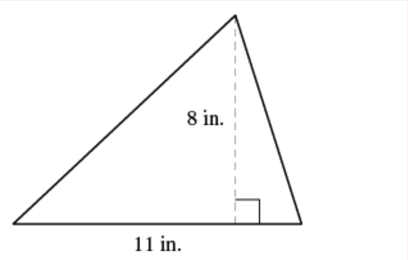
$A = \frac{1}{2}bh$



EXAMPLE 9

Find the area of a triangle whose base is 11 inches and whose height is 8 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the triangle
Step 3. **Name.** Choose a variable to represent it.	let A = area of the triangle
Step 4.**Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}_{A}\ \underbrace{=}_{=}\ \underbrace{\frac{1}{2}}_{\frac{1}{2}}\ \underbrace{\cdot}_{\cdot}\ \underbrace{b}_{11}\ \underbrace{\cdot}_{\cdot}\ \underbrace{h}_{8}$
Step 5. **Solve** the equation.	$A = 44$ square inches
Step 6. **Check:**	$A = \frac{1}{2} bh$ $44 \stackrel{?}{=} \frac{1}{2}(11)8$ $44 = 44$ ✓
Step 7. **Answer** the question.	The area is 44 square inches.

TRY IT 9.1

Find the area of a triangle with base 13 inches and height 2 inches.

Show answer
13 sq. in.

TRY IT 9.2

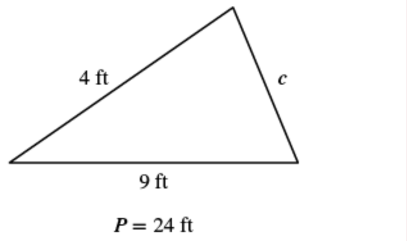
Find the area of a triangle with base 14 inches and height 7 inches.

Show answer
49 sq. in.

EXAMPLE 10

The perimeter of a triangular garden is 24 feet. The lengths of two sides are 4 feet and 9 feet. How long is the third side?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	length of the third side of a triangle
Step 3. **Name.** Choose a variable to represent it.	Let *c* = the third side
Step 4.**Translate.** Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{24} = \underbrace{a}_{4} + \underbrace{b}_{9} + \underbrace{c}_{c}$
Step 5. **Solve** the equation.	$24 = 13 + c$ $11 = c$
Step 6. **Check:**	$P = a + b + c$ $24 \overset{?}{=} 4 + 9 + 11$ $24 = 24$ ✓
Step 7. **Answer** the question.	The third side is 11 feet long.

TRY IT 10.1

The perimeter of a triangular garden is 24 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Show answer
8 ft

TRY IT 10.2

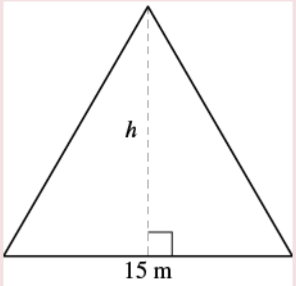
The lengths of two sides of a triangular window are 7 feet and 5 feet. The perimeter is 18 feet. How long is the third side?

Show answer
6 ft

EXAMPLE 11

The area of a triangular church window is 90 square metres. The base of the window is 15 metres. What is the window's height?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	height of a triangle
Step 3. **Name.** Choose a variable to represent it.	Let *h* = the height
Step 4.**Translate.** Write the appropriate formula. Substitute in the given information.	$\underbrace{A}_{90} \underbrace{=}_{=} \underbrace{\frac{1}{2}}_{\frac{1}{2}} \underbrace{\cdot}_{\cdot} \underbrace{b}_{15} \underbrace{\cdot}_{\cdot} \underbrace{h}_{h}$
Step 5. **Solve** the equation.	$90 = \frac{15}{2}h$ $12 = h$
Step 6. **Check:**	$A = \frac{1}{2}bh$ $90 \overset{?}{=} \frac{1}{2} \cdot 15 \cdot 12$ $90 = 90 \checkmark$
Step 7. **Answer** the question.	The height of the triangle is 12 metres.

TRY IT 11.1

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Show answer
14 in.

TRY IT 11.2

A triangular tent door has an area of 15 square feet. The height is 5 feet. What is the base?

Show answer

6 ft

Isosceles and Equilateral Triangles

Besides the right triangle, some other triangles have special names. A triangle with two sides of equal length is called an isosceles triangle. A triangle that has three sides of equal length is called an equilateral triangle. (Figure.12) shows both types of triangles.

In an isosceles triangle, two sides have the same length, and the third side is the base. In an equilateral triangle, all three sides have the same length.

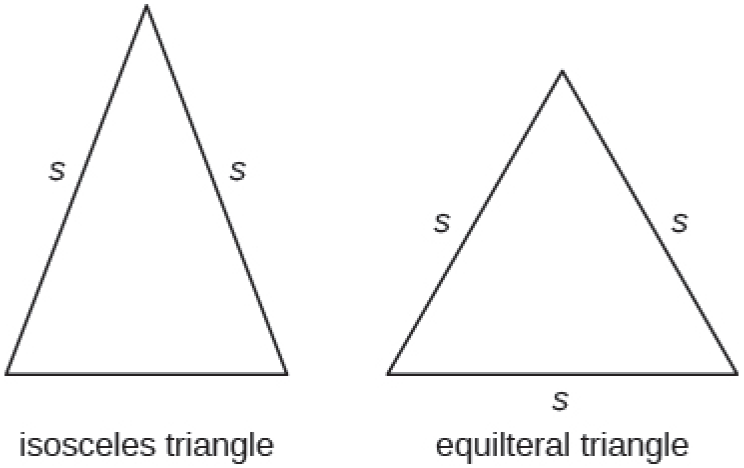


Figure.12

Isosceles and Equilateral Triangles

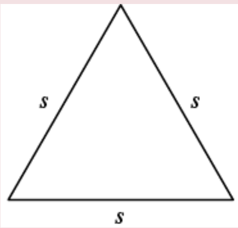
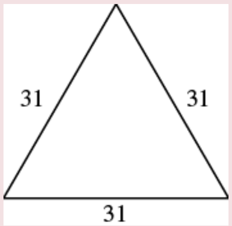
An **isosceles** triangle has two sides the same length.

An **equilateral** triangle has three sides of equal length.

EXAMPLE 12

The perimeter of an equilateral triangle is 93 inches. Find the length of each side.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	 Perimeter = 93 in.
Step 2. **Identify** what you are looking for.	length of the sides of an equilateral triangle
Step 3. **Name.** Choose a variable to represent it.	Let *s* = length of each side
Step 4.**Translate.** Write the appropriate formula. Substitute.	$\underbrace{P}_{93}\ \underbrace{=}_{=}\ \underbrace{a}_{s}\ \underbrace{+}_{+}\ \underbrace{b}_{s}\ \underbrace{+}_{+}\ \underbrace{c}_{s}$
Step 5. **Solve** the equation.	$93 = 3s$ $31 = s$
Step 6. **Check:**	 $93 \overset{?}{=} 31 + 31 + 31$ $93 = 93$ ✓
Step 7. **Answer** the question.	Each side is 31 inches

TRY IT 12.1

Find the length of each side of an equilateral triangle with perimeter 39 inches.

Show answer
13 in.

TRY IT 12.2

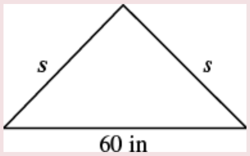
Find the length of each side of an equilateral triangle with perimeter 51 centimetres.

Show answer
17 cm

EXAMPLE 13

Arianna has 156 inches of beading to use as trim around a scarf. The scarf will be an isosceles triangle with a base of 60 inches. How long can she make the two equal sides?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	 *P* = 156 in.
Step 2. **Identify** what you are looking for.	the lengths of the two equal sides
Step 3. **Name.** Choose a variable to represent it.	Let *s* = the length of each side
Step 4.**Translate.** Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{156}\ \underbrace{=}_{=}\ \underbrace{a}_{s}\ \underbrace{+}_{+}\ \underbrace{b}_{60}\ \underbrace{+}_{+}\ \underbrace{c}_{s}$
Step 5. **Solve** the equation.	$156 = 2s + 60$ $96 = 2s$ $48 = s$
Step 6. **Check:**	$p = a + b + c$ $156 \overset{?}{=} 48 + 60 + 48$ $156 = 156\ \checkmark$
Step 7. **Answer** the question.	Arianna can make each of the two equal sides 48 inches l

TRY IT 13.1

A backyard deck is in the shape of an isosceles triangle with a base of 20 feet. The perimeter of the deck is 48 feet. How long is each of the equal sides of the deck?

Show answer
14 ft

TRY IT 13.2

A boat's sail is an isosceles triangle with base of 8 metres. The perimeter is 22 metres. How long is each of the equal sides of the sail?

Show answer
7 m

Use the Properties of Trapezoids

A trapezoid is four-sided figure, a *quadrilateral*, with two sides that are parallel and two sides that are not. The parallel sides are called the bases. We call the length of the smaller base b, and the length of the bigger base B. The height, h, of a trapezoid is the distance between the two bases as shown in (Figure.13).

A trapezoid has a larger base, B, and a smaller base, b. The height h is the distance between the bases.

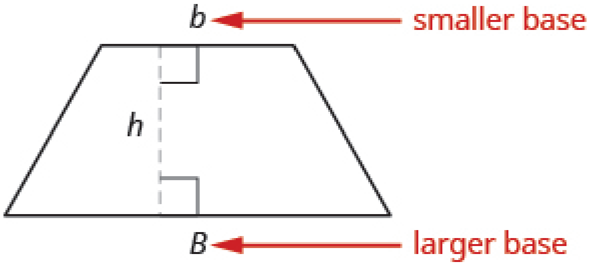


Figure.13

Formula for the Area of a Trapezoid

$$\text{Area}_{\text{trapezoid}} = \frac{1}{2}h\left(b+B\right)$$

Splitting the trapezoid into two triangles may help us understand the formula. The area of the trapezoid is the sum of the areas of the two triangles. See (Figure.14).
Splitting a trapezoid into two triangles may help you understand the formula for its area.

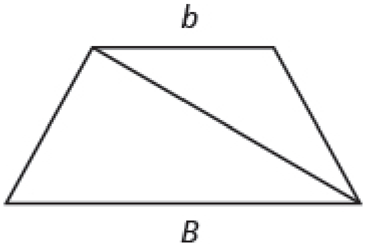


Figure.14

The height of the trapezoid is also the height of each of the two triangles. See (Figure.15).

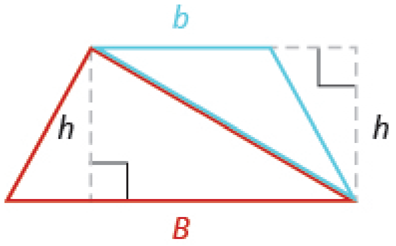


Figure.15

The formula for the area of a trapezoid is

$$\text{Area}_{\text{trapezoid}} = \frac{1}{2}h(b + B)$$

If we distribute, we get,

$$\text{Area}_{trapezoid} = \frac{1}{2}bh + \frac{1}{2}Bh$$

$$\text{Area}_{trapezoid} = A_{blue\Delta} + A_{red\Delta}$$

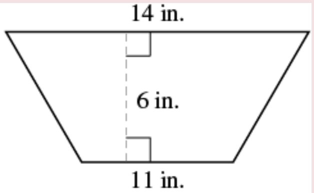
Properties of Trapezoids

- A trapezoid has four sides. See (Figure.13).
- Two of its sides are parallel and two sides are not.
- The area, A, of a trapezoid is $\mathrm{A} = \frac{1}{2}h\left(b + B\right)$.

EXAMPLE 14

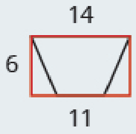
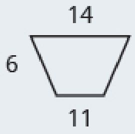
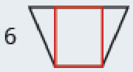
Find the area of a trapezoid whose height is 6 inches and whose bases are 14 and 11 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the trapezoid
Step 3. **Name.** Choose a variable to represent it.	Let $A = \text{the area}$
Step 4.**Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}_{A}\ \underbrace{=}_{=}\ \underbrace{\frac{1}{2}}_{\frac{1}{2}}\ \underbrace{\cdot}_{\cdot}\ \underbrace{h}_{6}\ \underbrace{\cdot}_{\cdot}\ \underbrace{(b+B)}_{(11+14)}$
Step 5. **Solve** the equation.	$A = \frac{1}{2} \cdot 6(25)$ $A = 3(25)$ $A = 75$ square inches
Step 6. **Check:** Is this answer reasonable?	

If we draw a rectangle around the trapezoid that has the same big base B and a height h, its area should be greater than that of the trapezoid.

If we draw a rectangle inside the trapezoid that has the same little base b and a height h, its area should be smaller than that of the trapezoid.

		
$A_{\text{rectangle}} = bh$	$A_{\text{trapezoid}} = \frac{1}{2}h(b+B)$	$A_{\text{rectangle}} = bh$
$A_{\text{rectangle}} = 14 \cdot 6$	$A_{\text{trapezoid}} = \frac{1}{2} \cdot 6(11+14)$	$A_{\text{rectangle}} = 11 \cdot 6$
$A_{\text{rectangle}} = 84$ sq. in.	$A_{\text{trapezoid}} = 75$ sq. in.	$A_{\text{rectangle}} = 66$ sq. in.

The area of the larger rectangle is 84 square inches and the area of the smaller rectangle is 66 square inches. So it makes sense that the area of the trapezoid is between 84 and 66 square inches

Step 7. **Answer** the question. The area of the trapezoid is 75 square inches.

TRY IT 14.1

The height of a trapezoid is 14 yards and the bases are 7 and 16 yards. What is the area?

Show answer
161 sq. yd

TRY IT 14.2

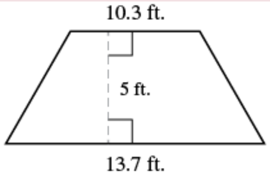
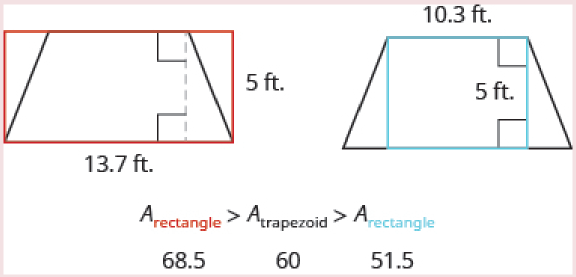
The height of a trapezoid is 18 centimetres and the bases are 17 and 8 centimetres. What is the area?

Show answer
225 sq. cm

EXAMPLE 15

Find the area of a trapezoid whose height is 5 feet and whose bases are 10.3 and 13.7 feet.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the trapezoid
Step 3. **Name.** Choose a variable to represent it.	Let *A* = the area
Step 4.**Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}_{A}\ \underbrace{=}_{=}\ \underbrace{\frac{1}{2}}_{\frac{1}{2}}\ \underbrace{\cdot}_{\cdot}\ \underbrace{h}_{5}\ \underbrace{\cdot}_{\cdot}\ \underbrace{(b+B)}_{(10.3+13.7)}$
Step 5. **Solve** the equation.	$A=\frac{1}{2}\cdot 5(24)$ $A=12\cdot 5$ $A=60$ square feet
Step 6. **Check:** Is this answer reasonable? The area of the trapezoid should be less than the area of a rectangle with base 13.7 and height 5, but more than the area of a rectangle with base 10.3 and height 5.	
Step 7. **Answer** the question.	The area of the trapezoid is 60 square feet.

TRY IT 15.1

The height of a trapezoid is 7 centimetres and the bases are 4.6 and 7.4 centimetres. What is the area?

Show answer
42 sq. cm

TRY IT 15.2

The height of a trapezoid is 9 metres and the bases are 6.2 and 7.8 metres. What is the area?

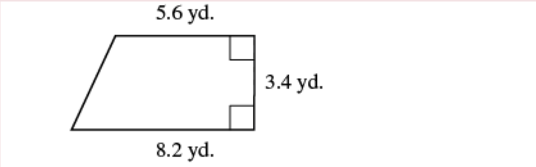
Show answer

63 sq. m

EXAMPLE 16

Vinny has a garden that is shaped like a trapezoid. The trapezoid has a height of 3.4 yards and the bases are 8.2 and 5.6 yards. How many square yards will be available to plant?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of a trapezoid
Step 3. **Name.** Choose a variable to represent it.	Let *A* = the area
Step 4.**Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}\ \underbrace{=}\ \underbrace{\frac{1}{2}}\ \underbrace{\cdot}\ \underbrace{h}\ \underbrace{\cdot}\ \underbrace{(b+B)}$ $A = \frac{1}{2} \cdot 3.4 \cdot (5.6+8.2)$
Step 5. **Solve** the equation.	$A = \frac{1}{2}(3.4)\,(13.8)$ $A = 23.46$ square yards

Step 6. **Check:** Is this answer reasonable?
Yes. The area of the trapezoid is less than the area of a rectangle with a base of 8.2 yd and height 3.4 yd, but more than the area of a rectangle with base 5.6 yd and height 3.4 yd.

$A_{\text{rectangle}} = Bh$ $= (8.2)(3.4)$ $= 27.88\text{ yd}^2$	$A_{\text{trapezoid}} = \frac{1}{2}(3.4\text{ yd})(5.6 = 8.2)$ $= 23.46\text{ yd}^2$	$A_{\text{rectangle}} = bh$ $= (5.6)(3.4)$ $= 19.04\text{ yd}^2$

$$A_{\text{rectangle}} > A_{\text{trapezoid}} > A_{\text{rectangle}}$$
$$27.88 \qquad 23.46 \qquad 19.04$$

Step 7. **Answer** the question.	Vinny has 23.46 square yards in which he can plan

TRY IT 16.1

Lin wants to sod his lawn, which is shaped like a trapezoid. The bases are 10.8 yards and 6.7 yards, and the height is 4.6 yards. How many square yards of sod does he need?

Show answer
40.25 sq. yd

TRY IT 16.2

Kira wants cover his patio with concrete pavers. If the patio is shaped like a trapezoid whose bases are 18 feet and 14 feet and whose height is 15 feet, how many square feet of pavers will he need?

Show answer
240 sq. ft.

Access Additional Online Resources

- Perimeter of a Rectangle
- Area of a Rectangle
- Perimeter and Area Formulas
- Area of a Triangle
- Area of a Triangle with Fractions
- Area of a Trapezoid

Key Concepts

- **Properties of Rectangles**
    - Rectangles have four sides and four right (90°) angles.
    - The lengths of opposite sides are equal.
    - The perimeter, P, of a rectangle is the sum of twice the length and twice the width.
        - $P = 2L + 2W$
    - The area, A, of a rectangle is the length times the width.
        - $A = L \cdot W$

- **Triangle Properties**
    - For any triangle ΔABC, the sum of the measures of the angles is 180°.
        - $m\angle A + m\angle B + m\angle C = 180°$
    - The perimeter of a triangle is the sum of the lengths of the sides.
        - $P = a + b + c$
    - The area of a triangle is one-half the base, b, times the height, h.
        - $A = \frac{1}{2}bh$

Glossary

area	The area is a measure of the surface covered by a figure.
equilateral triangle	A triangle with all three sides of equal length is called an equilateral triangle.
isosceles triangle	A triangle with two sides of equal length is called an isosceles triangle.
perimeter	The perimeter is a measure of the distance around a figure.
rectangle	A rectangle is a geometric figure that has four sides and four right angles.
trapezoid	A trapezoid is four-sided figure, a quadrilateral, with two sides that are parallel and two sides that are not.

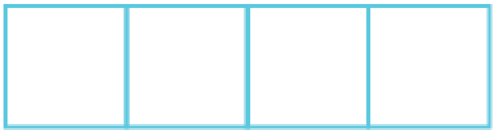
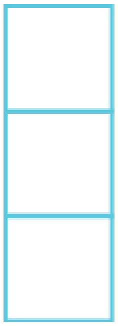
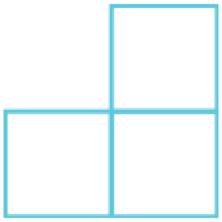
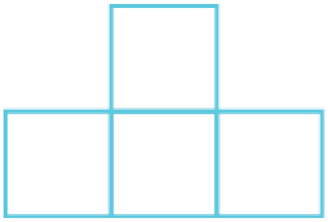
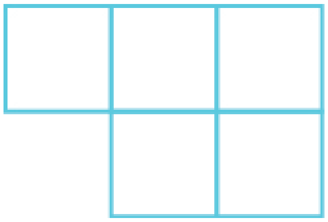
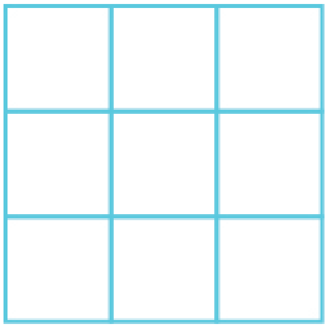
Practice Makes Perfect

Understand Linear, Square, and Cubic Measure

In the following exercises, determine whether you would measure each item using linear, square, or cubic units.

1. amount of water in a fish tank	2. length of dental floss
3. living area of an apartment	4. floor space of a bathroom tile
5. height of a doorway	6. capacity of a truck trailer

In the following exercises, find the a) perimeter and b) area of each figure. Assume each side of the square is 1 cm.

7. 	8. 
9. 	10. 
11. 	12. 

Use the Properties of Rectangles

In the following exercises, find the a) perimeter and b) area of each rectangle.

13. The length of a rectangle is 85 feet and the width is 45 feet.	14. The length of a rectangle is 26 inches and the width is 58 inches.
15. A rectangular room is 15 feet wide by 14 feet long.	16. A driveway is in the shape of a rectangle 20 feet wide by 35 feet long.

In the following exercises, solve.

17. Find the length of a rectangle with perimeter 124 inches and width 38 inches.

18. Find the length of a rectangle with perimeter 20.2 yards and width of 7.8 yards.

19. Find the width of a rectangle with perimeter 92 metres and length 19 metres.

20. Find the width of a rectangle with perimeter 16.2 metres and length 3.2 metres.

21. The area of a rectangle is 414 square metres. The length is 18 metres. What is the width?

22. The area of a rectangle is 782 square centimetres. The width is 17 centimetres. What is the length?

23. The length of a rectangle is 9 inches more than the width. The perimeter is 46 inches. Find the length and the width.

24. The width of a rectangle is 8 inches more than the length. The perimeter is 52 inches. Find the length and the width.

25. The perimeter of a rectangle is 58 metres. The width of the rectangle is 5 metres less than the length. Find the length and the width of the rectangle.

26. The perimeter of a rectangle is 62 feet. The width is 7 feet less than the length. Find the length and the width.

27. The width of the rectangle is 0.7 metres less than the length. The perimeter of a rectangle is 52.6 metres. Find the dimensions of the rectangle.

28. The length of the rectangle is 1.1 metres less than the width. The perimeter of a rectangle is 49.4 metres. Find the dimensions of the rectangle.

29. The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

30. The length of a rectangle is three times the width. The perimeter is 72 feet. Find the length and width of the rectangle.

31. The length of a rectangle is 3 metres less than twice the width. The perimeter is 36 metres. Find the length and width.

32. The length of a rectangle is 5 inches more than twice the width. The perimeter is 34 inches. Find the length and width.

33. The width of a rectangular window is 24 inches. The area is 624 square inches. What is the length?

34. The length of a rectangular poster is 28 inches. The area is 1316 square inches. What is the width?

35. The area of a rectangular roof is 2310 square metres. The length is 42 metres. What is the width?

36. The area of a rectangular tarp is 132 square feet. The width is 12 feet. What is the length?

37. The perimeter of a rectangular courtyard is 160 feet. The length is 10 feet more than the width. Find the length and the width.

38. The perimeter of a rectangular painting is 306 centimetres. The length is 17 centimetres more than the width. Find the length and the width.

39. The width of a rectangular window is 40 inches less than the height. The perimeter of the doorway is 224 inches. Find the length and the width.

40. The width of a rectangular playground is 7 metres less than the length. The perimeter of the playground is 46 metres. Find the length and the width.

Use the Properties of Triangles

In the following exercises, solve using the properties of triangles.

41. Find the area of a triangle with base 12 inches and height 5 inches.

42. Find the area of a triangle with base 45 centimetres and height 30 centimetres.

43. Find the area of a triangle with base 8.3 metres and height 6.1 metres.

44. Find the area of a triangle with base 24.2 feet and height 20.5 feet.

45. A triangular flag has base of 1 foot and height of 1.5 feet. What is its area?

46. A triangular window has base of 8 feet and height of 6 feet. What is its area?

47. If a triangle has sides of 6 feet and 9 feet and the perimeter is 23 feet, how long is the third side?

48. If a triangle has sides of 14 centimetres and 18 centimetres and the perimeter is 49 centimetres, how long is the third side?

49. What is the base of a triangle with an area of 207 square inches and height of 18 inches?

50. What is the height of a triangle with an area of 893 square inches and base of 38 inches?

51. The perimeter of a triangular reflecting pool is 36 yards. The lengths of two sides are 10 yards and 15 yards. How long is the third side?

52. A triangular courtyard has perimeter of 120 metres. The lengths of two sides are 30 metres and 50 metres. How long is the third side?

53. An isosceles triangle has a base of 20 centimetres. If the perimeter is 76 centimetres, find the length of each of the other sides.

54. An isosceles triangle has a base of 25 inches. If the perimeter is 95 inches, find the length of each of the other sides.

55. Find the length of each side of an equilateral triangle with a perimeter of 51 yards.

56. Find the length of each side of an equilateral triangle with a perimeter of 54 metres.

57. The perimeter of an equilateral triangle is 18 metres. Find the length of each side.

58. The perimeter of an equilateral triangle is 42 miles. Find the length of each side.

59. The perimeter of an isosceles triangle is 42 feet. The length of the shortest side is 12 feet. Find the length of the other two sides.

60. The perimeter of an isosceles triangle is 83 inches. The length of the shortest side is 24 inches. Find the length of the other two sides.

61. A dish is in the shape of an equilateral triangle. Each side is 8 inches long. Find the perimeter.

62. A floor tile is in the shape of an equilateral triangle. Each side is 1.5 feet long. Find the perimeter.

63. A road sign in the shape of an isosceles triangle has a base of 36 inches. If the perimeter is 91 inches, find the length of each of the other sides.

64. A scarf in the shape of an isosceles triangle has a base of 0.75 metres. If the perimeter is 2 metres, find the length of each of the other sides.

65. The perimeter of a triangle is 39 feet. One side of the triangle is 1 foot longer than the second side. The third side is 2 feet longer than the second side. Find the length of each side.

66. The perimeter of a triangle is 35 feet. One side of the triangle is 5 feet longer than the second side. The third side is 3 feet longer than the second side. Find the length of each side.

67. One side of a triangle is twice the smallest side. The third side is 5 feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

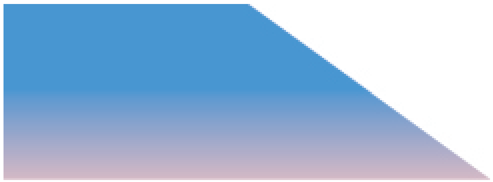
68. One side of a triangle is three times the smallest side. The third side is 3 feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

Use the Properties of Trapezoids

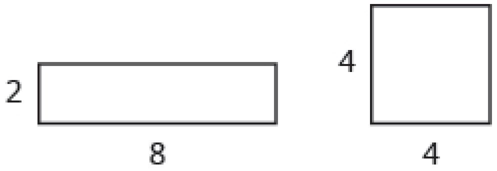
In the following exercises, solve using the properties of trapezoids.

69. The height of a trapezoid is 12 feet and the bases are 9 and 15 feet. What is the area?	70. The height of a trapezoid is 24 yards and the bases are 18 and 30 yards. What is the area?
71. Find the area of a trapezoid with a height of 51 metres and bases of 43 and 67 metres.	72. Find the area of a trapezoid with a height of 62 inches and bases of 58 and 75 inches.
73. The height of a trapezoid is 15 centimetres and the bases are 12.5 and 18.3 centimetres. What is the area?	74. The height of a trapezoid is 48 feet and the bases are 38.6 and 60.2 feet. What is the area?
75. Find the area of a trapezoid with a height of 4.2 metres and bases of 8.1 and 5.5 metres.	76. Find the area of a trapezoid with a height of 32.5 centimetres and bases of 54.6 and 41.4 centimetres.
77. Laurel is making a banner shaped like a trapezoid. The height of the banner is 3 feet and the bases are 4 and 5 feet. What is the area of the banner?	78. Niko wants to tile the floor of his bathroom. The floor is shaped like a trapezoid with width 5 feet and lengths 5 feet and 8 feet. What is the area of the floor?
79. Theresa needs a new top for her kitchen counter. The counter is shaped like a trapezoid with width 18.5 inches and lengths 62 and 50 inches. What is the area of the counter?	80. Elena is knitting a scarf. The scarf will be shaped like a trapezoid with width 8 inches and lengths 48.2 inches and 56.2 inches. What is the area of the scarf?

Everyday Math

81. **Fence** Jose just removed the children’s playset from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep out the dog. He has a 50 foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other side if he wants to use the entire roll of fence?	82. **Gardening** Lupita wants to fence in her tomato garden. The garden is rectangular and the length is twice the width. It will take 48 feet of fencing to enclose the garden. Find the length and width of her garden.
83. **Fence** Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are 6 feet, 8 feet, and 10 feet. The fence costs $10 per foot. How much will it cost for Christa to fence in her flowerbed?	84. **Painting** Caleb wants to paint one wall of his attic. The wall is shaped like a trapezoid with height 8 feet and bases 20 feet and 12 feet. The cost of the painting one square foot of wall is about ?0.05. About how much will it cost for Caleb to paint the attic wall? 

Writing Exercises

	86. If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.
87. Look at the two figures.  a) Which figure looks like it has the larger area? Which looks like it has the larger perimeter? b) Now calculate the area and perimeter of each figure. Which has the larger area? Which has the larger perimeter?	88. The length of a rectangle is 5 feet more than the width. The area is 50 square feet. Find the length and the width. a) Write the equation you would use to solve the problem. b) Why can't you solve this equation with the methods you learned in the previous chapter?

Answers

1. cubic	3. square	5. linear
7. a) 10 cm b) 4 sq. cm	9. a) 8 cm b) 3 sq. cm	11. a) 10 cm b) 5 sq. cm
13. a) 260 ft b) 3825 sq. ft	15. a) 58 ft b) 210 sq. ft	17. 24 inches
19. 27 metres	21. 23 m	23. 7 in., 16 in.
25. 17 m, 12 m	27. 13.5 m, 12.8 m	29. 25 ft, 50 ft
31. 7 m, 11 m	33. 26 in.	35. 55 m
37. 35 ft, 45 ft	39. 76 in., 36 in.	41. 60 sq. in.
43. 25.315 sq. m	45. 0.75 sq. ft	47. 8 ft
49. 23 in.	51. 11 ft	53. 28 cm
55. 17 ft	57. 6 m	59. 15 ft
61. 24 in.	63. 27.5 in.	65. 12 ft, 13 ft, 14 ft
67. 3 ft, 6 ft, 8 ft	69. 144 sq. ft	71. 2805 sq. m
73. 231 sq. cm	75. 28.56 sq. m	77. 13.5 sq. ft
79. 1036 sq. in.	81. 15 ft	83. $24
85. Answers will vary.	87. Answers will vary.	

Attributions

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3.3 Solve Geometry Applications: Volume and Surface Area

Learning Objectives

By the end of this section, you will be able to:

- Find volume and surface area of rectangular solids
- Find volume and surface area of spheres
- Find volume and surface area of cylinders
- Find volume of cone

In this section, we will find the volume and surface area of some three-dimensional figures. Since we will be solving applications, we will once again show our Problem-Solving Strategy for Geometry Applications.

Problem Solving Strategy for Geometry Applications

1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Find Volume and Surface Area of Rectangular Solids

A cheer leading coach is having the squad paint wooden crates with the school colors to stand on at the games. (See Figure.1). The amount of paint needed to cover the outside of each box is the surface area, a square measure of the total area of all the sides. The amount of space inside the crate is the volume, a cubic measure.

This wooden crate is in the shape of a rectangular solid.

Figure.1

Each crate is in the shape of a rectangular solid. Its dimensions are the length, width, and height. The rectangular solid shown in Figure.2 has length 4 units, width 2 units, and height 3 units. Can you tell how many cubic units there are altogether? Let's look layer by layer.

Breaking a rectangular solid into layers makes it easier to visualize the number of cubic units it contains. This 4 by 2 by 3 rectangular solid has 24 cubic units.

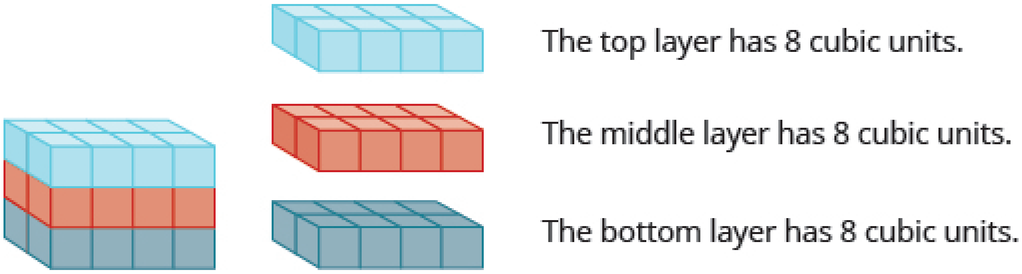


Figure.2

Altogether there are 24 cubic units. Notice that 24 is the $\text{length} \quad \times \quad \text{width} \quad \times \quad \text{height}.$

$$\underbrace{V}_{24} \;\underbrace{=}_{=}\; \underbrace{L}_{4} \;\underbrace{\cdot}_{\cdot}\; \underbrace{W}_{2} \;\underbrace{\cdot}_{\cdot}\; \underbrace{H}_{3}$$

The volume, V, of any rectangular solid is the product of the length, width, and height.

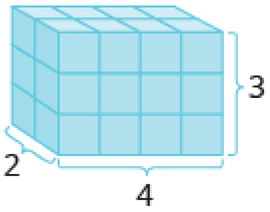
$$V = LWH$$

We could also write the formula for volume of a rectangular solid in terms of the area of the base. The area of the base, B, is equal to $\text{length} \times \text{width}.$
$\mathrm{B} = \mathrm{L} \cdot \mathrm{W}$

We can substitute B for $\mathrm{L} \cdot \mathrm{W}$ in the volume formula to get another form of the volume formula.

$$V = L \cdot W \cdot H$$
$$V = (L \cdot W) \cdot H$$
$$V = Bh$$

We now have another version of the volume formula for rectangular solids. Let’s see how this works with the $4 \times 2 \times 3$ rectangular solid we started with. See Figure.3.



$$V = Bh$$
$$V = \text{Base} \times \text{height}$$
$$V = (4 \cdot 2) \times \text{height}$$
$$V = (4 \cdot 2) \times 3$$
$$V = 8 \times 3$$
$$V = 24 \text{ cubic units}$$

Figure.3

To find the *surface area* of a rectangular solid, think about finding the area of each of its faces. How many faces does the rectangular solid above have? You can see three of them.

$A_{\text{front}} = L \times W$	$A_{\text{side}} = L \times W$	$A_{\text{top}} = L \times W$
$A_{\text{front}} = 4 \cdot 3$	$A_{\text{side}} = 2 \cdot 3$	$A_{\text{top}} = 4 \cdot 2$
$A_{\text{front}} = 12$	$A_{\text{side}} = 6$	$A_{\text{top}} = 8$

Notice for each of the three faces you see, there is an identical opposite face that does not show.

$$S = (\text{front} + \text{back}) + (\text{left side} + \text{right side}) + (\text{top} + \text{bottom})$$
$$S = (2 \cdot \text{front}) + (2 \cdot \text{left side}) + (2 \cdot \text{top})$$
$$S = 2 \cdot 12 + 2 \cdot 6 + 2 \cdot 8$$
$$S = 24 + 12 + 16$$
$$S = 52 \,\text{sq. units}$$

The surface area S of the rectangular solid shown in (Figure.3) is 52 square units.

In general, to find the surface area of a rectangular solid, remember that each face is a rectangle, so its area is the product of its length and its width (see Figure.4). Find the area of each face that you see and then multiply each area by two to account for the face on the opposite side.

$$S = 2LH + 2LW + 2WH$$

For each face of the rectangular solid facing you, there is another face on the opposite side. There are 6 faces in all.

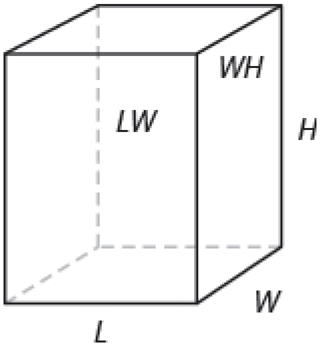
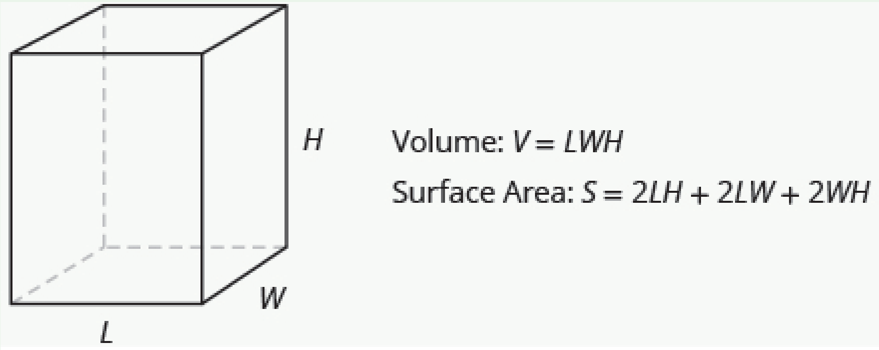


Figure.4

Volume and Surface Area of a Rectangular Solid

For a rectangular solid with length L, width W, and height H :



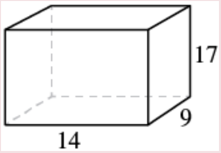
Volume: $V = LWH$

Surface Area: $S = 2LH + 2LW + 2WH$

EXAMPLE 1

For a rectangular solid with length 14 cm, height 17 cm, and width 9 cm, find the a) volume and b) surface area.

Solution
Step 1 is the same for both a) and b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the rectangular solid
Step 3. **Name.** Choose a variable to represent it.	Let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute.	$V=LWH$ $V=14\cdot 9\cdot 17$
Step 5. **Solve** the equation.	$V=2,142$
Step 6. **Check** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The surface area is $1{,}034$ square centimetres.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the solid
Step 3. **Name.** Choose a variable to represent it.	Let S= surface area
Step 4. **Translate.** Write the appropriate formula. Substitute.	$S=2LH+2LW+2WH$ $S=2\left(14\cdot 17\right)+2\left(14\cdot 9\right)+2\left(9\cdot 17\right)$
Step 5. **Solve the equation.**	$S=1,034$
Step 6. **Check:** Double-check with a calculator.	
Step 7. **Answer** the question.	The surface area is 1,034 square centimetres.

TRY IT 1.1

Find the a) volume and b) surface area of rectangular solid with the: length 8 feet, width 9 feet, and height 11 feet.

Show answer

a. 792 cu. ft
b. 518 sq. ft

TRY IT 1.2

Find the a) volume and b) surface area of rectangular solid with the: length 15 feet, width 12 feet, and height 8 feet.

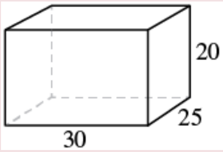
Show answer

a. 1,440 cu. ft
b. 792 sq. ft

EXAMPLE 2

A rectangular crate has a length of 30 inches, width of 25 inches, and height of 20 inches. Find its a) volume and b) surface area.

Solution
Step 1 is the same for both a) and b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the crate
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute.	$V = LWH$ $V = 30 \cdot 25 \cdot 20$
Step 5. **Solve** the equation.	$V = 15,000$
Step 6. **Check:** Double check your math.	
Step 7. **Answer** the question.	The volume is 15,000 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the crate
Step 3. **Name.** Choose a variable to represent it.	let S= surface area
Step 4. **Translate.** Write the appropriate formula. Substitute.	$S = 2LH + 2LW + 2WH$ $S = 2\left(30 \cdot 20\right) + 2\left(30 \cdot 25\right) + 2\left(25 \cdot 20\right)$
Step 5. **Solve** the equation.	$S = 3,700$
Step 6. **Check:** Check it yourself!	
Step 7. **Answer** the question.	The surface area is 3,700 square inches.

TRY IT 2.1

A rectangular box has length 9 feet, width 4 feet, and height 6 feet. Find its a) volume and b) surface area.

Show answer

a. 216 cu. ft
b. 228 sq. ft

TRY IT 2.2

A rectangular suitcase has length 22 inches, width 14 inches, and height 9 inches. Find its a) volume and b) surface area.

Show answer

a. 2,772 cu. in.
b. 1,264 sq. in.

Volume and Surface Area of a Cube

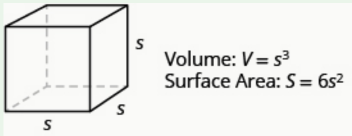
A cube is a rectangular solid whose length, width, and height are equal. See Volume and Surface Area of a Cube, below. Substituting, *s* for the length, width and height into the formulas for volume and surface area of a rectangular solid, we get:

$$\begin{array}{ll} V = LWH & S = 2LH + 2LW + 2WH \\ V = \mathrm{s} \cdot \mathrm{s} \cdot \mathrm{s} & S = 2\mathrm{s} \cdot \mathrm{s} + 2\mathrm{s} \cdot \mathrm{s} + 2\mathrm{s} \cdot \mathrm{s} \\ V = \mathrm{s}^3 & S = 2s^2 + 2s^2 + 2s^2 \\ & S = 6s^2 \end{array}$$

So for a cube, the formulas for volume and surface area are $V = s^3$ and $S = 6s^2$.

Volume and Surface Area of a Cube

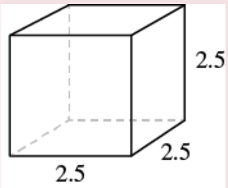
For any cube with sides of length s,



EXAMPLE 3

A cube is 2.5 inches on each side. Find its a) volume and b) surface area.

Solution
Step 1 is the same for both a) and b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the cube
Step 3. **Name.** Choose a variable to represent it.	let *V* = volume
Step 4. **Translate.** Write the appropriate formula.	$V = s^3$
Step 5. **Solve.** Substitute and solve.	$V = (2.5)^3$ $V = 15.625$
Step 6. **Check:** Check your work.	
Step 7. **Answer** the question.	The volume is 15.625 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cube
Step 3. **Name.** Choose a variable to represent it.	let *S* = surface area
Step 4. **Translate.** Write the appropriate formula.	$S=6s^2$
Step 5. **Solve.** Substitute and solve.	$S=6\cdot(2.5)^2$ $S=37.5$
Step 6. **Check:** The check is left to you.	
Step 7. **Answer** the question.	The surface area is 37.5 square inches.

TRY IT 3.1

For a cube with side 4.5 metres, find the a) volume and b) surface area of the cube.

Show answer

a. 91.125 cu. m
b. 121.5 sq. m

TRY IT 3.2

For a cube with side 7.3 yards, find the a) volume and b) surface area of the cube.

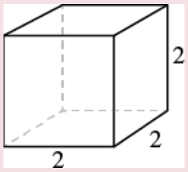
Show answer

a. 389.017 cu. yd.
b. 319.74 sq. yd.

EXAMPLE 4

A notepad cube measures 2 inches on each side. Find its a) volume and b) surface area.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the cube
Step 3. **Name.** Choose a variable to represent it.	let *V* = volume
Step 4. **Translate.** Write the appropriate formula.	$V = s^3$
Step 5. **Solve** the equation.	$V = 2^3$ $V = 8$
Step 6. **Check:** Check that you did the calculations correctly.	
Step 7. **Answer** the question.	The volume is 8 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cube
Step 3. **Name.** Choose a variable to represent it.	let *S* = surface area
Step 4. **Translate.** Write the appropriate formula.	$S = 6s^2$
Step 5. **Solve** the equation.	$S = 6 \cdot 2^2$ $S = 24$
Step 6. **Check:** The check is left to you.	
Step 7. **Answer** the question.	The surface area is 24 square inches.

TRY IT 4.1

A packing box is a cube measuring 4 feet on each side. Find its a) volume and b) surface area.

Show answer

a. 64 cu. ft

b. 96 sq. ft

TRY IT 4.2

A packing box is a cube measuring 4 feet on each side. Find its a) volume and b) surface area.

Show answer

a. 64 cu. ft
b. 96 sq. ft

Find the Volume and Surface Area of Spheres

A sphere is the shape of a basketball, like a three-dimensional circle. Just like a circle, the size of a sphere is determined by its radius, which is the distance from the centre of the sphere to any point on its surface. The formulas for the volume and surface area of a sphere are given below.

Showing where these formulas come from, like we did for a rectangular solid, is beyond the scope of this course. We will approximate π with 3.14.

Volume and Surface Area of a Sphere

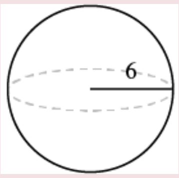
For a sphere with radius r:

Volume: $V = \frac{4}{3}\pi r^3$
Surface Area: $S = 4\pi r^2$

EXAMPLE 5

A sphere has a radius 6 inches. Find its a) volume and b) surface area.

Solution
Step 1 is the same for both a) and b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the sphere
Step 3. **Name.** Choose a variable to represent it.	let *V* = volume
Step 4. **Translate.** Write the appropriate formula.	$V = \frac{4}{3}\pi r^3$
Step 5. **Solve.**	$V \approx \frac{4}{3}(3.14)\,6^3$ $V \approx 904.32 \text{ cubic inches}$
Step 6. **Check:** Double-check your math on a calculator.	
Step 7. **Answer** the question.	The volume is approximately 904.32 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cube
Step 3. **Name.** Choose a variable to represent it.	let *S* = surface area
Step 4. **Translate.** Write the appropriate formula.	$S = 4\pi r^2$
Step 5. **Solve.**	$S \approx 4\,(3.14)\,6^2$ $S \approx 452.16$
Step 6. **Check:** Double-check your math on a calculator	
Step 7. **Answer** the question.	The surface area is approximately 452.16 square inches.

TRY IT 5.1

Find the a) volume and b) surface area of a sphere with radius 3 centimetres.

Show answer

a. 113.04 cu. cm
b. 113.04 sq. cm

TRY IT 5.2

Find the a) volume and b) surface area of each sphere with a radius of 1 foot

Show answer

a. 4.19 cu. ft

b. 12.56 sq. ft

EXAMPLE 6

A globe of Earth is in the shape of a sphere with radius 14 centimetres. Find its a) volume and b) surface area. Round the answer to the nearest hundredth.

Solution

Step 1. **Read** the problem. Draw a figure with the given information and label it.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the sphere
Step 3. **Name.** Choose a variable to represent it.	let *V* = volume
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \frac{4}{3}\pi r^3$ $V \approx \frac{4}{3}(3.14)\,14^3$
Step 5. **Solve.**	$V \approx 11,488.21$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The volume is approximately 11,488.21 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the sphere
Step 3. **Name.** Choose a variable to represent it.	let S = surface area
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$S = 4\pi r^2$ $S \approx 4\,(3.14)\,14^2$
Step 5. **Solve.**	$S \approx 2461.76$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The surface area is approximately 2461.76 square inches.

TRY IT 6.1

A beach ball is in the shape of a sphere with radius of 9 inches. Find its a) volume and b) surface area.

Show answer

a. 3052.08 cu. in.
b. 1017.36 sq. in.

TRY IT 6.2

A Roman statue depicts Atlas holding a globe with radius of 1.5 feet. Find the a) volume and b) surface area of the globe.

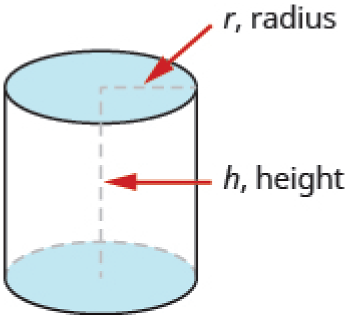
Show answer

a. 14.13 cu. ft
b. 28.26 sq. ft

Find the Volume and Surface Area of a Cylinder

If you have ever seen a can of soda, you know what a cylinder looks like. A cylinder is a solid figure with two parallel circles of the same size at the top and bottom. The top and bottom of a cylinder are called the bases. The height h of a cylinder is the distance between the two bases. For all the cylinders we will work with here, the sides and the height, h , will be perpendicular to the bases.

A cylinder has two circular bases of equal size. The height is the distance between the bases.



Rectangular solids and cylinders are somewhat similar because they both have two bases and a height. The formula for the volume of a rectangular solid, $V = Bh$, can also be used to find the volume of a cylinder.

For the rectangular solid, the area of the base, B , is the area of the rectangular base, length × width. For a cylinder, the area of the base, B, is the area of its circular base, πr^2. (Figure.5) compares how the formula $V = Bh$ is used for rectangular solids and cylinders.

Seeing how a cylinder is similar to a rectangular solid may make it easier to understand the formula for the volume of a cylinder.

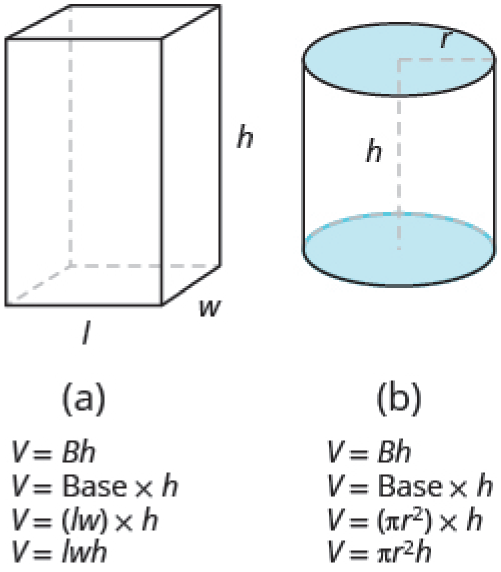


Figure.5

To understand the formula for the surface area of a cylinder, think of a can of vegetables. It has three surfaces: the top, the bottom, and the piece that forms the sides of the can. If you carefully cut the label off the side of the can and unroll it, you will see that it is a rectangle. See (Figure.6).
By cutting and unrolling the label of a can of vegetables, we can see that the surface of a cylinder is a rectangle. The length of the rectangle is the circumference of the cylinder's base, and the width is the height of the cylinder.

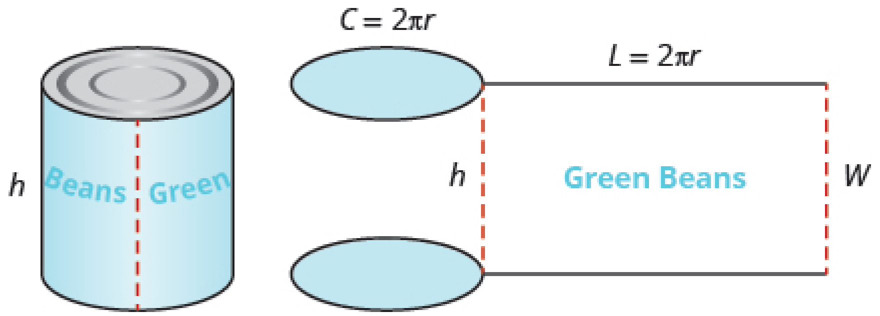
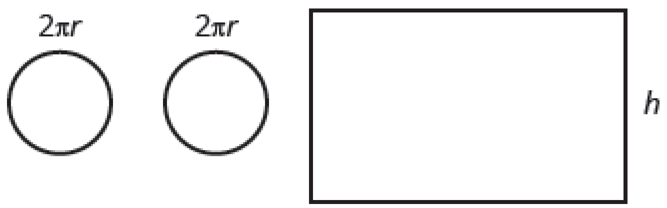


Figure.6

The distance around the edge of the can is the circumference of the cylinder’s base it is also the length L of the rectangular label. The height of the cylinder is the width W of the rectangular label. So the area of the label can be represented as

$$\begin{array}{l} A = L \cdot W \\ \downarrow\ \ \searrow \\ A = 2\pi r \cdot h \end{array}$$

To find the total surface area of the cylinder, we add the areas of the two circles to the area of the rectangle.

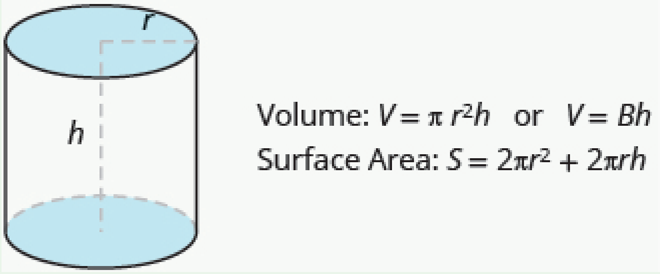


$$\begin{aligned} S &= A_{\text{top circle}} + A_{\text{bottom circle}} + A_{\text{rectangle}} \\ S &= \underbrace{\pi r^2 + \pi r^2} + 2\pi r \cdot h \\ S &= 2 \cdot \pi r^2 + 2\pi r h \\ S &= 2\pi r^2 + 2\pi r h \end{aligned}$$

The surface area of a cylinder with radius r and height h, is
$S = 2\pi r^2 + 2\pi r h$

Volume and Surface Area of a Cylinder

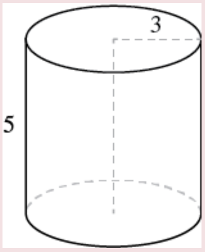
For a cylinder with radius r and height h :



EXAMPLE 7

A cylinder has height 5 centimetres and radius 3 centimetres. Find the a) volume and b) surface area.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the cylinder
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \pi r^2 h$ $V \approx (3.14)\, 3^2 \cdot 5$
Step 5. **Solve.**	$V \approx 141.3$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The volume is approximately 141.3 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cylinder
Step 3. **Name.** Choose a variable to represent it.	let S = surface area
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$S = 2\pi r^2 + 2\pi rh$ $S \approx 2\left(3.14\right)3^2 + 2\left(3.14\right)\left(3\right)5$
Step 5. **Solve.**	$S \approx 150.72$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The surface area is approximately 150.72 square inches.

TRY IT 7.1

Find the a) volume and b) surface area of the cylinder with radius 4 cm and height 7cm.

Show answer

a. 351.68 cu. cm
b. 276.32 sq. cm

TRY IT 7.2

Find the a) volume and b) surface area of the cylinder with given radius 2 ft and height 8 ft.

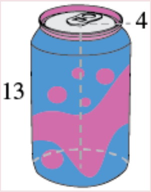
Show answer

a. 100.48 cu. ft
b. 125.6 sq. ft

EXAMPLE 8

Find the a) volume and b) surface area of a can of soda. The radius of the base is 4 centimetres and the height is 13 centimetres. Assume the can is shaped exactly like a cylinder.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the cylinder
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \pi r^2 h$ $V \approx (3.14)\,4^2 \cdot 13$
Step 5. **Solve.**	$V \approx 653.12$
Step 6. **Check:** We leave it to you to check.	
Step 7. **Answer** the question.	The volume is approximately 653.12 cubic centimetres.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cylinder
Step 3. **Name.** Choose a variable to represent it.	let S = surface area
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$S = 2\pi r^2 + 2\pi r h$ $S \approx 2\,(3.14)\,4^2 + 2\,(3.14)\,(4)\,13$
Step 5. **Solve.**	$S \approx 427.04$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The surface area is approximately 427.04 square centimetres.

TRY IT 8.1

Find the a) volume and b) surface area of a can of paint with radius 8 centimetres and height 19 centimetres. Assume the can is shaped exactly like a cylinder.

Show answer

a. 3,818.24 cu. cm

b. 1,356.48 sq. cm

TRY IT 8.2

Find the a) volume and b) surface area of a cylindrical drum with radius 2.7 feet and height 4 feet. Assume the drum is shaped exactly like a cylinder.

Show answer

a. 91.5624 cu. ft

b. 113.6052 sq. ft

Find the Volume of Cones

The first image that many of us have when we hear the word ‘cone’ is an ice cream cone. There are many other applications of cones (but most are not as tasty as ice cream cones). In this section, we will see how to find the volume of a cone.

In geometry, a cone is a solid figure with one circular base and a vertex. The height of a cone is the distance between its base and the vertex.The cones that we will look at in this section will always have the height perpendicular to the base. See (Figure.6).
The height of a cone is the distance between its base and the vertex.

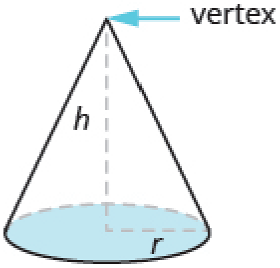


Figure.6

Earlier in this section, we saw that the volume of a cylinder is $V = \pi r^2 h$. We can think of a cone as part of a cylinder. Figure.7 shows a cone placed inside a cylinder with the same height and same base. If we compare the volume of the cone and the cylinder, we can see that the volume of the cone is less than that of the cylinder.
The volume of a cone is less than the volume of a cylinder with the same base and height.

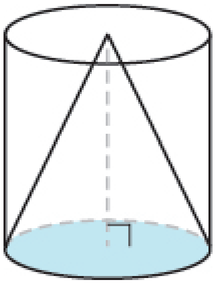


Figure.7

In fact, the volume of a cone is exactly one-third of the volume of a cylinder with the same base and height. The volume of a cone is

$$V=\frac{1}{3}Bh$$

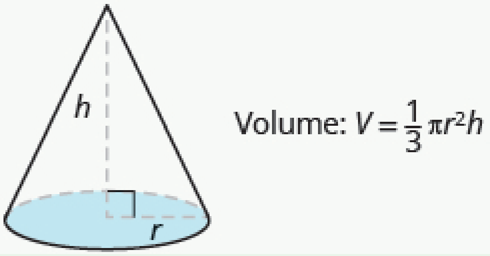
Since the base of a cone is a circle, we can substitute the formula of area of a circle, πr^2 , for B *to get the formula for volume of a cone.*

$$V=\frac{1}{3}\pi r^2 h$$

In this book, we will only find the volume of a cone, and not its surface area.

Volume of a Cone

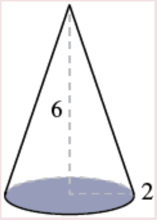
For a cone with radius r and height h.



EXAMPLE 9

Find the volume of a cone with height 6 inches and radius of its base 2 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the volume of the cone
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \frac{1}{3} \quad \pi \quad r^2 \quad h$ $V \approx \frac{1}{3} \quad 3.14 \quad (2)^2 \quad (6)$
Step 5. **Solve.**	$V \approx 25.12$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The volume is approximately 25.12 cubic inches.

TRY IT 9.1

Find the volume of a cone with height 7 inches and radius 3 inches

Show answer
65.94 cu. in.

TRY IT 9.2

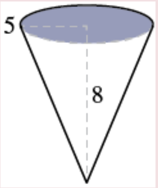
Find the volume of a cone with height 9 centimetres and radius 5 centimetres

Show answer
235.5 cu. cm

EXAMPLE 10

Marty’s favorite gastro pub serves french fries in a paper wrap shaped like a cone. What is the volume of a conic wrap that is 8 inches tall and 5 inches in diametre? Round the answer to the nearest hundredth.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information. Notice here that the base is the circle at the top of the cone.	
Step 2. **Identify** what you are looking for.	the volume of the cone
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π, and notice that we were given the distance across the circle, which is its diameter. The radius is 2.5 inches.)	$V = \frac{1}{3} \quad \pi \quad r^2 \quad h$ $V \approx \frac{1}{3} \quad 3.14 \quad (2.5)^2 \quad (8)$
Step 5. **Solve.**	$V \approx 52.33$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The volume of the wrap is approximately 52.33 cubic inches.

TRY IT 10.1

How many cubic inches of candy will fit in a cone-shaped piñata that is 18 inches long and 12 inches across its base? Round the answer to the nearest hundredth.

Show answer
678.24 cu. in.

TRY IT 10.2

What is the volume of a cone-shaped party hat that is 10 inches tall and 7 inches across at the base? Round the answer to the nearest hundredth.

Show answer
128.2 cu. in.

ACCESS ADDITIONAL ONLINE RESOURCES

- Volume of a Cone

Key Concepts

- **Volume and Surface Area of a Rectangular Solid**
    - $V = LWH$
    - $S = 2LH + 2LW + 2WH$
- **Volume and Surface Area of a Cube**
    - $V = s^3$
    - $S = 6s^2$
- **Volume and Surface Area of a Sphere**
    - $V = \frac{4}{3}\pi r^3$
    - $S = 4\pi r^2$
- **Volume and Surface Area of a Cylinder**
    - $V = \pi r^2 h$
    - $S = 2\pi r^2 + 2\pi rh$
- **Volume of a Cone**
    - For a cone with radius r and height h:
      Volume: $V = \frac{1}{3}\pi r^2 h$

Glossary

cone
A cone is a solid figure with one circular base and a vertex.
cube
A cube is a rectangular solid whose length, width, and height are equal.
cylinder
A cylinder is a solid figure with two parallel circles of the same size at the top and bottom.

Practice Makes Perfect

Find Volume and Surface Area of Rectangular Solids

In the following exercises, find a) the volume and b) the surface area of the rectangular solid with the given dimensions.

1. length 2 metres, width 1.5 metres, height 3 metres	2. length 5 feet, width 8 feet, height 2.5 feet
3. length 3.5 yards, width 2.1 yards, height 2.4 yards	4. length 8.8 centimetres, width 6.5 centimetres, height 4.2 centimetres

In the following exercises, solve.

5. **Moving van** A rectangular moving van has length 16 feet, width 8 feet, and height 8 feet. Find its a) volume and b) surface area.	6. **Gift box** A rectangular gift box has length 26 inches, width 16 inches, and height 4 inches. Find its a) volume and b) surface area.
7. **Carton** A rectangular carton has length 21.3 cm, width 24.2 cm, and height 6.5 cm. Find its a) volume and b) surface area.	8.**Shipping container** A rectangular shipping container has length 22.8 feet, width 8.5 feet, and height 8.2 feet. Find its a) volume and b) surface area.

In the following exercises, find a) the volume and b) the surface area of the cube with the given side length.

9. 5 centimetres	10. 6 inches
11. 10.4 feet	12. 12.5 metres

In the following exercises, solve.

13. **Science center** Each side of the cube at the Discovery Science Center in Santa Ana is 64 feet long. Find its a) volume and b) surface area.	14. **Museum** A cube-shaped museum has sides 45 metres long. Find its a) volume and b) surface area.
15. **Base of statue** The base of a statue is a cube with sides 2.8 metres long. Find its a) volume and b) surface area.	16. **Tissue box** A box of tissues is a cube with sides 4.5 inches long. Find its a) volume and b) surface area.

Find the Volume and Surface Area of Spheres

In the following exercises, find a) the volume and b) the surface area of the sphere with the given radius. Round answers to the nearest hundredth.

17. 3 centimetres	18. 9 inches
19. 7.5 feet	20. 2.1 yards

In the following exercises, solve. Round answers to the nearest hundredth.

21. **Exercise ball** An exercise ball has a radius of 15 inches. Find its a) volume and b) surface area.	22. **Balloon ride** The Great Park Balloon is a big orange sphere with a radius of 36 feet . Find its a) volume and b) surface area.
23. **Golf ball** A golf ball has a radius of 4.5 centimetres. Find its a) volume and b) surface area.	24. **Baseball** A baseball has a radius of 2.9 inches. Find its a) volume and b) surface area.

Find the Volume and Surface Area of a Cylinder

In the following exercises, find a) the volume and b) the surface area of the cylinder with the given radius and height. Round answers to the nearest hundredth.

25. radius 3 feet, height 9 feet	26. radius 5 centimetres, height 15 centimetres
27. radius 1.5 metres, height 4.2 metres	28. radius 1.3 yards, height 2.8 yards

In the following exercises, solve. Round answers to the nearest hundredth.

29. **Coffee can** A can of coffee has a radius of 5 cm and a height of 13 cm. Find its a) volume and b) surface area.	30. **Snack pack** A snack pack of cookies is shaped like a cylinder with radius 4 cm and height 3 cm. Find its a) volume and b) surface area.
31. **Barber shop pole** A cylindrical barber shop pole has a diametre of 6 inches and height of 24 inches. Find its a) volume and b) surface area.	32. **Architecture** A cylindrical column has a diametre of 8 feet and a height of 28 feet. Find its a) volume and b) surface area.

Find the Volume of Cones

In the following exercises, find the volume of the cone with the given dimensions. Round answers to the nearest hundredth.

33. height 9 feet and radius 2 feet	34. height 8 inches and radius 6 inches
35. height 12.4 centimetres and radius 5 cm	36. height 15.2 metres and radius 4 metres

In the following exercises, solve. Round answers to the nearest hundredth.

37. **Teepee** What is the volume of a cone-shaped teepee tent that is 10 feet tall and 10 feet across at the base?	38. **Popcorn cup** What is the volume of a cone-shaped popcorn cup that is 8 inches tall and 6 inches across at the base?
39. **Silo** What is the volume of a cone-shaped silo that is 50 feet tall and 70 feet across at the base?	40. **Sand pile** What is the volume of a cone-shaped pile of sand that is 12 metres tall and 30 metres across at the base?

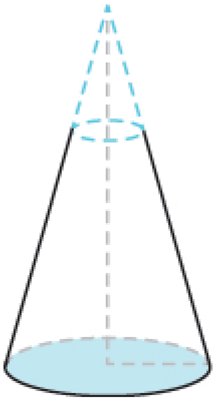
Everyday Math

41. **Street light post** The post of a street light is shaped like a truncated cone, as shown in the picture below. It is a large cone minus a smaller top cone. The large cone is 30 feet tall with base radius 1 foot. The smaller cone is 10 feet tall with base radius of 0.5 feet. To the nearest tenth,

a) find the volume of the large cone.

b) find the volume of the small cone.

c) find the volume of the post by subtracting the volume of the small cone from the volume of the large cone.



42. **Ice cream cones** A regular ice cream cone is 4 inches tall and has a diametre of 2.5 inches. A waffle cone is 7 inches tall and has a diametre of 3.25 inches. To the nearest hundredth,

a) find the volume of the regular ice cream cone.

b) find the volume of the waffle cone.

c) how much more ice cream fits in the waffle cone compared to the regular cone?

Writing Exercises

43. The formulas for the volume of a cylinder and a cone are similar. Explain how you can remember which formula goes with which shape.

44. Which has a larger volume, a cube of sides of 8 feet or a sphere with a diametre of 8 feet? Explain your reasoning.

Answers

1. a) 9 cu. m b) 27 sq. m	3. a) 17.64 cu. yd. b) 41.58 sq. yd.	5. a) 1,024 cu. ft b) 640 sq. ft
7. a) 3,350.49 cu. cm b) 1,622.42 sq. cm	9. a) 125 cu. cm b) 150 sq. cm	11. a) 1124.864 cu. ft. b) 648.96 sq. ft
13. a) 262,144 cu. ft b) 24,576 sq. ft	15. a) 21.952 cu. m b) 47.04 sq. m	17. a) 113.04 cu. cm b) 113.04 sq. cm
19. a) 1,766.25 cu. ft b) 706.5 sq. ft	21. a) 14,130 cu. in. b) 2,826 sq. in.	23. a) 381.51 cu. cm b) 254.34 sq. cm
25. a) 254.34 cu. ft b) 226.08 sq. ft	27. a) 29.673 cu. m b) 53.694 sq. m	29. a) 1,020.5 cu. cm b) 565.2 sq. cm
31. a) 678.24 cu. in. b) 508.68 sq. in.	33. 37.68 cu. ft	35. 324.47 cu. cm
37. 261.67 cu. ft	39. 64,108.33 cu. ft	41. a) 31.4 cu. ft b) 2.6 cu. ft c) 28.8 cu. ft
43. Answers will vary.		

Attributions

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3.4 Solve Geometry Applications: Circles and Irregular Figures

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of circles
- Find the area of irregular figures

In this section, we'll continue working with geometry applications. We will add several new formulas to our collection of formulas. To help you as you do the examples and exercises in this section, we will show the Problem Solving Strategy for Geometry Applications here.

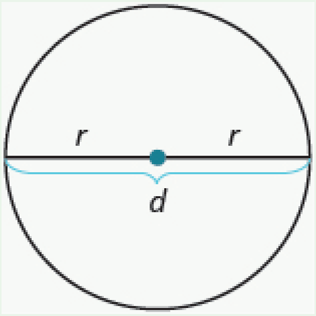
Problem Solving Strategy for Geometry Applications

1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Use the Properties of Circles

We'll refer to the properties of circles as we use them to solve applications.

Properties of Circles



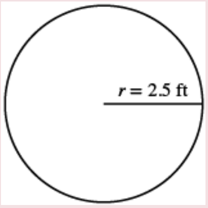
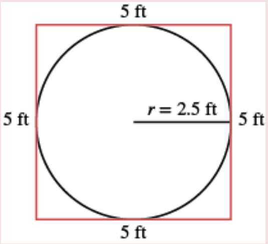
- r is the length of the radius
- d is the length of the diametre
- $d = 2r$
- Circumference is the perimeter of a circle. The formula for circumference is
  $C = 2\pi r$
- The formula for area of a circle is
  $A = \pi r^2$

Remember, that we approximate π with 3.14 or $\frac{22}{7}$ depending on whether the radius of the circle is given as a decimal or a fraction. If you use the π key on your calculator to do the calculations in this section, your answers will be slightly different from the answers shown. That is because the π key uses more than two decimal places.

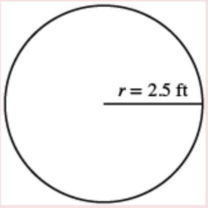
EXAMPLE 1

A circular sandbox has a radius of 2.5 feet. Find the a) circumference and b) area of the sandbox.

Solution
a)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the circumference of the circle
Step 3. **Name.** Choose a variable to represent it.	Let *c* = circumference of the circle
Step 4. **Translate.** Write the appropriate formula Substitute	$C = 2\pi r$ $C = 2\pi\left(2.5\right)$
Step 5. **Solve** the equation.	$C \approx 2\left(3.14\right)\left(2.5\right)$ $C \approx 15\text{ft}$
Step 6. **Check.** Does this answer make sense? Yes. If we draw a square around the circle, its sides would be 5 ft (twice the radius), so its perimeter would be 20 ft. This is slightly more than the circle's circumference, 15.7 ft.	
Step 7. **Answer** the question.	The circumference of the sandbox is 15.7 feet.

b)

Step 1. **Read** the problem. Draw the figure and label it with the given information	
Step 2. **Identify** what you are looking for.	the area of the circle
Step 3. **Name.** Choose a variable to represent it.	Let *A* = the area of the circle
Step 4. **Translate.** Write the appropriate formula Substitute	$A=\pi r^2$ $A=\pi(2.5)^2$
Step 5. **Solve** the equation.	$A\approx(3.14)\,(2.5)^2$ $A\approx 19.625\,\text{sq. ft}$
Step 6. **Check.** Yes. If we draw a square around the circle, its sides would be 5 ft, as shown in part a). So the area of the square would be 25 sq. ft. This is slightly more than the circle's area, 19.625 sq. ft.	
Step 7. **Answer** the question.	The area of the circle is 19.625 square feet.

TRY IT 1.1

A circular mirror has radius of 5 inches. Find the a) circumference and b) area of the mirror.

Show answer

a. 31.4 in.
b. 78.5 sq. in.

TRY IT 1.2

A circular spa has radius of 4.5 feet. Find the a) circumference and b) area of the spa.

Show answer

a. 28.26 ft
b. 63.585 sq. ft

We usually see the formula for circumference in terms of the radius r of the circle:
$C = 2\pi r$

But since the diametre of a circle is two times the radius, we could write the formula for the circumference in terms of d.

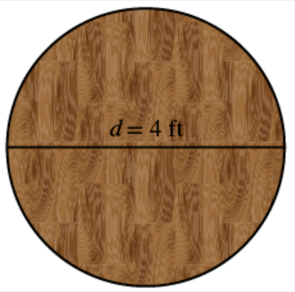
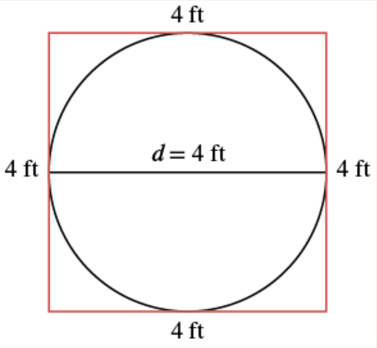
	$C = 2\pi r$
Using the commutative property, we get	$C = \pi \cdot 2r$
Then substituting $d = 2r$	$C = \pi \cdot \text{d}$
So	$C = \pi d$

We will use this form of the circumference when we're given the length of the diametre instead of the radius.

EXAMPLE 2

A circular table has a diametre of four feet. What is the circumference of the table?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the circumference of the table
Step 3. **Name.** Choose a variable to represent it.	Let *c* = the circumference of the table
Step 4. **Translate.** Write the appropriate formula for the situation. Substitute.	$C = \pi d$ $C = \pi \left(4\right)$
Step 5. **Solve** the equation, using 3.14 for π.	$C \approx \left(3.14\right)\left(4\right)$ $C \approx 12.56 \text{ feet}$
Step 6. **Check:** If we put a square around the circle, its side would be 4. The perimeter would be 16. It makes sense that the circumference of the circle, 12.56, is a little less than 16.	
Step 7. **Answer** the question.	The diametre of the table is 12.56 square feet

TRY IT 2.1

Find the circumference of a circular fire pit whose diametre is 5.5 feet.

Show answer
17.27 ft

TRY IT 2.2

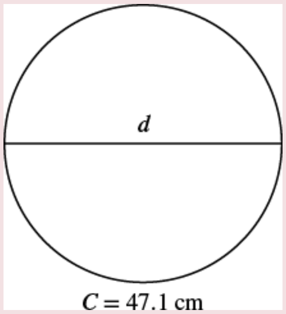
If the diametre of a circular trampoline is 12 feet, what is its circumference?

Show answer
37.68 ft

EXAMPLE 3

Find the diametre of a circle with a circumference of 47.1 centimetres.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the diametre of the circle
Step 3. **Name.** Choose a variable to represent it.	Let *d* = the diametre of the circle
Step 4. **Translate.**	
Write the formula. Substitute, using 3.14 to approximate π.	$C = \pi d$ $47.1 \approx 3.14d$
Step 5. **Solve.**	$\frac{47.1}{3.14} \approx \frac{3.14d}{3.14}$ $15 \approx d$
Step 6. **Check:** $47.1 \overset{?}{=} (3.14)(15)$ $47.1 = 47.1$?	$C = \pi d$
Step 7. **Answer the question.**	The diametre of the circle is approximately 15 centimetres.

TRY IT 3.1

Find the diametre of a circle with circumference of 94.2 centimetres.

Show answer
30 cm

TRY IT 3.2

Find the diametre of a circle with circumference of 345.4 feet.

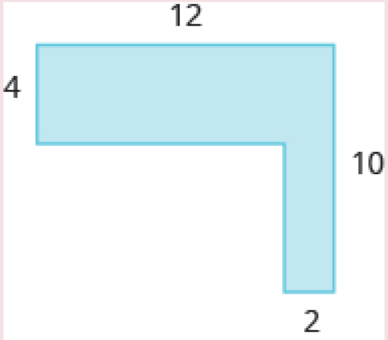
Show answer
110 ft

Find the Area of Irregular Figures

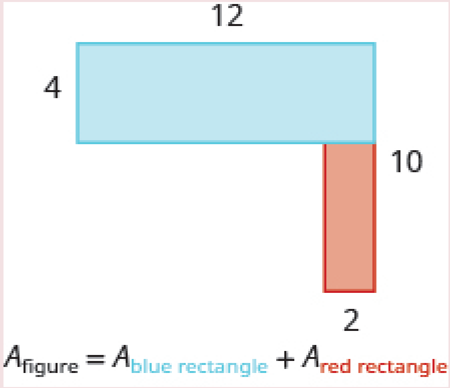
So far, we have found area for rectangles, triangles, trapezoids, and circles. An irregular figure is a figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas. But some irregular figures are made up of two or more standard geometric shapes. To find the area of one of these irregular figures, we can split it into figures whose formulas we know and then add the areas of the figures.

EXAMPLE 4

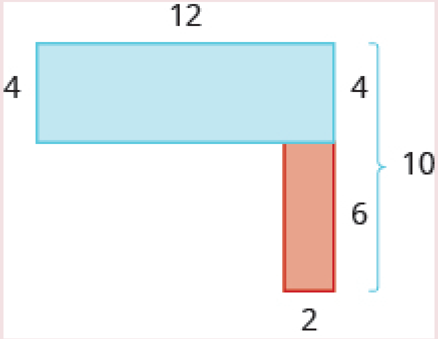
Find the area of the shaded region.



Solution
The given figure is irregular, but we can break it into two rectangles. The area of the shaded region will be the sum of the areas of both rectangles.



The blue rectangle has a width of 12 and a length of 4. The red rectangle has a width of 2, but its length is not labeled. The right side of the figure is the length of the red rectangle plus the length of the blue rectangle. Since the right side of the blue rectangle is 4 units long, the length of the red rectangle must be 6 units.



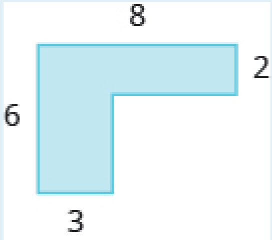
$$A_{\text{figure}} = A_{\text{rectangle}} + A_{\text{rectangle}}$$
$$A_{\text{figure}} = bh + bh$$
$$A_{\text{figure}} = 12 \cdot 4 + 2 \cdot 6$$
$$A_{\text{figure}} = 48 + 12$$
$$A_{\text{figure}} = 60$$

The area of the figure is 60 square units.

Is there another way to split this figure into two rectangles? Try it, and make sure you get the same area.

TRY IT 4.1

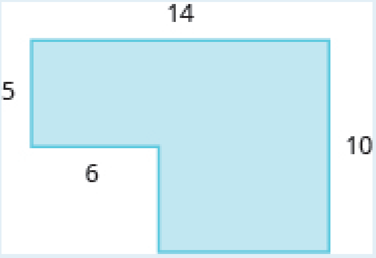
Find the area of each shaded region:



Show answer
28 sq. units

TRY IT 4.2

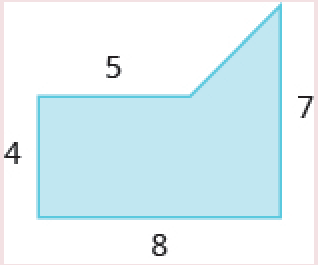
Find the area of each shaded region:



Show answer
110 sq. units

EXAMPLE 5

Find the area of the shaded region.

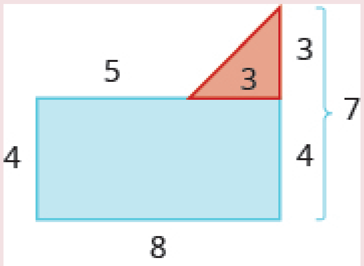


Solution
We can break this irregular figure into a triangle and rectangle. The area of the figure will be the sum of the areas of triangle and rectangle.

The rectangle has a length of 8 units and a width of 4 units.

We need to find the base and height of the triangle.

Since both sides of the rectangle are 4, the vertical side of the triangle is 3, which is $7-4$.
The length of the rectangle is 8, so the base of the triangle will be 3, which is $8-4$.



Now we can add the areas to find the area of the irregular figure.

$$A_{\text{figure}} = A_{\text{rectangle}} + A_{\text{triangle}}$$
$$A_{\text{figure}} = lw + \frac{1}{2}bh$$
$$A_{\text{figure}} = 8 \cdot 4 + \frac{1}{2} \cdot 3 \cdot 3$$
$$A_{\text{figure}} = 32 + 4.5$$
$$A_{\text{figure}} = 36.5 \text{ sq. units}$$

The area of the figure is 36.5 square units.

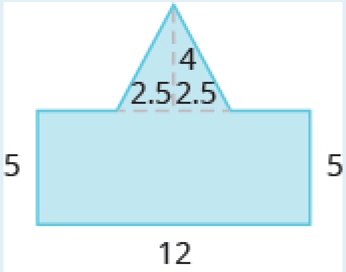
TRY IT 5.1

Find the area of each shaded region.

Show answer
36.5 sq. units

TRY IT 5.2

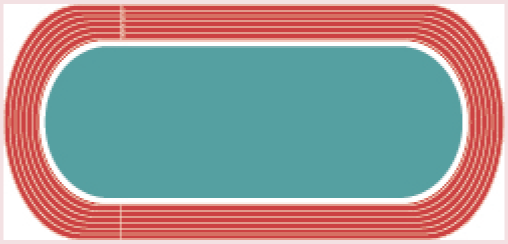
Find the area of each shaded region.



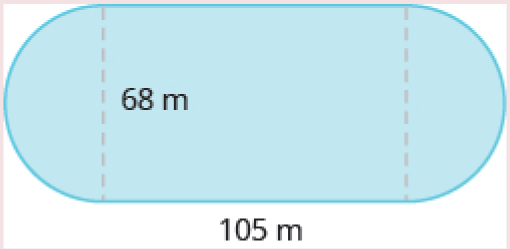
Show answer
70 sq. units

EXAMPLE 6

A high school track is shaped like a rectangle with a semi-circle (half a circle) on each end. The rectangle has length 105 metres and width 68 metres. Find the area enclosed by the track. Round your answer to the nearest hundredth.



Solution
We will break the figure into a rectangle and two semi-circles. The area of the figure will be the sum of the areas of the rectangle and the semicircles.



The rectangle has a length of 105 m and a width of 68 m. The semi-circles have a diametre of 68 m, so each has a radius of 34 m.

$$A_{\text{figure}} = A_{\text{rectangle}} + A_{\text{semicircles}}$$

$$A_{\text{figure}} = bh + 2\left(\frac{1}{2}\pi \cdot r^2\right)$$

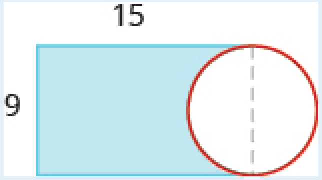
$$A_{\text{figure}} \approx 105 \cdot 68 + 2\left(\frac{1}{2} \cdot 3.14 \cdot 34^2\right)$$

$$A_{\text{figure}} \approx 7140 + 3629.84$$

$$A_{\text{figure}} \approx 10{,}769.84 \text{ square meters}$$

TRY IT 6.1

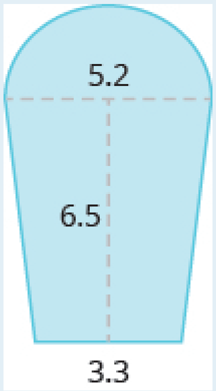
Find the area:



Show answer
103.2 sq. units

TRY IT 6.2

Find the area:



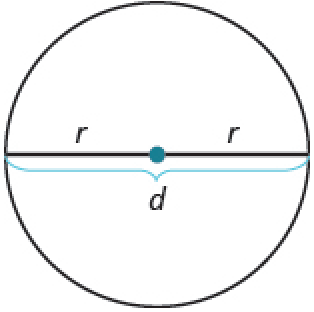
Show answer
38.24 sq. units

Access Additional Online Resources

- Circumference of a Circle
- Area of a Circle
- Area of an L-shaped polygon
- Area of an L-shaped polygon with Decimals
- Perimeter Involving a Rectangle and Circle
- Area Involving a Rectangle and Circle

Key Concepts

- **Problem Solving Strategy for Geometry Applications**
    1. Read the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
    2. Identify what you are looking for.
    3. Name what you are looking for. Choose a variable to represent that quantity.
    4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
    5. Solve the equation using good algebra techniques.
    6. Check the answer in the problem and make sure it makes sense.
    7. Answer the question with a complete sentence.
- **Properties of Circles**



- $d = 2r$
- **Circumference:** $C = 2\pi r$ or $C = \pi d$
- **Area:** $A = \pi r^2$

Glossary

irregular figure

An irregular figure is a figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas.

Practice Makes Perfect

Use the Properties of Circles

In the following exercises, solve using the properties of circles.

1. The lid of a paint bucket is a circle with radius 7 inches. Find the a) circumference and b) area of the lid.	2. An extra-large pizza is a circle with radius 8 inches. Find the a) circumference and b) area of the pizza.
3. A farm sprinkler spreads water in a circle with radius of 8.5 feet. Find the a) circumference and b) area of the watered circle.	4. A circular rug has radius of 3.5 feet. Find the a) circumference and b) area of the rug.
5. A reflecting pool is in the shape of a circle with diametre of 20 feet. What is the circumference of the pool?	6. A turntable is a circle with diametre of 10 inches. What is the circumference of the turntable?
7. A circular saw has a diametre of 12 inches. What is the circumference of the saw?	8. A round coin has a diametre of 3 centimetres. What is the circumference of the coin?
9. A barbecue grill is a circle with a diametre of 2.2 feet. What is the circumference of the grill?	10. The top of a pie tin is a circle with a diametre of 9.5 inches. What is the circumference of the top?
11. A circle has a circumference of 163.28 inches. Find the diametre.	12. A circle has a circumference of 59.66 feet. Find the diametre.
13. A circle has a circumference of 17.27 metres. Find the diametre.	14. A circle has a circumference of 80.07 centimetres. Find the diametre.

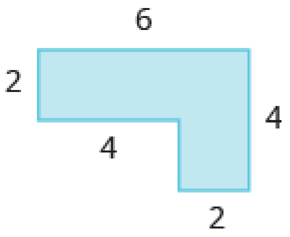
In the following exercises, find the radius of the circle with given circumference.

15. A circle has a circumference of 150.72 feet.	16. A circle has a circumference of 251.2 centimetres.
17. A circle has a circumference of 40.82 miles.	18. A circle has a circumference of 78.5 inches.

Find the Area of Irregular Figures

In the following exercises, find the area of the irregular figure. Round your answers to the nearest hundredth.

19.



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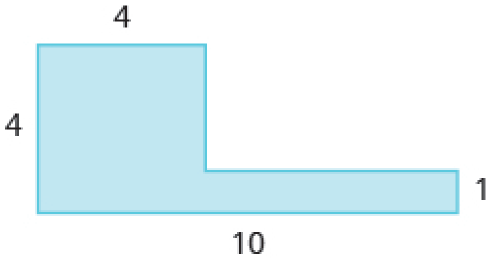

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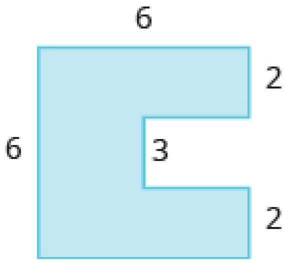

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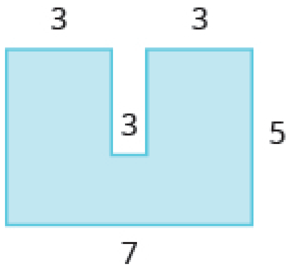

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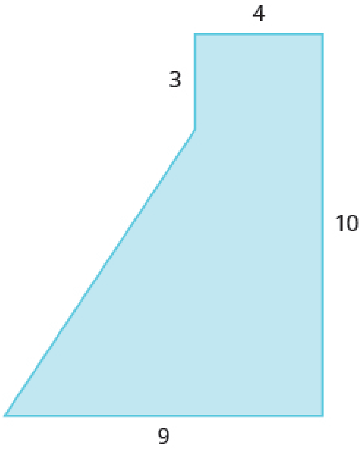

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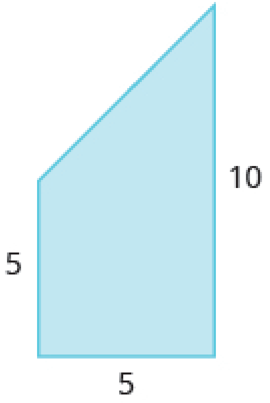

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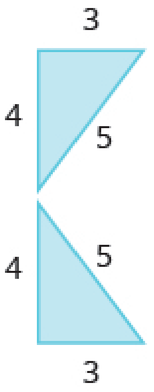

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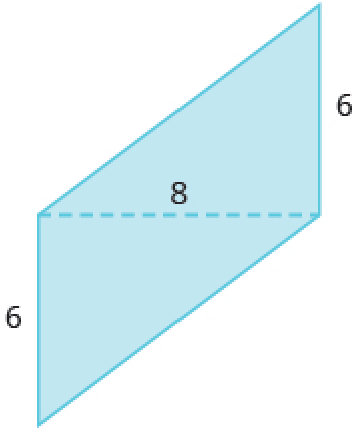

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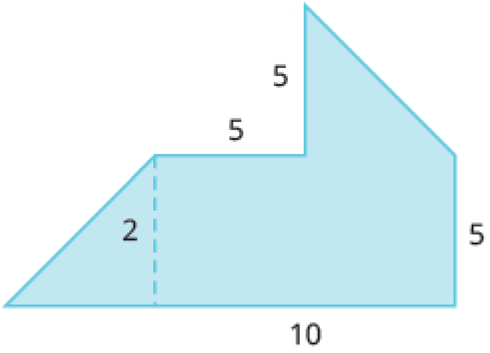

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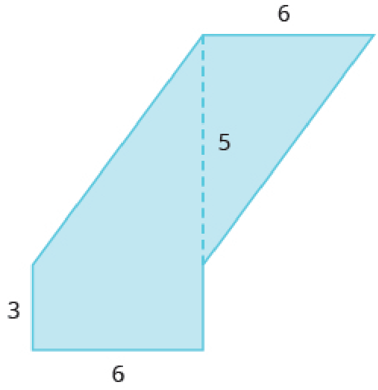

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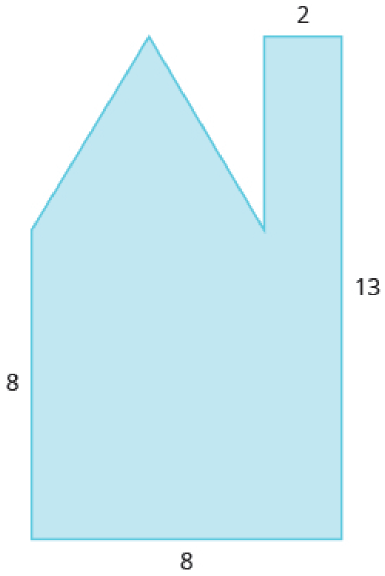

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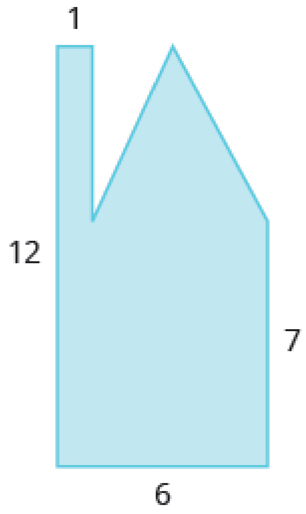

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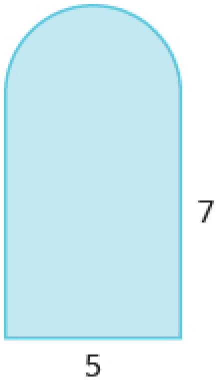

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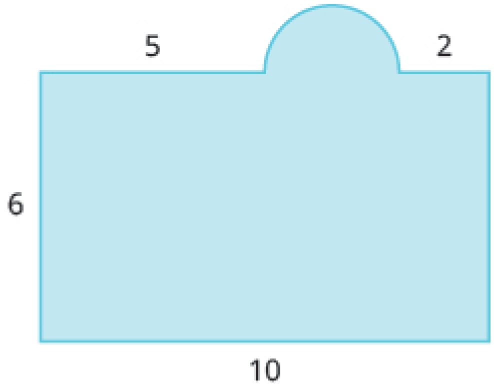

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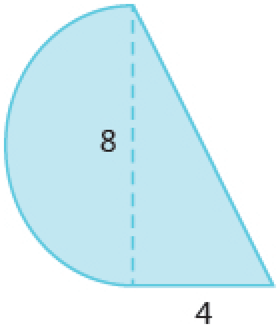

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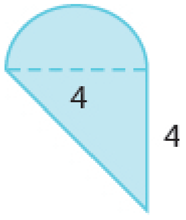

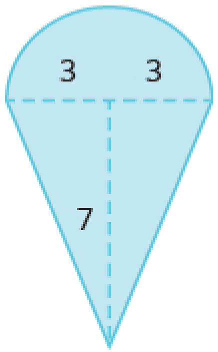

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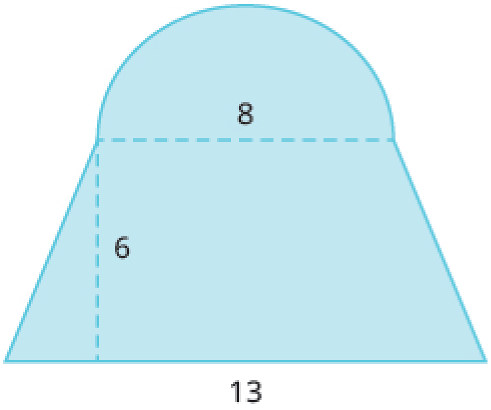

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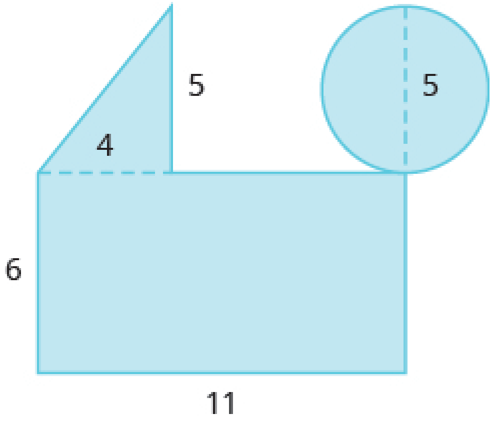

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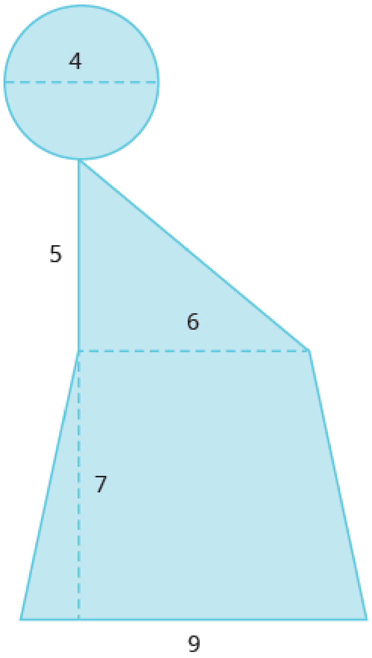
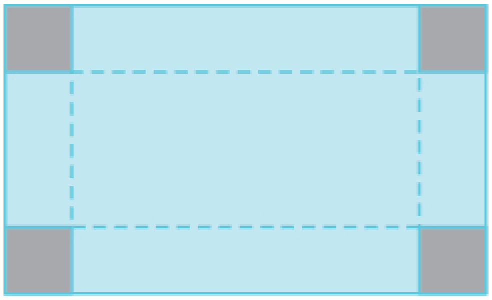


38.

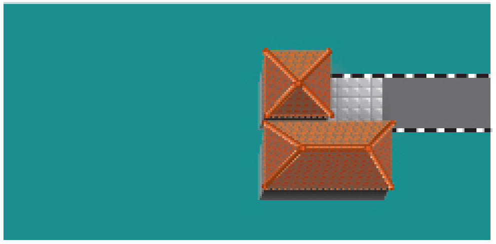
In the following exercises, solve.

39. A city park covers one block plus parts of four more blocks, as shown. The block is a square with sides 250 feet long, and the triangles are isosceles right triangles. Find the area of the park.

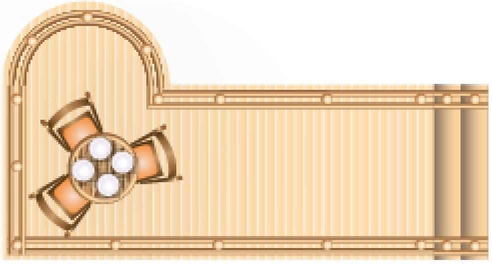
40. A gift box will be made from a rectangular piece of cardboard measuring 12 inches by 20 inches, with squares cut out of the corners of the sides, as shown. The sides of the squares are 3 inches. Find the area of the cardboard after the corners are cut out.



41. Perry needs to put in a new lawn. His lot is a rectangle with a length of 120 feet and a width of 100 feet. The house is rectangular and measures 50 feet by 40 feet. His driveway is rectangular and measures 20 feet by 30 feet, as shown. Find the area of Perry’s lawn.



42. Denise is planning to put a deck in her back yard. The deck will be a 20-ft by 12-ft rectangle with a semicircle of diametre 6 feet, as shown below. Find the area of the deck.



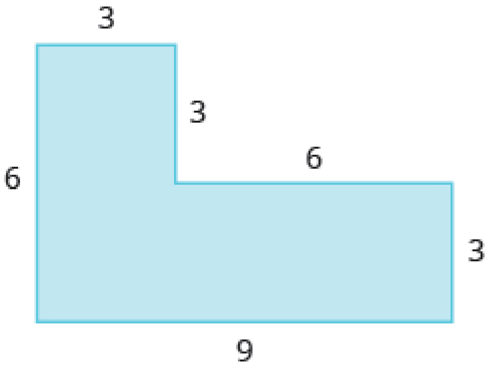
Everyday Math

43. **Area of a Tabletop** Yuki bought a drop-leaf kitchen table. The rectangular part of the table is a 1-ft by 3-ft rectangle with a semicircle at each end, as shown. a) Find the area of the table with one leaf up. b) Find the area of the table with both leaves up.

44. **Painting** Leora wants to paint the nursery in her house. The nursery is an 8-ft by 10-ft rectangle, and the ceiling is 8 feet tall. There is a 3-ft by 6.5-ft door on one wall, a 3-ft by 6.5-ft closet door on another wall, and one 4-ft by 3.5-ft window on the third wall. The fourth wall has no doors or windows. If she will only paint the four walls, and not the ceiling or doors, how many square feet will she need to paint?

Writing Exercises

45. Describe two different ways to find the area of this figure, and then show your work to make sure both ways give the same area.



46. A circle has a diametre of 14 feet. Find the area of the circle a) using 3.14 for π b) using $\frac{22}{7}$ for π. c) Which calculation to do prefer? Why?

Answers

1. a) 43.96 in. b) 153.86 sq. in.	3. a) 53.38 ft b) 226.865 sq. ft	5. 62.8 ft
7. 37.68 in.	9. 6.908 ft	11. 52 in.
13. 5.5 m	15. 24 ft	17. 6.5 mi
19. 16 sq. units	21. 30 sq. units	23. 57.5 sq. units
25. 12 sq. units	27. 67.5 sq. units	29. 89 sq. units
31. 44.81 sq. units	33. 41.12 sq. units	35. 35.13 sq. units
37. 95.625 sq. units	39. 187,500 sq. ft	41. 9400 sq. ft
43. a) 6.5325 sq. ft b) 10.065 sq. ft	45. Answers will vary.	

Attributions

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3.5 Chapter Review

Review Exercises

Systems of Measurement

In the following exercises, convert between Imperial units. Round to the nearest tenth.

1. A picture frame is 42 inches wide. Convert the width to feet.	2. A floral arbor is 7 feet tall. Convert the height to inches.
3. A playground is 45 feet wide. Convert the width to yards.	4. Kelly is 5 feet 4 inches tall. Convert her height to inches.
5. An orca whale in the Salish Sea weighs 4.5 tons. Convert the weight to pounds.	6. The height of Mount Shasta is $14,179$ feet. Convert the height to miles.
7. How many tablespoons are in a quart?	8. The play lasted $1\frac{3}{4}$ hours. Convert the time to minutes.
9. Trinh needs 30 cups of paint for her class art project. Convert the volume to gallons.	10. Naomi's baby weighed 5 pounds 14 ounces at birth. Convert the weight to ounces.

In the following exercises, solve, and state your answer in mixed units.

11. Every day last week, Pedro recorded the amount of time he spent reading. He read for $50, 25, 83, 45, 32, 60, \text{and } 135$ minutes. How much time, in hours and minutes, did Pedro spend reading?	12. John caught 4 lobsters. The weights of the lobsters were 1 pound 9 ounces, 1 pound 12 ounces, 4 pounds 2 ounces, and 2 pounds 15 ounces. What was the total weight of the lobsters?
13. Dalila wants to make pillow covers. Each cover takes 30 inches of fabric. How many yards and inches of fabric does she need for 4 pillow covers?	14. Fouad is 6 feet 2 inches tall. If he stands on a rung of a ladder 8 feet 10 inches high, how high off the ground is the top of Fouad's head?

In the following exercises, convert between metric units.

15. Mount Everest is $8,850$ metres tall. Convert the height to kilometres.	16. Donna is 1.7 metres tall. Convert her height to centimetres.
17. One cup of yogurt contains 13 grams of protein. Convert this to milligrams.	18. One cup of yogurt contains 488 milligrams of calcium. Convert this to grams.
19. A bottle of water contained 650 millilitre s. Convert this to litres.	20. Sergio weighed 2.9 kilograms at birth. Convert this to grams.

In the following exercises, solve.

21. Selma had a 1-liter bottle of water. If she drank 145 millilitres, how much water, in millilitres, was left in the bottle?	22. Minh is 2 metres tall. His daughter is 88 centimetres tall. How much taller, in metres, is Minh than his daughter?
23. One ounce of tofu provides 2 grams of protein. How many milligrams of protein are provided by 5 ounces of tofu?	24. One serving of cranberry juice contains 30 grams of sugar. How many kilograms of sugar are in 30 servings of cranberry juice?

In the following exercises, convert between Imperial and metric units. Round to the nearest tenth.

25. A college basketball court is 84 feet long. Convert this length to metres.	26. Majid is 69 inches tall. Convert his height to centimetres.
27. Lucas weighs 78 kilograms. Convert his weight to pounds.	28. Caroline walked 2.5 kilometres. Convert this length to miles.
29. A box of books weighs 25 pounds. Convert this weight to kilograms.	30. Steve's car holds 55 litres of gas. Convert this to gallons.

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

31. 23°F	32. 95°F
33. 64°F	34. 20°F

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

35. -5°C	36. 30°C
37. 24°C	38. -12°C

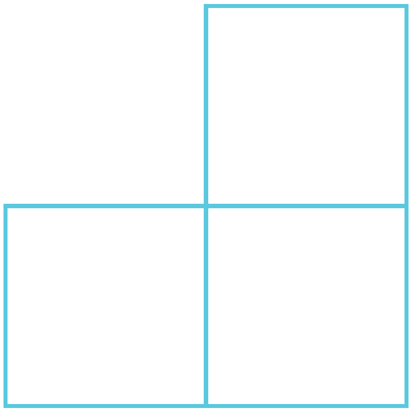
Understand Linear, Square, Cubic Measure

In the following exercises, would you measure each item using linear, square, or cubic measure?

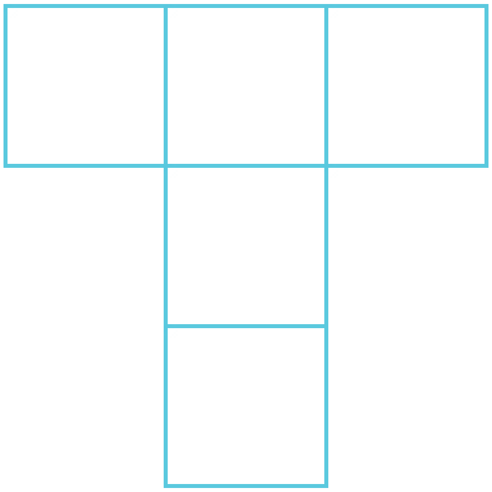
39. amount of sand in a sandbag	40. height of a tree
41. size of a patio	42. length of a highway

In the following exercises, find a) the perimeter b) the area of each figure

43.



44.



Use Properties of Rectangles

In the following exercises, find the a) perimeter b) area of each rectangle

45. The length of a rectangle is 42 metres and the width is 28 metres.

46. The length of a rectangle is 36 feet and the width is 19 feet.

47. A sidewalk in front of Kathy's house is in the shape of a rectangle 4 feet wide by 45 feet long.

48. A rectangular room is 16 feet wide by 12 feet long.

In the following exercises, solve.

49. Find the length of a rectangle with perimeter of 220 centimetres and width of 85 centimetres.

50. Find the width of a rectangle with perimeter 39 and length 11.

51. The area of a rectangle is 2356 square metres. The length is 38 metres. What is the width?

52. The width of a rectangle is 45 centimetres. The area is 2700 square centimetres. What is the length?

53. The length of a rectangle is 12 centimetres more than the width. The perimeter is 74 centimetres. Find the length and the width.

54. The width of a rectangle is 3 more than twice the length. The perimeter is 96 inches. Find the length and the width.

Use Properties of Triangles

In the following exercises, solve using the properties of triangles.

55. Find the area of a triangle with base 18 inches and height 15 inches.	56. Find the area of a triangle with base 33 centimetres and height 21 centimetres.
57. A triangular road sign has base 30 inches and height 40 inches. What is its area?	58. If a triangular courtyard has sides 9 feet and 12 feet and the perimeter is 32 feet, how long is the third side?
59. A tile in the shape of an isosceles triangle has a base of 6 inches. If the perimeter is 20 inches, find the length of each of the other sides.	60. Find the length of each side of an equilateral triangle with perimeter of 81 yards.
61. The perimeter of a triangle is 59 feet. One side of the triangle is 3 feet longer than the shortest side. The third side is 5 feet longer than the shortest side. Find the length of each side.	62. One side of a triangle is three times the smallest side. The third side is 9 feet more than the shortest side. The perimeter is 39 feet. Find the lengths of all three sides.

Use Properties of Trapezoids

In the following exercises, solve using the properties of trapezoids.

63. The height of a trapezoid is 8 feet and the bases are 11 and 14 feet. What is the area?	64. The height of a trapezoid is 5 yards and the bases are 7 and 10 yards. What is the area?
65. Find the area of the trapezoid with height 25 metres and bases 32.5 and 21.5 metres.	66. A flag is shaped like a trapezoid with height 62 centimetres and the bases are 91.5 and 78.1 centimetres. What is the area of the flag?

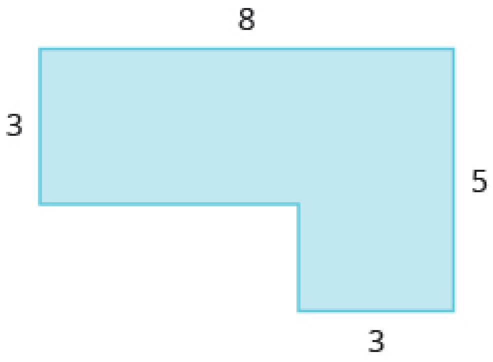
Use Properties of Circles

In the following exercises, solve using the properties of circles. Round answers to the nearest hundredth.

67. A circular mosaic has radius 3 metres. Find the a) circumference b) area of the mosaic	68. A circular fountain has radius 8 feet. Find the a) circumference b) area of the fountain
69. Find the diametre of a circle with circumference 150.72 inches.	70. Find the radius of a circle with circumference 345.4 centimetres

Find the Area of Irregular Figures

In the following exercises, find the area of each shaded region.



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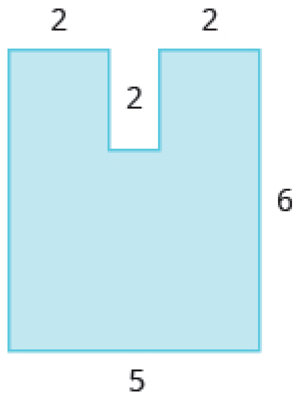

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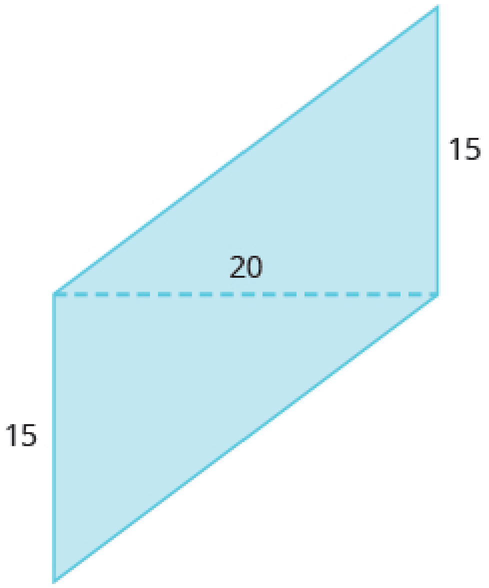

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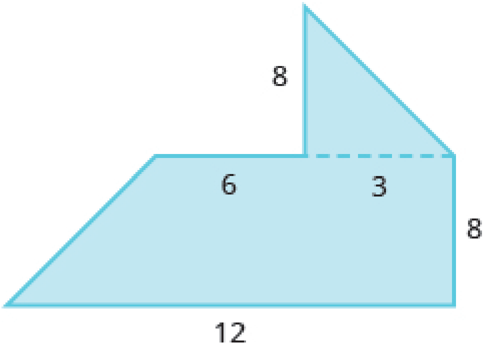

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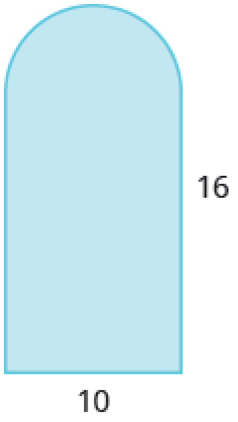

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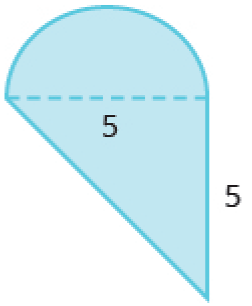


76.

Find Volume and Surface Area of Rectangular Solids

In the following exercises, find the a) volume b) surface area of the rectangular solid

77. A rectangular solid with length 14 centimetres, width 4.5 centimetres, and height 10 centimetres

78. A cube with sides that are 3 feet long

79. A cube of tofu with sides 2.5 inches

80. A rectangular carton with length 32 inches, width 18 inches, and height 10 inches

Find Volume and Surface Area of Spheres

In the following exercises, find the a) volume b) surface area of the sphere.

81. a sphere with radius 4 yards

82. a sphere with radius 12 metres

83. a baseball with radius 1.45 inches

84. a soccer ball with radius 22 centimetres

Find Volume and Surface Area of Cylinders

In the following exercises, find the a) volume b) surface area of the cylinder

85. A cylinder with radius 2 yards and height 6 yards

86. A cylinder with diametre 18 inches and height 40 inches

87. A juice can with diametre 8 centimetres and height 15 centimetres

88. A cylindrical pylon with diametre 0.8 feet and height 2.5 feet

Find Volume of Cones

In the following exercises, find the volume of the cone.

89. A cone with height 5 metres and radius 1 metre

90. A cone with height 24 feet and radius 8 feet

91. A cone-shaped water cup with diametre 2.6 inches and height 2.6 inches

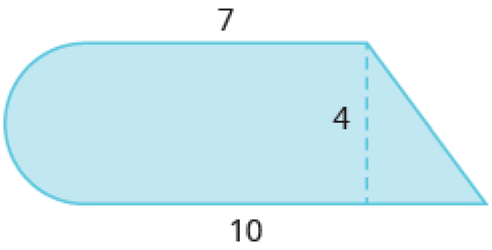
92. A cone-shaped pile of gravel with diametre 6 yards and height 5 yards

Review Answers

1. 3.5 feet	3. 15 yards	5. 9000 pounds
7. 64 tablespoons	9. 1.9 gallons	11. 7 hours 10 minutes
13. 3 yards, 12 inches	15. 8.85 kilometres	17. 13,000 milligrams
19. 0.65 litres	21. 855 millilitre s	23. 10,000 milligrams
25. 25.6 metres	27. 171.6 pounds	29. 11.4 kilograms
31. -5°C	33. 17.8°C	35. 23°F
37. 75.2°F	39. cubic	41. square
43. a) 8 units b) 3 sq. units	45. a) 140 m b) 1176 sq. m	47. a) 98 ft. b) 180 sq. ft.
49. 25 cm	51. 62 m	53. 24.5 in., 12.5 in.
55. 135 sq. in.	57. 600 sq. in.	59. 7 in., 7 in.
61. 17 ft., 20 ft., 22 ft.	63. 100 sq. ft.	65. 675 sq. m
67. a) 18.84 m b) 28.26 sq. m	69. 48 in.	71. 30 sq. units
73. 300 sq. units	75. 199.25 sq. units	77. a) 630 cu. cm b) 496 sq. cm
79. a) 15.625 cu. in. b) 37.5 sq. in.	81. a) 267.95 cu. yd. b) 200.96 sq. yd.	83. a) 12.76 cu. in. b) 26.41 sq. in.
85. a) 75.36 cu. yd. b) 100.48 sq. yd.	87. a) 753.6 cu. cm b) 477.28 sq. cm	89. 5.233 cu. m
91. 4.599 cu. in.		

Practice Test

In the following exercises, solve using the appropriate unit conversions.

1. One cup of milk contains 276 milligrams of calcium. Convert this to grams. $(1 \text{ milligram} = 0.001 \text{ gram})$	2. Azize walked $4\frac{1}{2}$ miles. Convert this distance to feet. $(1 \text{ mile} = 5{,}280 \text{ feet})$.
3. Janice ran 15 kilometres. Convert this distance to miles. Round to the nearest hundredth of a mile. $(1 \text{ mile} = 1.61 \text{ kilometres})$	4. Larry had 5 phone customer phone calls yesterday. The calls lasted 28, 44, 9, 75, and 55 minutes. How much time, in hours and minutes, did Larry spend on the phone? $(1 \text{ hour} = 60 \text{ minutes})$
5. Use the formula $F = \frac{9}{5}C + 32$ to convert 35°C to degrees F	6. Yolie is 63 inches tall. Convert her height to centimetres. Round to the nearest centimetre. $(1 \text{ inch} = 2.54 \text{ centimetres})$
7. A triangular poster has base 80 centimetres and height 55 centimetres. Find the area of the poster.	8. The length of a rectangle is 2 feet more than five times the width. The perimeter is 40 feet. Find the dimensions of the rectangle.
9. A circular pool has diametre 90 inches. What is its circumference? Round to the nearest *tenth.*	10. A trapezoid has height 14 inches and bases 20 inches and 23 inches. Find the area of the trapezoid.
11. Find the volume of a rectangular room with width 12 feet, length 15 feet, and height 8 feet.	12. Find the area of the shaded region. Round to the nearest tenth. 
13. A traffic cone has height 75 centimetres. The radius of the base is 20 centimetres. Find the volume of the cone. Round to the nearest tenth.	14. A coffee can is shaped like a cylinder with height 7 inches and radius 5 inches. Find (a) the surface area and (b) the volume of the can. Round to the nearest tenth.

Practice Test Answers

1. .276 grams	2. 23760 feet	3. 9.317 miles
4. 211 minutes, 3 hours and 31 minutes	5. 95°F	6. 160 centimetres
7. 2,200 square centimetres	8. 11 feet, 9 feet	9. 282.6 inches
10. 201 feet	11. 1,440 cubic feet	12. 10.3 square inches
13. 31,400 cubic inches	14. a) 534.1 square inches b) 1335 cubic inches	

CHAPTER 4 Ratio, Proportion, and Percent

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When you apply for a mortgage, the loan officer will compare your total debt to your total income to decide if you qualify for the loan. This comparison is called the debt-to-income ratio. A ratio compares two quantities that are measured with the same unit. If we compare a and b, the ratio is written as a to b, $\frac{a}{b}$, or a:b.

4.1 Ratios and Rate

Learning Objectives

By the end of this section, you will be able to:

- Write a ratio as a fraction
- Find unit rates
- Find unit price
- Translate phrases to expressions with fractions

Write a Ratio as a Fraction

Ratios

A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written $a \text{ to } b, \ \frac{a}{b}, \text{ or a:b.}$

In this section, we will use the fraction notation. When a ratio is written in fraction form, the fraction should be simplified. If it is an improper fraction, we do not change it to a mixed number. Because a ratio compares two quantities, we would leave a ratio as $\frac{4}{1}$ instead of simplifying it to 4 so that we can see the two parts of the ratio.

EXAMPLE 1

Write each ratio as a fraction: a)$15 \text{ to } 27$b)$45 \text{ to } 18.$

Solution

a)

	$15 \text{ to } 27$
Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{15}{27}$
Simplify the fraction.	$\frac{5}{9}$

We leave the ratio in b) as an improper fraction.

b)

	$45 \text{ to } 18$
Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{45}{18}$
Simplify.	$\frac{5}{2}$

TRY IT 1.1

Write each ratio as a fraction: a) $21 \text{ to } 56$ b) $48 \text{ to } 32$.

Show answer

a. $\frac{3}{8}$
b. $\frac{3}{2}$

TRY IT 1.2

Write each ratio as a fraction: a)$27 \text{ to } 72$ b) $51 \text{ to } 34$.

Show answer

a. $\frac{3}{8}$
b. $\frac{3}{2}$

Ratios Involving Decimals

We will often work with ratios of decimals, especially when we have ratios involving money. In these cases, we can eliminate the decimals by using the Equivalent Fractions Property to convert the ratio to a fraction with whole numbers in the numerator and denominator.

For example, consider the ratio 0.8 to 0.05. We can write it as a fraction with decimals and then multiply the numerator and denominator by 100 to eliminate the decimals.

$$\frac{0.8}{0.05}$$

$$\frac{(0.8)100}{(0.05)100}$$

$$\frac{80}{5}$$

Do you see a shortcut to find the equivalent fraction? Notice that $0.8 = \frac{8}{10}$ and $0.05 = \frac{5}{100}$. The least common denominator of $\frac{8}{10}$ and $\frac{5}{100}$ is 100. By multiplying the numerator and denominator of $\frac{0.8}{0.05}$ by 100, we 'moved' the decimal two places to the right to get the equivalent fraction with no decimals. Now that we understand the math behind the process, we can find the fraction with no decimals like this:

	
"Move" the decimal 2 places.	$\frac{80}{5}$
Simplify.	$\frac{16}{1}$

You do not have to write out every step when you multiply the numerator and denominator by powers of ten. As long as you move both decimal places the same number of places, the ratio will remain the same.

EXAMPLE 2

Write each ratio as a fraction of whole numbers:

a) 4.8 to 11.2

b) 2.7 to 0.54

Solution

a) $4.8 \text{ to } 11.2$	
Write as a fraction.	$\frac{4.8}{11.2}$
Rewrite as an equivalent fraction without decimals, by moving both decimal points 1 place to the right.	$\frac{48}{112}$
Simplify.	$\frac{3}{7}$

So $4.8 \text{ to } 11.2$ is equivalent to $\frac{3}{7}$.

b) The numerator has one decimal place and the denominator has 2. To clear both decimals we need to move the decimal 2 places to the right. $2.7 \text{ to } 0.54$	
Write as a fraction.	$\frac{2.7}{0.54}$
Move both decimals right two places.	$\frac{270}{54}$
Simplify.	$\frac{5}{1}$

So $2.7 \text{ to } 0.54$ is equivalent to $\frac{5}{1}$.

TRY IT 2.1

Write each ratio as a fraction: a) $4.6 \text{ to } 11.5$ b) $2.3 \text{ to } 0.69$.

Show answer

a. $\frac{2}{5}$
b. $\frac{10}{3}$

TRY IT 2.2

Write each ratio as a fraction: a) $3.4 \text{ to } 15.3$ b) $3.4 \text{ to } 0.68$.

Show answer

a. $\frac{2}{9}$
b. $\frac{5}{1}$

Some ratios compare two mixed numbers. Remember that to divide mixed numbers, you first rewrite them as improper fractions.

EXAMPLE 3

Write the ratio of $1\frac{1}{4}$ to $2\frac{3}{8}$ as a fraction.

Solution

	$1\frac{1}{4}$ to $2\frac{3}{8}$
Write as a fraction.	$\frac{1\frac{1}{4}}{2\frac{3}{8}}$
Convert the numerator and denominator to improper fractions.	$\frac{\frac{5}{4}}{\frac{19}{8}}$
Rewrite as a division of fractions.	$\frac{5}{4} \div \frac{19}{8}$
Invert the divisor and multiply.	$\frac{5}{4} \cdot \frac{8}{19}$
Simplify.	$\frac{10}{19}$

TRY IT 3.1

Write each ratio as a fraction: $1\frac{3}{4}$ to $2\frac{5}{8}$.

Show answer
$\frac{2}{3}$

TRY IT 3.2

Write each ratio as a fraction: $1\frac{1}{8}$ to $2\frac{3}{4}$.

Show answer
$\frac{9}{22}$

Applications of Ratios

One real-world application of ratios that affects many people involves measuring cholesterol in blood. The ratio of total cholesterol to HDL cholesterol is one way doctors assess a person's overall health. A ratio of less than 5 to 1 is considered good.

EXAMPLE 4

Hector's total cholesterol is 249 mg/dl and his HDL cholesterol is 39 mg/dl. a) Find the ratio of his total cholesterol to his HDL cholesterol. b) Assuming that a ratio less than 5 to 1 is considered good, what would you suggest to Hector?

Solution
a) First, write the words that express the ratio. We want to know the ratio of Hector's total cholesterol to his HDL cholesterol.

Write as a fraction.	$\frac{\text{total cholesterol}}{\text{HDL cholesterol}}$
Substitute the values.	$\frac{249}{39}$
Simplify.	$\frac{83}{13}$

b) Is Hector's cholesterol ratio ok? If we divide 83 by 13 we obtain approximately 6.4, so $\frac{83}{13} \approx \frac{6.4}{1}$. Hector's cholesterol ratio is high! Hector should either lower his total cholesterol or raise his HDL cholesterol.

TRY IT 4.1

Find the patient's ratio of total cholesterol to HDL cholesterol using the given information.

Total cholesterol is 185 mg/dL and HDL cholesterol is 40 mg/dL.

Show answer

$\frac{37}{8}$

TRY IT 4.2

Find the patient’s ratio of total cholesterol to HDL cholesterol using the given information.

Total cholesterol is 204 mg/dL and HDL cholesterol is 38 mg/dL.

Show answer
$\frac{102}{19}$

Ratios of Two Measurements in Different Units

To find the ratio of two measurements, we must make sure the quantities have been measured with the same unit. If the measurements are not in the same units, we must first convert them to the same units.

We know that to simplify a fraction, we divide out common factors. Similarly in a ratio of measurements, we divide out the common unit.

EXAMPLE 5

The Canadian National Building Code (CNBC) Guidelines for wheel chair ramps require a maximum vertical rise of 1 inch for every 1 foot of horizontal run. What is the ratio of the rise to the run?

Solution
In a ratio, the measurements must be in the same units. We can change feet to inches, or inches to feet. It is usually easier to convert to the smaller unit, since this avoids introducing more fractions into the problem.

Write the words that express the ratio.

	Ratio of the rise to the run
Write the ratio as a fraction.	$\frac{\text{rise}}{\text{run}}$
Substitute in the given values.	$\frac{1\text{ inch}}{1\text{ foot}}$
Convert 1 foot to inches.	$\frac{1\text{ inch}}{12\text{ inches}}$
Simplify, dividing out common factors and units.	$\frac{1}{12}$

So the ratio of rise to run is 1 to 12. This means that the ramp should rise 1 inch for every 12 inches of horizontal run to comply with the guidelines.

TRY IT 5.1

Find the ratio of the first length to the second length: 32 inches to 1 foot.

Show answer
$\frac{8}{3}$

TRY IT 5.2

Find the ratio of the first length to the second length: 1 foot to 54 inches.

Show answer
$\frac{2}{9}$

Write a Rate as a Fraction

Frequently we want to compare two different types of measurements, such as miles to gallons. To make this comparison, we use a rate. Examples of rates are 120 miles in 2 hours, 160 words in 4 minutes, and $\$5$ dollars per 64 ounces.

Rate

A rate compares two quantities of different units. A rate is usually written as a fraction.

When writing a fraction as a rate, we put the first given amount with its units in the numerator and the second amount with its units in the denominator. When rates are simplified, the units remain in the numerator and denominator.

EXAMPLE 6

Bob drove his car 525 miles in 9 hours. Write this rate as a fraction.

Solution

	$525 \text{ miles in } 9 \text{ hours}$
Write as a fraction, with 525 miles in the numerator and 9 hours in the denominator.	$\frac{525 \text{ miles}}{9 \text{ hours}}$
	$\frac{175 \text{ miles}}{3 \text{ hours}}$

So 525 miles in 9 hours is equivalent to $\frac{175 \text{ miles}}{3 \text{ hours}}$.

TRY IT 6.1

Write the rate as a fraction: 492 miles in 8 hours.

Show answer
$\frac{123 \text{ miles}}{2 \text{ hours}}$

TRY IT 6.2

Write the rate as a fraction: 242 miles in 6 hours.

Show answer
$\frac{121 \text{ miles}}{3 \text{ hours}}$

Find Unit Rates

In the last example, we calculated that Bob was driving at a rate of $\frac{175 \text{ miles}}{3 \text{ hours}}$. This tells us that every three hours, Bob will travel 175 miles. This is correct, but not very useful. We usually want the rate to reflect the number of miles in one hour. A rate that has a denominator of 1 unit is referred to as a unit rate.

Unit Rate

A unit rate is a rate with denominator of 1 unit.

Unit rates are very common in our lives. For example, when we say that we are driving at a speed of 68 miles per hour we mean that we travel 68 miles in 1 hour. We would write this rate as 68 miles/hour (read

68 miles per hour). The common abbreviation for this is 68 mph. Note that when no number is written before a unit, it is assumed to be 1.

So 68 miles/hour really means $68 \text{ miles}/1 \text{ hour}$.

Two rates we often use when driving can be written in different forms, as shown:

Example	**Rate**	**Write**	**Abbreviate**	**Read**
68 miles in 1 hour	$\frac{68 \text{ miles}}{1 \text{ hour}}$	68 miles/hour	68 mph	$68 \text{ miles per hour}$
36 miles to 1 gallon	$\frac{36 \text{ miles}}{1 \text{ gallon}}$	36 miles/gallon	36 mpg	$36 \text{ miles per gallon}$

Another example of unit rate that you may already know about is hourly pay rate. It is usually expressed as the amount of money earned for one hour of work. For example, if you are paid $\$12.50$ for each hour you work, you could write that your hourly (unit) pay rate is $\$12.50/\text{hour}$ (read $\$12.50$ per hour.)

To convert a rate to a unit rate, we divide the numerator by the denominator. This gives us a denominator of 1.

EXAMPLE 7

Anita was paid $\$384$ last week for working 32 hours. What is Anita's hourly pay rate?

Solution

Start with a rate of dollars to hours. Then divide.	$\$384 \text{ last week for } 32 \text{ hours}$
Write as a rate.	$\frac{\$384}{32 \text{ hours}}$
Divide the numerator by the denominator.	$\frac{\$12}{1 \text{ hour}}$
Rewrite as a rate.	$\$12/\text{hour}$

Anita's hourly pay rate is $\$12$ per hour.

TRY IT 7.1

Find the unit rate: $\$630$ for 35 hours.

Show answer
$18.00/hour

TRY IT 7.2

Find the unit rate: $\$684$ for 36 hours.

Show answer
$19.00/hour

EXAMPLE 8

Sven drives his car 455 miles, using 14 gallons of gasoline. How many miles per gallon does his car get?

Solution
Start with a rate of miles to gallons. Then divide.

	$455 \text{ miles to } 14 \text{ gallons of gas}$
Write as a rate.	$\frac{455 \text{ miles}}{14 \text{ gallons}}$
Divide 455 by 14 to get the unit rate.	$\frac{32.5 \text{ miles}}{1 \text{ gallon}}$

Sven's car gets 32.5 miles/gallon, or 32.5 mpg.

TRY IT 8.1

Find the unit rate: 423 miles to 18 gallons of gas.

Show answer
23.5 mpg

TRY IT 8.2

Find the unit rate: 406 miles to 14.5 gallons of gas.

Show answer
28 mpg

Find Unit Price

Sometimes we buy common household items ‘in bulk’, where several items are packaged together and sold for one price. To compare the prices of different sized packages, we need to find the unit price. To find the unit price, divide the total price by the number of items. A unit price is a unit rate for one item.

Unit price

A unit price is a unit rate that gives the price of one item.

EXAMPLE 9

The grocery store charges $\$3.99$ for a case of 24 bottles of water. What is the unit price?

Solution
What are we asked to find? We are asked to find the unit price, which is the price per bottle.

Write as a rate.	$\frac{\$3.99}{24 \text{ bottles}}$
Divide to find the unit price.	$\frac{\$0.16625}{1 \text{ bottle}}$
Round the result to the nearest penny.	$\frac{\$0.17}{1 \text{ bottle}}$

The unit price is approximately $\$0.17$ per bottle. Each bottle costs about $\$0.17$.

TRY IT 9.1

Find the unit price. Round your answer to the nearest cent if necessary.

24-pack of juice boxes for $\$6.99$

Show answer
$0.29/box

TRY IT 9.2

Find the unit price. Round your answer to the nearest cent if necessary.

24-pack of bottles of ice tea for $\$12.72$

Show answer
$0.53/bottle

Unit prices are very useful if you comparison shop. The *better buy* is the item with the lower unit price. Most grocery stores list the unit price of each item on the shelves.

EXAMPLE 10

Paul is shopping for laundry detergent. At the grocery store, the liquid detergent is priced at $\$14.99$ for 64 loads of laundry and the same brand of powder detergent is priced at $\$15.99$ for 80 loads.

Which is the better buy, the liquid or the powder detergent?

Solution
To compare the prices, we first find the unit price for each type of detergent.

	Liquid	Powder
Write as a rate.	$\frac{\$14.99}{64\text{ loads}}$	$\frac{\$15.99}{80\text{ loads}}$
Find the unit price.	$\frac{\$0.234\ldots}{1\text{ load}}$	$\frac{\$0.199\ldots}{1\text{ load}}$
Round to the nearest cent.	$\$0.23/\text{load}$ $(23\text{ cents per load.})$	$\$0.20/\text{load}$ $(20\text{ cents per load})$

Now we compare the unit prices. The unit price of the liquid detergent is about $\$0.23$ per load and the unit price of the powder detergent is about $\$0.20$ per load. The powder is the better buy.

TRY IT 10.1

Find each unit price and then determine the better buy. Round to the nearest cent if necessary.

Brand A Storage Bags, $\$4.59$ for 40 count, or Brand B Storage Bags, $\$3.99$ for 30 count

Show answer
Brand A costs $0.12 per bag. Brand B costs $0.13 per bag. Brand A is the better buy.

TRY IT 10.2

Find each unit price and then determine the better buy. Round to the nearest cent if necessary.

Brand C Chicken Noodle Soup, $\$1.89$ for 26 ounces, or Brand D Chicken Noodle Soup, $\$0.95$ for 10.75 ounces

Show answer

Brand C costs \$0.07 per ounce. Brand D costs \$0.09 per ounce. Brand C is the better buy.

Notice in the above example that we rounded the unit price to the nearest cent. Sometimes we may need to carry the division to one more place to see the difference between the unit prices.

Translate Phrases to Expressions with Fractions

Have you noticed that the examples in this section used the comparison words *ratio of, to, per, in, for, on,* and *from*? When you translate phrases that include these words, you should think either ratio or rate. If the units measure the same quantity (length, time, etc.), you have a ratio. If the units are different, you have a rate. In both cases, you write a fraction.

EXAMPLE 11

Translate the word phrase into an algebraic expression:

a) 427 miles per h hours

b) x students to 3 teachers

c) y dollars for 18 hours

Solution

a)	
	$427 \text{ miles per } h \text{ hours}$
Write as a rate.	$\frac{427 \text{ miles}}{h \text{ hours}}$

b)	
	$x \text{ students to } 3 \text{ teachers}$
Write as a rate.	$\frac{x \text{ students}}{3 \text{ teachers}}$

c)	
	y dollars for 18 hours
Write as a rate.	$\frac{\$y}{18 \text{ hours}}$

TRY IT 11.1

Translate the word phrase into an algebraic expression.

a) 689 miles per h hours b) y parents to 22 students c) d dollars for 9 minutes

Show answer

a. 689 mi/*h* hours
b. *y* parents/22 students
c. $*d*/9 min

TRY IT 11.2

Translate the word phrase into an algebraic expression.

a)m miles per 9 hours b) x students to 8 buses c) y dollars for 40 hours

Show answer

a. *m* mi/9 h
b. *x* students/8 buses
c. $*y*/40 h

Access to Additional Online R

- Ratios
- Write Ratios as a Simplified Fractions Involving Decimals and Fractions
- Write a Ratio as a Simplified Fraction

- Rates and Unit Rates
- Unit Rate for Cell Phone Plan

Glossary

ratio

A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or $a : b$.

rate

A rate compares two quantities of different units. A rate is usually written as a fraction.

unit rate

A unit rate is a rate with denominator of 1 unit.

unit price

A **unit price** is a unit rate that gives the price of one item.

Practice Makes Perfect

Write a Ratio as a Fraction

In the following exercises, write each ratio as a fraction.

1. 20 to 36	2. 20 to 32
3. 42 to 48	4. 45 to 54
5. 49 to 21	6. 56 to 16
7. 84 to 36	8. 6.4 to 0.8
9. 0.56 to 2.8	10. 1.26 to 4.2
11. $1\frac{2}{3}$ to $2\frac{5}{6}$	12. $1\frac{3}{4}$ to $2\frac{5}{8}$
13. $4\frac{1}{6}$ to $3\frac{1}{3}$	14. $5\frac{3}{5}$ to $3\frac{3}{5}$
15. $\$18$ to $\$63$	16. $\$16$ to $\$72$
17. $\$1.21$ to $\$0.44$	18. $\$1.38$ to $\$0.69$
19. 28 ounces to 84 ounces	20. 32 ounces to 128 ounces
21. 12 feet to 46 feet	22. 15 feet to 57 feet
23. 246 milligrams to 45 milligrams	24. 304 milligrams to 48 milligrams
25. total cholesterol of 175 to HDL cholesterol of 45	26. total cholesterol of 215 to HDL cholesterol of 55
27. 27 inches to 1 foot	28. 28 inches to 1 foot

Write a Rate as a Fraction

In the following exercises, write each rate as a fraction.

29. 140 calories per 12 ounces	30. 180 calories per 16 ounces
31. 8.2 pounds per 3 square inches	32. 9.5 pounds per 4 square inches
33. 488 miles in 7 hours	34. 527 miles in 9 hours
35. $\$595$ for 40 hours	36. $\$798$ for 40 hours

Find Unit Rates

In the following exercises, find the unit rate. Round to two decimal places, if necessary.

37. 140 calories per 12 ounces	38. 180 calories per 16 ounces
39. 8.2 pounds per 3 square inches	40. 9.5 pounds per 4 square inches
41. 488 miles in 7 hours	42. 527 miles in 9 hours
43. $\$595$ for 40 hours	44. $\$798$ for 40 hours
45. 576 miles on 18 gallons of gas	46. 435 miles on 15 gallons of gas
47. 43 pounds in 16 weeks	48. 57 pounds in 24 weeks
49. 46 beats in 0.5 minute	50. 54 beats in 0.5 minute
51. The bindery at a printing plant assembles $96,000$ magazines in 12 hours. How many magazines are assembled in one hour?	52. The pressroom at a printing plant prints $540,000$ sections in 12 hours. How many sections are printed per hour?

Find Unit Price

In the following exercises, find the unit price. Round to the nearest cent.

53. Soap bars at 8 for $\$8.69$	54. Soap bars at 4 for $\$3.39$
55. Women's sports socks at 6 pairs for $\$7.99$	56. Men's dress socks at 3 pairs for $\$8.49$
57. Snack packs of cookies at 12 for $\$5.79$	58. Granola bars at 5 for $\$3.69$
59. CD-RW discs at 25 for $\$14.99$	60. CDs at 50 for $\$4.49$
61. The grocery store has a special on macaroni and cheese. The price is $\$3.87$ for 3 boxes. How much does each box cost?	62. The pet store has a special on cat food. The price is $\$4.32$ for 12 cans. How much does each can cost?

In the following exercises, find each unit price and then identify the better buy. Round to three decimal places.

63. Mouthwash, 50.7-ounce size for $\$6.99$ or 33.8-ounce size for $\$4.79$	64. Toothpaste, 6 ounce size for $\$3.19$ or $7.8-ounce$ size for $\$5.19$
65. Breakfast cereal, 18 ounces for $\$3.99$ or 14 ounces for $\$3.29$	66. Breakfast Cereal, 10.7 ounces for $\$2.69$ or 14.8 ounces for $\$3.69$
67. Ketchup, 40-ounce regular bottle for $\$2.99$ or 64-ounce squeeze bottle for $\$4.39$	68. Mayonnaise 15-ounce regular bottle for $\$3.49$ or 22-ounce squeeze bottle for $\$4.99$
69. Cheese $\$6.49$ for 1 lb. block or $\$3.39$ for $\frac{1}{2}$ lb. block	70. Candy $\$10.99$ for a 1 lb. bag or $\$2.89$ for $\frac{1}{4}$ lb. of loose candy

Translate Phrases to Expressions with Fractions

In the following exercises, translate the English phrase into an algebraic expression.

71. 793 miles per p hours	72. 78 feet per r seconds
73. $\$3$ for 0.5 lbs.	74. j beats in 0.5 minutes
75. 105 calories in x ounces	76. 400 minutes for m dollars
77. the ratio of y and $5x$	78. the ratio of $12x$ and y

Everyday Math

79. One elementary school in Saskatchewan has 684 students and 45 teachers. Write the student-to-teacher ratio as a unit rate.	80. The average Canadian produces about 350 pounds of paper trash per year (365 days). How many pounds of paper trash does the average Canadian produce each day? (Round to the nearest tenth of a pound.)
81. A popular fast food burger weighs 7.5 ounces and contains 540 calories, 29 grams of fat, 43 grams of carbohydrates, and 25 grams of protein. Find the unit rate of a) calories per ounce b) grams of fat per ounce c) grams of carbohydrates per ounce d) grams of protein per ounce. Round to two decimal places.	82. A $16-ounce$ chocolate mocha coffee with whipped cream contains 470 calories, 18 grams of fat, 63 grams of carbohydrates, and 15 grams of protein. Find the unit rate of a) calories per ounce b) grams of fat per ounce c) grams of carbohydrates per ounce d) grams of protein per ounce.

Writing Exercises

83. Would you prefer the ratio of your income to your friend's income to be $3/1$ or $1/3$? Explain your reasoning.	84. The parking lot at the airport charges $\$0.75$ for every 15 minutes. a) How much does it cost to park for 1 hour? b) Explain how you got your answer to part a). Was your reasoning based on the unit cost or did you use another method?
85. Kathryn ate a $4-ounce$ cup of frozen yogurt and then went for a swim. The frozen yogurt had 115 calories. Swimming burns 422 calories per hour. For how many minutes should Kathryn swim to burn off the calories in the frozen yogurt? Explain your reasoning.	86. Arjun had a $16-ounce$ cappuccino at his neighbourhood coffee shop. The cappuccino had 110 calories. If Arjun walks for one hour, he burns 246 calories. For how many minutes must Arjun walk to burn off the calories in the cappuccino? Explain your reasoning.

Answers

1. $\frac{5}{9}$	3. $\frac{7}{8}$	5. $\frac{7}{3}$
7. $\frac{7}{3}$	9. $\frac{1}{5}$	11. $\frac{10}{17}$
13. $\frac{5}{4}$	15. $\frac{2}{7}$	17. $\frac{11}{4}$
19. $\frac{1}{3}$	21. $\frac{6}{23}$	23. $\frac{82}{15}$
25. $\frac{35}{9}$	27. $\frac{9}{4}$	29. $\frac{35\text{ calories}}{3\text{ ounces}}$
31. $\frac{41\text{ lbs}}{15\text{ sq. in.}}$	33. $\frac{488\text{ miles}}{7\text{ hours}}$	35. $\frac{\$119}{8\text{ hours}}$
37. 11.67 calories/ounce	39. 2.73 lbs./sq. in.	41. 69.71 mph
43. $14.88/hour	45. 32 mpg	47. 2.69 lbs./week
49. 92 beats/minute	51. 8,000	53. $1.09/bar
55. $1.33/pair	57. $0.48/pack	59. $0.60/disc
61. $1.29/box	63. The 50.7-ounce size costs $0.138 per ounce. The 33.8-ounce size costs $0.142 per ounce. The 50.7-ounce size is the better buy.	65. The 18-ounce size costs $0.222 per ounce. The 14-ounce size costs $0.235 per ounce. The 18-ounce size is a better buy.
67. The regular bottle costs $0.075 per ounce. The squeeze bottle costs $0.069 per ounce. The squeeze bottle is a better buy.	69. The half-pound block costs $6.78/lb, so the 1-lb. block is a better buy.	71. $\frac{793\text{ miles}}{p\text{ hours}}$
73. $\frac{?3}{0.5\text{ lbs.}}$	75. $\frac{105\text{ calories}}{x\text{ ounces}}$	77. $\frac{y}{5x}$
79. 15.2 students per teacher	81. a) 72 calories/ounce b) 3.87 grams of fat/ounce c) 5.73 grams carbs/once d) 3.33 grams protein/ounce	83. Answers will vary.
85. Answers will vary.		

Attributions

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4.2 Understand Percent

Learning Objectives

By the end of this section, you will be able to:

- Use the definition of percent
- Convert percents to fractions and decimals
- Convert decimals and fractions to percents

Use the Definition of Percent

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A percent is a ratio whose denominator is 100. We use the percent symbol $\%$, to show percent.

Percent

A percent is a ratio whose denominator is 100.

According to data from the Statistics Canada, 57% of Canadian Internet users reported a cyber security incident, including being redirected to fraudulent websites that asked for personal information or getting a virus or other computer infection. This means 57 out of every 100 Canadian internet users reported cyber security incidents as (Figure 1) shows. Out of the 100 squares on the grid, 57 are shaded, which we write as the ratio $\frac{57}{100}$.

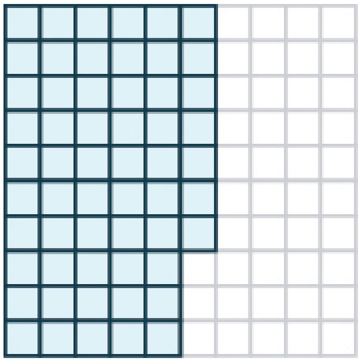


Figure 1

Similarly, 25% means a ratio of $\frac{25}{100}$, 3% means a ratio of $\frac{3}{100}$ and 100% means a ratio of $\frac{100}{100}$. In words, "one hundred percent" means the total 100% is $\frac{100}{100}$, and since $\frac{100}{100} = 1$, we see that 100% means 1 whole.

EXAMPLE 1

According to a survey done by Universities Canada (2017), 71% of Canada's Universities are working to include Indigenous representation within their governance or leadership structures.Write this percent as a ratio.

Solution

The amount we want to convert is 71%.	71%
Write the percent as a ratio. Remember that *percent* means per 100.	$\frac{71}{100}$

TRY IT 1.1

Write the percent as a ratio.

According to a survey, 89% of college students have a smartphone.

Show answer

$\frac{89}{100}$

TRY IT 1.2

Write the percent as a ratio.

A study found that 72% of Canadian teens send text messages regularly.

Show answer

$\frac{72}{100}$

EXAMPLE 2

In 2018, according to a Universities Canada survey, 56 out of every 100 of today's undergraduates benefit from experiential learning such as co-ops, internships and service learning. Write this as a ratio and then as a percent.

Solution

The amount we want to convert is 56 out of 100.	56 out of 100
Write as a ratio.	$\frac{56}{100}$
Convert the 56 per 100 to percent.	56%

TRY IT 2.1

Write as a ratio and then as a percent: According to Statistics Canada, only 10 out of 100 young Canadians cross a provincial border to complete their university degree.

Show answer

$\frac{10}{100}, 10\%$

TRY IT 2.2

Write as a ratio and then as a percent: According to an international comparison done by the British Council, 55 out of 100 current professional leaders across 30 countries and in all sectors, are liberal arts grads with bachelor's degrees in the social sciences or humanities.

Show answer

$\frac{55}{100}, 55\%$

Convert Percents to Fractions and Decimals

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means per 100, so the denominator of the fraction is 100.

Convert a Percent to a Fraction.

1. Write the percent as a ratio with the denominator 100.
2. Simplify the fraction if possible.

EXAMPLE 3

Convert each percent to a fraction:

a. 36%
b. 125%

Solution

a)	
	36%
Write as a ratio with denominator 100.	$\frac{36}{100}$
Simplify.	$\frac{9}{25}$

b)	
	125%
Write as a ratio with denominator 100.	$\frac{125}{100}$
Simplify.	$\frac{5}{4}$

TRY IT 3.1

Convert each percent to a fraction:

a. 48%
b. 110%

Show answer

a. $\frac{12}{25}$
b. $\frac{11}{10}$

TRY IT 3.2

Convert each percent to a fraction:

a. 64%
b. 150%

Show answer

a. $\frac{16}{25}$
b. $\frac{3}{2}$

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. These are improper fractions, and their values are greater than one.

EXAMPLE 4

Convert each percent to a fraction:

a. 24.5%
b. $33\frac{1}{3}\%$

Solution

a)	
	24.5%
Write as a ratio with denominator 100.	$\frac{24.5}{100}$
Clear the decimal by multiplying numerator and denominator by 10.	$\frac{24.5(10)}{100(10)}$
Multiply.	$\frac{245}{1000}$
Rewrite showing common factors.	$\frac{5 \cdot 49}{5 \cdot 200}$
Simplify.	$\frac{49}{200}$

b)	
	$33\frac{1}{3}\%$
Write as a ratio with denominator 100.	$\frac{33\frac{1}{3}}{100}$
Write the numerator as an improper fraction.	$\frac{\frac{100}{3}}{100}$
Rewrite as fraction division, replacing 100 with $\frac{100}{1}$.	$\frac{100}{3} \div \frac{100}{1}$
Multiply by the reciprocal.	$\frac{100}{3} \cdot \frac{1}{100}$
Simplify.	$\frac{1}{3}$

TRY IT 4.1

Convert each percent to a fraction:

a. 64.4%
b. $66\frac{2}{3}\%$

Show answer

a. $\frac{161}{250}$
b. $\frac{2}{3}$

TRY IT 4.2

Convert each percent to a fraction:

a. 42.5%
b. $8\frac{3}{4}\%$

Show answer

a. $\frac{113}{250}$
b. $\frac{7}{80}$

To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

HOW TO: Convert a Percent to a Decimal

1. Write the percent as a ratio with the denominator 100.
2. Convert the fraction to a decimal by dividing the numerator by the denominator.

EXAMPLE 5

Convert each percent to a decimal:

a. 6%
b. 78%

Solution
Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

a)	
	6%
Write as a ratio with denominator 100.	$\frac{6}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06

b)	
	78%
Write as a ratio with denominator 100.	$\frac{78}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.78

TRY IT 5.1

Convert each percent to a decimal:

a. 9%
b. 87%

Show answer

a. 0.09
b. 0.87

TRY IT 5.2

Convert each percent to a decimal:

a. 3%
b. 91%

Show answer

a. 0.03
b. 0.91

EXAMPLE 6

Convert each percent to a decimal:

a. 135%
b. 12.5%

Solution

a)	
	135%
Write as a ratio with denominator 100.	$\frac{135}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	1.35

b)	
	12.5%
Write as a ratio with denominator 100.	$\frac{12.5}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.125

TRY IT 6.1

Convert each percent to a decimal:

a. 115%
b. 23.5%

Show answer

a. 1.15
b. 0.235

TRY IT 6.2

Convert each percent to a decimal:

a. 123%
b. 16.8%

Show answer

a. 1.23

b. 0.168

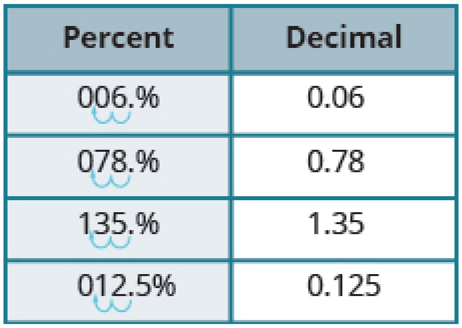
Let’s summarize the results from the previous examples in the table below, and look for a pattern we could use to quickly convert a percent number to a decimal number.

Percent	**Decimal**
6%	0.06
78%	0.78
135%	1.35
12.5%	0.125

Do you see the pattern?

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the $\%$ sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

(Figure 2) uses the percents in the table above and shows visually how to convert them to decimals by moving the decimal point two places to the left.



Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure 2

EXAMPLE 7

Among a group of business leaders, 77% believe that poor math and science education in Canada will lead to higher unemployment rates.

Convert the percent to: a) a fraction b) a decimal

Solution

a)	
	77%
Write as a ratio with denominator 100.	$\frac{77}{100}$

b)	
	$\frac{77}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.77

TRY IT 7.1

Convert the percent to: a) a fraction and b) a decimal

Twitter’s share of web traffic jumped 24% when one celebrity tweeted live on air.

Show answer

a. $\frac{6}{25}$
b. 0.24

TRY IT 7.2

Convert the percent to: a) a fraction and b) a decimal

Statistics Canada shows that in 2016, 29% of adults aged 25 to 64 had a bachelor degree.

Show answer

a. $\frac{29}{100}$
b. 0.29

EXAMPLE 8

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to:

a. a fraction
b. a decimal

(credit: Riles32807, Wikimedia Commons)

Solution

a)	
	25%
Write as a ratio with denominator 100.	$\frac{25}{100}$
Simplify.	$\frac{1}{4}$

b)	$\frac{1}{4}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

TRY IT 8.1

Convert the percent to: a) a fraction, and b) a decimal

The probability that it will rain Monday is 30%.

Show answer

a. $\frac{3}{10}$
b. 0.3

TRY IT 8.2

Convert the percent to: a) a fraction, and b) a decimal

The probability of getting heads three times when tossing a coin three times is 12.5%.

Show answer

a. $\frac{12.5}{100}$
b. 0.125

Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

HOW TO: Convert a Decimal to a Percent

1. Write the decimal as a fraction.
2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
3. Write this ratio as a percent.

EXAMPLE 9

Convert each decimal to a percent: a) 0.05 b) 0.83

Solution

a)	
	0.05
Write as a fraction. The denominator is 100.	$\frac{5}{100}$
Write this ratio as a percent.	5%

b)	
	0.83
The denominator is 100.	$\frac{83}{100}$
Write this ratio as a percent.	83%

TRY IT 9.1

Convert each decimal to a percent: a)0.01 b)0.17.

Show answer

a. 1%
b. 17%

TRY IT 9.2

Convert each decimal to a percent: a)0.04 b)0.41

Show answer

a. 4%
b. 41%

To convert a mixed number to a percent, we first write it as an improper fraction.

EXAMPLE 10

Convert each decimal to a percent: a) 1.05 b) 0.075

Solution

a)	
	0.05
Write as a fraction.	$1\frac{5}{100}$
Write as an improper fraction. The denominator is 100.	$\frac{105}{100}$
Write this ratio as a percent.	105%

Notice that since $1.05 > 1$, the result is more than 100%.

b)	
	0.075
Write as a fraction. The denominator is 1,000.	$\frac{75}{1{,}000}$
Divide the numerator and denominator by 10, so that the denominator is 100.	$\frac{7.5}{100}$
Write this ratio as a percent.	7.5%

TRY IT 10.1

Convert each decimal to a percent: a)1.75 b)0.0825

Show answer

a. 175%
b. 8.25%

TRY IT 10.2

Convert each decimal to a percent: a)2.25 b)0.0925

Show answer

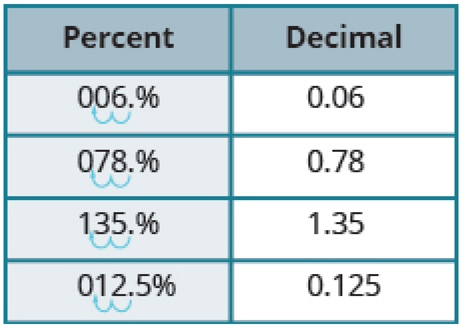
a. 225%
b. 9.25%

Let’s summarize the results from the previous examples in the table below so we can look for a pattern.

Decimal	**Percent**
0.05	5%
0.83	83%
1.05	105%
0.075	7.5%

Do you see the pattern? To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

(Figure.3) uses the decimal numbers in the table above and shows visually to convert them to percents by moving the decimal point two places to the right and then writing the $\%$ sign.



Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure. 3

Now we also know how to change decimals to percents. So to convert a fraction to a percent, we first change it to a decimal and then convert that decimal to a percent.

HOW TO: Convert a Fraction to a Percent

1. Convert the fraction to a decimal.
2. Convert the decimal to a percent.

EXAMPLE 11

Convert each fraction or mixed number to a percent: a) $\frac{3}{4}$ b) $\frac{11}{8}$ c) $2\frac{1}{5}$

Solution
To convert a fraction to a decimal, divide the numerator by the denominator.

a)	
Change to a decimal.	$\frac{3}{4}$
Write as a percent by moving the decimal two places.	 0.75
	75%

b)	
Change to a decimal.	$\frac{11}{8}$
Write as a percent by moving the decimal two places.	 1.375
	137.5%

c)	
Write as an improper fraction.	$2\frac{1}{5}$
Change to a decimal.	$\frac{11}{5}$
Write as a percent.	 2.20
	220%

Notice that we needed to add zeros at the end of the number when moving the decimal two places to the right.

TRY IT 11.1

Convert each fraction or mixed number to a percent: a) $\frac{5}{8}$ b) $\frac{11}{4}$ c) $3\frac{2}{5}$

Show answer

a. 62.5%
b. 275%
c. 340%

TRY IT 11.2

Convert each fraction or mixed number to a percent: a)$\frac{7}{8}$ b)$\frac{9}{4}$ c)$1\frac{3}{5}$

Show answer

a. 87.5%
b. 225%
c. 160%

Sometimes when changing a fraction to a decimal, the division continues for many decimal places and we will round off the quotient. The number of decimal places we round to will depend on the situation. If the decimal involves money, we round to the hundredths place. For most other cases in this book we will round the number to the nearest thousandth, so the percent will be rounded to the nearest tenth.

EXAMPLE 12

Convert $\frac{5}{7}$ to a percent.

Solution
To change a fraction to a decimal, we divide the numerator by the denominator.

	$\frac{5}{7}$
Change to a decimal—rounding to the nearest thousandth.	0.714
Write as a percent.	71.4%

TRY IT 12.1

Convert the fraction to a percent: $\frac{3}{7}$

Show answer
42.9%

TRY IT 12.2

Convert the fraction to a percent: $\frac{4}{7}$

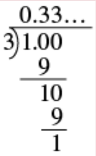
Show answer
57.1%

When we first looked at fractions and decimals, we saw that fractions converted to a repeating decimal. When we converted the fraction $\frac{4}{3}$ to a decimal, we wrote the answer as $1.\overline{3}$. We will use this same notation, as well as fraction notation, when we convert fractions to percents in the next example.

EXAMPLE 13

Statistics Canada reported in 2018 that approximately $\frac{1}{3}$ of Canadian adults are obese. Convert the fraction $\frac{1}{3}$ to a percent.

Solution

	$\frac{1}{3}$
Change to a decimal.	
Write as a repeating decimal.	$0.333\ldots$
Write as a percent.	$33\frac{1}{3}\%$

We could also write the percent as $33.\overline{3}\%$.

TRY IT 13.1

Convert the fraction to a percent:

According to the Canadian Census 2016, about $\frac{33}{50}$ people within the population of Canada are between the ages of 15 and 64.

Show answer

$66.\overline{6}\%$, or $11\frac{6}{25}\%$

TRY IT 13.2

Convert the fraction to a percent:

According to the Canadian Census 2015, about $\frac{1}{6}$ of Canadian residents under age 18 are low income.

Show answer

$16.\overline{6}\ \%$, or $16\frac{2}{3}\%$

Key Concepts

- **Convert a percent to a fraction.**
    1. Write the percent as a ratio with the denominator 100.
    2. Simplify the fraction if possible.
- **Convert a percent to a decimal.**
    1. Write the percent as a ratio with the denominator 100.
    2. Convert the fraction to a decimal by dividing the numerator by the denominator.
- **Convert a decimal to a percent.**
    1. Write the decimal as a fraction.
    2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
    3. Write this ratio as a percent.
- **Convert a fraction to a percent.**
    1. Convert the fraction to a decimal.
    2. Convert the decimal to a percent.

Glossary

percent
A percent is a ratio whose denominator is 100.

Practice Makes Perfect

Use the Definition of Percents

In the following exercises, write each percent as a ratio.

1. In 2014, the unemployment rate for those with only a high school degree was 6.0%.	2. In 2015, among the unemployed, 29% were long-term unemployed.
3. The unemployment rate for those with Bachelor's degrees was 3.2% in 2014.	4. The unemployment rate in Canada in 2019 was 13.7% . In the following exercises, write as a) a ratio and b) a percent
5. 57 out of 100 nursing candidates received their degree at a community college.	6. 80 out of 100 firefighters and law enforcement officers were educated at a community college.
7. 42 out of 100 first-time freshmen students attend a community college.	8. 71 out of 100 full-time community college faculty have a master's degree.

Convert Percents to Fractions and Decimals

In the following exercises, convert each percent to a fraction and simplify all fractions.

9. 4%	10. 8%
11. 17%	12. 19%
13. 52%	14. 78%
15. 125%	16. 135%
17. 37.5%	18. 42.5%
19. 18.4%	20. 46.4%
21. $9\frac{1}{2}\%$	22. $8\frac{1}{2}\%$
23. $5\frac{1}{3}\%$	24. $6\frac{2}{3}\%$

In the following exercises, convert each percent to a decimal.

25. 5%	26. 9%
27. 1%	28. 2%
29. 63%	30. 71%
31. 40%	32. 50%
33. 115%	34. 125%
35. 150%	36. 250%
37. 21.4%	38. 39.3%
39. 7.8%	40. 6.4%

In the following exercises, convert each percent to

a) a simplified fraction and

b) a decimal

41. In $2010, 1.5\%$ of home sales had owner financing. (Source: Bloomberg Businessweek, 5/23–29/2011)	42. In $2016, 22.3\%$ of the Canadian population was a visible minority. (Source: www12.statcan.gc.ca)
43. According to government data, in 2013 the number of cell phones in India was 70.23% of the population.	44. According to the Survey of Earned Doctorates, among Canadians age 25 or older who had doctorate degrees in $2006, 44\%$ are women.
45. A couple plans to have two children. The probability they will have two girls is 25%.	46. Javier will choose one digit at random from 0 through 9. The probability he will choose 3 is 10%.
47. According to the local weather report, the probability of thunderstorms in New York City on July 15 is 60%.	48. A club sells 50 tickets to a raffle. Osbaldo bought one ticket. The probability he will win the raffle is 2%.

Convert Decimals and Fractions to Percents

In the following exercises, convert each decimal to a percent.

49. 0.01	50. 0.03
51. 0.18	52. 0.15
53. 1.35	54. 1.56
55. 3	56. 4
57. 0.009	58. 0.008
59. 0.0875	60. 0.0625
61. 1.5	62. 2.2
63. 2.254	64. 2.317

In the following exercises, convert each fraction to a percent.

65. $\frac{1}{4}$	66. $\frac{1}{5}$
67. $\frac{3}{8}$	68. $\frac{5}{8}$
69. $\frac{7}{4}$	70. $\frac{9}{8}$
71. $6\frac{4}{5}$	72. $5\frac{1}{4}$
73. $\frac{5}{12}$	74. $\frac{11}{12}$
75. $2\frac{2}{3}$	76. $1\frac{2}{3}$
77. $\frac{3}{7}$	78. $\frac{6}{7}$
79. $\frac{5}{9}$	80. $\frac{4}{9}$

In the following exercises, convert each fraction to a percent.

81. $\frac{1}{4}$ of washing machines needed repair.	82. $\frac{1}{5}$ of dishwashers needed repair.

In the following exercises, convert each fraction to a percent.

83. According to the Government of Canada, in 2017, $\frac{16}{25}$ of Canadian adults were overweight or obese.	84. Statistics Canada showed that in 2016, 15.4% of Canadian workers are using more than one language at work.

In the following exercises, complete the table.

85.

Fraction	**Decimal**	**Percent**
$\frac{1}{2}$		
	0.45	
		18%
$\frac{1}{3}$		
	0.0008	
2		

86.

Fraction	**Decimal**	**Percent**
$\frac{1}{4}$		
	0.65	
		22%
$\frac{2}{3}$		
	0.0004	
3		

Everyday Math

87. Sales tax Felipa says she has an easy way to estimate the sales tax when she makes a purchase. The sales tax in her city is 9.05%. She knows this is a little less than 10%.

a) Convert 10% to a fraction

b) Use your answer from a) to estimate the sales tax Felipa would pay on a $\$95$ dress.

88. Savings Ryan has 25% of each paycheck automatically deposited in his savings account.

a) Write 25% as a fraction.

b) Use your answer from a) to find the amount that goes to savings from Ryan's $\$2,400$ paycheck.

Amelio is shopping for textbooks online. He found three sellers that are offering a book he needs for the same price, including shipping. To decide which seller to buy from he is comparing their customer satisfaction ratings. The ratings are given in the chart.

Seller	Rating
A	$4/5$
B	$3.5/4$
C	85%

89. Write seller C's rating as a fraction and a decimal.

90. Write seller B's rating as a percent and a decimal.

91. Write seller A's rating as a percent and a decimal.

92. Which seller should Amelio buy from and why?

Writing Exercises

93. Convert $25\%, 50\%, 75\%, \text{and } 100\%$ to fractions. Do you notice a pattern? Explain what the pattern is.	94. Convert $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}$, and $\frac{9}{10}$ to percents. Do you notice a pattern? Explain what the pattern is.
95. When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.	96. According to cnn.com, cell phone use in 2008 was 600% of what it had been in 2001. Explain what 600% means in this context.

Answers

1. $\frac{6}{100}$	3. $\frac{32}{1000}$	5. a) $\frac{57}{100}$ b) 57%
7. a) $\frac{42}{100}$ b) 42%	9. $\frac{1}{25}$	11. $\frac{17}{100}$
13. $\frac{13}{25}$	15. $\frac{5}{4}$	17. $\frac{3}{8}$
19. $\frac{23}{125}$	21. $\frac{19}{200}$	23. $\frac{4}{75}$
25. 0.05	27. 0.01	29. 0.63
31. 0.4	33. 1.15	35. 1.5
37. 0.214	39. 0.078	41. a) $\frac{3}{200}$ b) 0.015
43. a) $\frac{7023}{10{,}000}$ b) 0.7023	45. a) $\frac{1}{4}$ b) 0.25	47. a) $\frac{3}{5}$ b) 0.6
49. 1%	51. 18%	53. 135%
55. 300%	57. 0.9%	59. 8.75%
61. 150%	63. 225.4%	65. 25%
67. 37.5%	69. 175%	71. 680%
73. 41.7%	75. $266.\overline{6}\%$	77. 42.9%
79. 55.6%	81. 25%	83. 64%

85.

Fraction	**Decimal**	**Percent**
$\frac{1}{2}$	0.5	50%
$\frac{9}{20}$	0.45	45%
$\frac{9}{50}$	0.18	18%
$\frac{1}{3}$	0.33	$33\frac{1}{3}\%$
$\frac{2}{25}$	0.0008	0.08%
2	2.0	200%

87.

a) $\frac{1}{10}$

b) $\text{approximately } \$9.50$

89. $\frac{17}{20}$; 0.85

91. 80%; 0.8

93. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1.$

95. The Szetos sold their home for five times what they paid 30 years ago.

Attributions

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4.3 Solve Proportions and their Applications

Learning Objectives

By the end of this section, you will be able to:

- Use the definition of proportion
- Solve proportions
- Solve applications using proportions
- Write percent equations as proportions
- Translate and solve percent proportions

Use the Definition of Proportion

When two ratios or rates are equal, the equation relating them is called a proportion.

Proportion

A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.

The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d".

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2} = \frac{4}{8}$ is read "1 is to 2 as 4 is to 8".

If we compare quantities with units, we have to be sure we are comparing them in the right order. For example, in the proportion $\frac{20 \text{ students}}{1 \text{ teacher}} = \frac{60 \text{ students}}{3 \text{ teachers}}$ we compare the number of students to the number of teachers. We put students in the numerators and teachers in the denominators.

EXAMPLE 1

Write each sentence as a proportion:

a. 3 is to 7 as 15 is to 35.
b. 5 hits in 8 at bats is the same as 30 hits in 48 at-bats.
c. $\$1.50$ for 6 ounces is equivalent to $\$2.25$ for 9 ounces.

Solution

a)	
	3 is to 7 as 15 is to 35.
Write as a proportion.	$\frac{3}{7} = \frac{15}{35}$

b)	
	5 hits in 8 at-bats is the same as 30 hits in 48 at-bats.
Write each fraction to compare hits to at-bats.	$\frac{\text{hits}}{\text{at-bats}} = \frac{\text{hits}}{\text{at-bats}}$
Write as a proportion.	$\frac{5}{8} = \frac{30}{48}$

c)	
	\$1.50 for 6 ounces is equivalent to \$2.25 for 9 ounces.
Write each fraction to compare dollars to ounces.	$\frac{?}{\text{ounces}} = \frac{?}{\text{ounces}}$
Write as a proportion.	$\frac{1.50}{6} = \frac{2.25}{9}$

TRY IT 1.1

Write each sentence as a proportion:

a. 5 is to 9 as 20 is to 36.
b. 7 hits in 11 at-bats is the same as 28 hits in 44 at-bats.
c. $\$2.50$ for 8 ounces is equivalent to $\$3.75$ for 12 ounces.

Show answer

a. $\frac{5}{9} = \frac{20}{36}$
b. $\frac{7}{11} = \frac{28}{44}$
c. $\frac{2.50}{8} = \frac{3.75}{12}$

TRY IT 1.2

Write each sentence as a proportion:

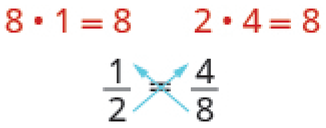
a. 6 is to 7 as 36 is to 42.
b. 8 adults for 36 children is the same as 12 adults for 54 children.
c. $\$3.75$ for 6 ounces is equivalent to $\$2.50$ for 4 ounces.

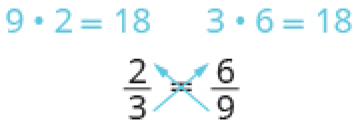
Show answer

a. $\frac{6}{7} = \frac{36}{42}$
b. $\frac{8}{36} = \frac{12}{54}$
c. $\frac{3.75}{6} = \frac{2.50}{4}$

Look at the proportions $\frac{1}{2} = \frac{4}{8}$ and $\frac{2}{3} = \frac{6}{9}$. From our work with equivalent fractions we know these equations are true. But how do we know if an equation is a proportion with equivalent fractions if it contains fractions with larger numbers?

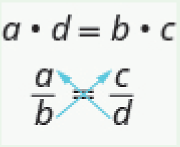
To determine if a proportion is true, we find the **cross products** of each proportion. To find the cross products, we multiply each denominator with the opposite numerator (diagonally across the equal sign). The results are called a cross products because of the cross formed. The cross products of a proportion are equal.





Cross Products of a Proportion

For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$, its cross products are equal.



Cross products can be used to test whether a proportion is true. To test whether an equation makes a proportion, we find the cross products. If they are the equal, we have a proportion.

EXAMPLE 2

Determine whether each equation is a proportion:

a. $\frac{4}{9} = \frac{12}{28}$
b. $\frac{17.5}{37.5} = \frac{7}{15}$

Solution
To determine if the equation is a proportion, we find the cross products. If they are equal, the equation is a proportion.

a)	
	$\frac{4}{9} = \frac{12}{28}$
Find the cross products.	$28 \cdot 4 = 112 \qquad 9 \cdot 12 = 108$ $\frac{4}{9} = \frac{12}{28}$

Since the cross products are not equal, $28 \cdot 4 \neq 9 \cdot 12$, the equation is not a proportion.

b)	
	$\frac{17.5}{37.5} = \frac{7}{15}$
Find the cross products.	$15 \cdot 17.5 = 262.5 \qquad 37.5 \cdot 7 = 262.5$ $\frac{17.5}{37.5} = \frac{7}{15}$

Since the cross products are equal, $15 \cdot 17.5 = 37.5 \cdot 7$, the equation is a proportion.

TRY IT 2.1

Determine whether each equation is a proportion:

a. $\frac{7}{9} = \frac{54}{72}$
b. $\frac{24.5}{45.5} = \frac{7}{13}$

Show answer

a. no
b. yes

TRY IT 2.2

Determine whether each equation is a proportion:

a. $\frac{8}{9} = \frac{56}{73}$
b. $\frac{28.5}{52.5} = \frac{8}{15}$

Show answer

a. no
b. no

Solve Proportions

To solve a proportion containing a variable, we remember that the proportion is an equation. All of the techniques we have used so far to solve equations still apply. In the next example, we will solve a proportion by multiplying by the Least Common Denominator (LCD) using the Multiplication Property of Equality.

EXAMPLE 3

Solve: $\frac{x}{63} = \frac{4}{7}$.

Solution

		$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.		$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.		$x = \frac{9 \cdot \cancel{7} \cdot 4}{\cancel{7}}$
Divide the common factors.		$x = 36$
Check: To check our answer, we substitute into the original proportion.		
	$\frac{x}{63} = \frac{4}{7}$	
Substitute $x = 36$	$\frac{36}{63} \overset{?}{=} \frac{4}{7}$	
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \overset{?}{=} \frac{4}{7}$	
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓	

TRY IT 3.1

Solve the proportion: $\frac{n}{84} = \frac{11}{12}$.

Show answer
77

TRY IT 3.2

Solve the proportion: $\frac{y}{96} = \frac{13}{12}$.

Show answer
104

When the variable is in a denominator, we’ll use the fact that the cross products of a proportion are equal to solve the proportions.

We can find the cross products of the proportion and then set them equal. Then we solve the resulting equation using our familiar techniques.

EXAMPLE 4

Solve: $\frac{144}{a} = \frac{9}{4}$.

Solution

Notice that the variable is in the denominator, so we will solve by finding the cross products and setting them equal.

	$\frac{144}{a} = \frac{9}{4}$
Find the cross products and set them equal.	$4 \cdot 144 = a \cdot 9$
Simplify.	$576 = 9a$
Divide both sides by 9.	$\frac{576}{9} = \frac{9a}{9}$
Simplify.	$64 = a$
Check your answer:	
	$\frac{144}{a} = \frac{9}{4}$
Substitute a = 64	$\frac{144}{64} \overset{?}{=} \frac{9}{4}$
Show common factors.	$\frac{9 \cdot 16}{4 \cdot 16} \overset{?}{=} \frac{9}{4}$
Simplify.	$\frac{9}{4} = \frac{9}{4}$ ✓

Another method to solve this would be to multiply both sides by the LCD, $4a$. Try it and verify that you get the same solution.

TRY IT 4.1

Solve the proportion: $\frac{91}{b} = \frac{7}{5}$.

Show answer
65

TRY IT 4.2

Solve the proportion: $\frac{39}{c} = \frac{13}{8}$.

Show answer
24

EXAMPLE 5

Solve: $\frac{52}{91} = \frac{-4}{y}$.

Solution

Find the cross products and set them equal.	$\frac{52}{91} = \frac{-4}{y}$
	$y \cdot 52 = 91(-4)$
Simplify.	$52y = -364$
Divide both sides by 52.	$\frac{52y}{52} = \frac{-364}{52}$
Simplify.	$y = -7$
Check:	
	$\frac{52}{91} = \frac{-4}{y}$
Substitute y = −7	$\frac{52}{91} \overset{?}{=} \frac{-4}{-7}$
Show common factors.	$\frac{13 \cdot 4}{13 \cdot 4} \overset{?}{=} \frac{-4}{-7}$
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓

TRY IT 5.1

Solve the proportion: $\frac{84}{98} = \frac{-6}{x}$.

Show answer
−7

TRY IT 5.2

Solve the proportion: $\frac{-7}{y} = \frac{105}{135}$.

Show answer
−9

Solve Applications Using Proportions

The strategy for solving applications that we have used earlier in this chapter, also works for proportions, since proportions are equations. When we set up the proportion, we must make sure the units are correct—the units in the numerators match and the units in the denominators match.

EXAMPLE 6

When pediatricians prescribe acetaminophen to children, they prescribe 5 millilitre s (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many millilitre s of acetaminophen will her doctor prescribe?

Solution

Identify what you are asked to find.	How many ml of acetaminophen the doctor will prescribe
Choose a variable to represent it.	Let a = ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion.	$\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$
Substitute given values—be careful of the units.	$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by 80.	$80 \cdot \frac{5}{25} = 80 \cdot \frac{a}{80}$
Multiply and show common factors.	$\frac{16 \cdot 5 \cdot 5}{5 \cdot 5} = \frac{80a}{80}$
Simplify.	$16 = a$
Check if the answer is reasonable.	
Yes. Since 80 is about 3 times 25, the medicine should be about 3 times 5.	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

You could also solve this proportion by setting the cross products equal.

TRY IT 6.1

Pediatricians prescribe 5 millilitre s (ml) of acetaminophen for every 25 pounds of a child’s weight. How many millilitre s of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Show answer
12 ml

TRY IT 6.2

For every 1 kilogram (kg) of a child’s weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Show answer
180 mg

EXAMPLE 7

One brand of microwave popcorn has 120 calories per serving. A whole bag of this popcorn has 3.5 servings. How many calories are in a whole bag of this microwave popcorn?

Solution

Identify what you are asked to find.	How many calories are in a whole bag of microwave popcorn?
Choose a variable to represent it.	Let $c =$ number of calories.
Write a sentence that gives the information to find it.	If there are 120 calories per serving, how many calories are in a whole bag with 3.5 servings?
Translate into a proportion.	$\frac{\text{calories}}{\text{serving}} = \frac{\text{calories}}{\text{serving}}$
Substitute given values.	$\frac{120}{1} = \frac{c}{3.5}$
Multiply both sides by 3.5.	$(3.5)\left(\frac{120}{1}\right) = (3.5)\left(\frac{c}{3.5}\right)$
Multiply.	$420 = c$
Check if the answer is reasonable.	
Yes. Since 3.5 is between 3 and 4, the total calories should be between 360 (3·120) and 480 (4·120).	
Write a complete sentence.	The whole bag of microwave popcorn has 420 calories.

TRY IT 7.1

Marissa loves the Caramel Macchiato at the coffee shop. The 16 oz. medium size has 240 calories. How many calories will she get if she drinks the large 20 oz. size?

Show answer
300

TRY IT 7.2

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many pieces can she have in her snack?

Show answer
5

EXAMPLE 8

Josiah went to Mexico for spring break and changed $\$325$ dollars into Mexican pesos. At that time, the exchange rate had $\$1$ U.S. is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution

Identify what you are asked to find.	How many Mexican pesos did Josiah get?
Choose a variable to represent it.	Let $p =$ number of pesos.
Write a sentence that gives the information to find it.	If \$1 U.S. is equal to 12.54 Mexican pesos, then \$325 is how many pesos?
Translate into a proportion.	$\frac{\$}{\text{pesos}} = \frac{\$}{\text{pesos}}$
Substitute given values.	$\frac{1}{12.54} = \frac{325}{p}$
The variable is in the denominator, so find the cross products and set them equal.	$p \cdot 1 = 12.54(325)$
Simplify.	$c = 4{,}075.5$
Check if the answer is reasonable.	
Yes, \$100 would be \$1,254 pesos. \$325 is a little more than 3 times this amount.	
Write a complete sentence.	Josiah has 4075.5 pesos for his spring break trip.

TRY IT 8.1

Yurianna is going to Europe and wants to change $\$800$ dollars into Euros. At the current exchange rate, $\$1$ Canadian dollar is equal to 0.65 Euro. How many Euros will she have for her trip?

Show answer
520 Euros

TRY IT 8.2

Corey and Nicole are traveling to Japan and need to exchange $\$600$ into Japanese yen. If each dollar is 75.7 yen, how many yen will they get?

Show answer
45,421.43 yen

Write Percent Equations As Proportions

Previously, we solved percent equations by applying the properties of equality we have used to solve equations throughout this text. Some people prefer to solve percent equations by using the proportion method. The proportion method for solving percent problems involves a percent proportion. A **percent proportion** is an equation where a percent is equal to an equivalent ratio.

For example, $60\% = \frac{60}{100}$ and we can simplify $\frac{60}{100} = \frac{3}{5}$. Since the equation $\frac{60}{100} = \frac{3}{5}$ shows a percent equal to an equivalent ratio, we call it a percent proportion. Using the vocabulary we used earlier:

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$
$$\frac{3}{5} = \frac{60}{100}$$

Percent Proportion

The amount is to the base as the percent is to 100.

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$

If we restate the problem in the words of a proportion, it may be easier to set up the proportion:

$$\text{The amount is to the base as the percent is to one hundred.}$$

We could also say:

$$\text{The amount out of the base is the same as the percent out of one hundred.}$$

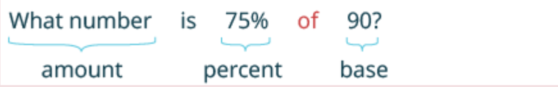
First we will practice translating into a percent proportion. Later, we'll solve the proportion.

EXAMPLE 9

Translate to a proportion. What number is 75% of 90?

Solution

If you look for the word “of”, it may help you identify the base.

Identify the parts of the percent proportion.	 $\underbrace{\text{What number}}_{\text{amount}}$ is $\underbrace{75\%}_{\text{percent}}$ of $\underbrace{90?}_{\text{base}}$
Restate as a proportion.	 What number out of 90 is the same as 75 out of 100?
Set up the proportion. Let $n = \text{number}$.	$\frac{n}{90} = \frac{75}{100}$

TRY IT 9.1

Translate to a proportion: What number is 60% of 105?

Show answer

$\frac{n}{105} = \frac{60}{100}$

TRY IT 9.2

Translate to a proportion: What number is 40% of 85?

Show answer

$\frac{n}{85} = \frac{40}{100}$

EXAMPLE 10

Translate to a proportion. 19 is 25% of what number?

Solution

Identify the parts of the percent proportion.	$\underbrace{19}_{\text{amount}}$ is $\underbrace{25\%}_{\text{percent}}$ of $\underbrace{\text{what number}}_{\text{base}}$?
Restate as a proportion.	19 out of what number is the same as 25 out of 100?
Set up the proportion. Let $n = \text{number}$.	$\frac{19}{n} = \frac{25}{100}$

TRY IT 10.1

Translate to a proportion: 36 is 25% of what number?

Show answer
$\frac{36}{n} = \frac{25}{100}$

TRY IT 10.2

Translate to a proportion: 27 is 36% of what number?

Show answer
$\frac{27}{n} = \frac{36}{100}$

EXAMPLE 11

Translate to a proportion. What percent of 27 is 9?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What percent}}_{\text{percent}}$ of $\underbrace{27}_{\text{base}}$ is $\underbrace{9?}_{\text{amount}}$
Restate as a proportion.	9 out of 27 is the same as what number out of 100?
Set up the proportion. Let $p = \text{percent}$.	$\frac{9}{27} = \frac{p}{100}$

TRY IT 11.1

Translate to a proportion: What percent of 52 is 39?

Show answer

$\frac{n}{100} = \frac{39}{52}$

TRY IT 11.2

Translate to a proportion: What percent of 92 is 23?

Show answer

$\frac{n}{100} = \frac{23}{92}$

Translate and Solve Percent Proportions

Now that we have written percent equations as proportions, we are ready to solve the equations.

EXAMPLE 12

Translate and solve using proportions: What number is 45% of 80?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What number}}_{\text{amount}}\ \text{is}\ \underbrace{45\%}_{\text{percent}}\ \text{of}\ \underbrace{80?}_{\text{base}}$
Restate as a proportion.	What number out of 80 is the same as 45 out of 100?
Set up the proportion. Let n = number.	$\frac{n}{80}=\frac{45}{100}$
Find the cross products and set them equal.	$100\cdot n=80\cdot 45$
Simplify.	$100n=3{,}600$
Divide both sides by 100.	$\frac{100n}{100}=\frac{3{,}600}{100}$
Simplify.	$n=36$
Check if the answer is reasonable.	
Yes. 45 is a little less than half of 100 and 36 is a little less than half 80.	
Write a complete sentence that answers the question.	36 is 45% of 80.

TRY IT 12.1

Translate and solve using proportions: What number is 65% of 40?

Show answer
26

TRY IT 12.2

Translate and solve using proportions: What number is 85% of 40?

Show answer
34

In the next example, the percent is more than 100, which is more than one whole. So the unknown number will be more than the base.

EXAMPLE 13

Translate and solve using proportions: 125% of 25 is what number?

Solution

Identify the parts of the percent proportion.	$\underbrace{125\%}_{\text{percent}}$ is $\underbrace{25}_{\text{base}}$ of $\underbrace{\text{what number ?}}_{\text{amount}}$
Restate as a proportion.	What number out of 25 is the same as 125 out of 100?
Set up the proportion. Let n = number.	$\frac{n}{25} = \frac{125}{100}$
Find the cross products and set them equal.	$100 \cdot n = 25 \cdot 125$
Simplify.	$100n = 3{,}125$
Divide both sides by 100.	$\frac{100n}{100} = \frac{3{,}125}{100}$
Simplify.	$n = 31.25$
Check if the answer is reasonable.	
Yes. 125 is more than 100 and 31.25 is more than 25.	
Write a complete sentence that answers the question.	125% of 25 is 31.25.

TRY IT 13.1

Translate and solve using proportions: 125% of 64 is what number?

Show answer

80

TRY IT 13.2

Translate and solve using proportions: 175% of 84 is what number?

Show answer

147

Percents with decimals and money are also used in proportions.

EXAMPLE 14

Translate and solve: 6.5% of what number is $\$1.56$?

Solution

Identify the parts of the percent proportion.	$\underbrace{6.5\%}_{\text{percent}}$ of $\underbrace{\text{what number}}_{\text{base}}$ is $\underbrace{\$1.56?}_{\text{amount}}$
Restate as a proportion.	\$1.56 out of what number is the same as 6.5 out of 100?
Set up the proportion. Let n $=$ number.	$\frac{1.56}{n} = \frac{6.5}{100}$
Find the cross products and set them equal.	$100(1.56) = n \cdot 6.5$
Simplify.	$156 = 6.5n$
Divide both sides by 6.5 to isolate the variable.	$\frac{156}{6.5} = \frac{6.5n}{6.5}$
Simplify.	$24 = n$
Check if the answer is reasonable.	
Yes. 6.5% is a small amount and \$1.56 is much less than \$24.	
Write a complete sentence that answers the question.	6.5% of \$24 is \$1.56.

TRY IT 14.1

Translate and solve using proportions: 8.5% of what number is $\$3.23$?

Show answer
38

TRY IT 14.2

Translate and solve using proportions: 7.25% of what number is $\$4.64$?

Show answer
64

EXAMPLE 15

Translate and solve using proportions: What percent of 72 is 9?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What percent}}_{\text{percent}}$ of $\underbrace{72}_{\text{base}}$ is $\underbrace{9?}_{\text{amount}}$
Restate as a proportion.	9 out of 72 is the same as what number out of 100?
Set up the proportion. Let n $=$ number.	$\frac{9}{72} = \frac{n}{100}$
Find the cross products and set them equal.	$72 \cdot n = 100 \cdot 9$
Simplify.	$72n = 900$
Divide both sides by 72.	$\frac{72n}{72} = \frac{900}{72}$
Simplify.	$n = 12.5$
Check if the answer is reasonable.	
Yes. 9 is $\frac{1}{8}$ of 72 and $\frac{1}{8}$ is 12.5%.	
Write a complete sentence that answers the question.	12.5% of 72 is 9.

TRY IT 15.1

Translate and solve using proportions: What percent of 72 is 27?

Show answer
37.5%

TRY IT 15.2

Translate and solve using proportions: What percent of 92 is 23?

Show answer
25%

Key Concepts

- **Proportion**
    - A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d".
- **Cross Products of a Proportion**
    - For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$, its cross products are equal: $a \cdot d = b \cdot c$.
- **Percent Proportion**
    - The amount is to the base as the percent is to 100. $\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$

Glossary

proportion
A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d".

Practice Makes Perfect

Use the Definition of Proportion

In the following exercises, write each sentence as a proportion.

1. 4 is to 15 as 36 is to 135.	2. 7 is to 9 as 35 is to 45.
3. 12 is to 5 as 96 is to 40.	4. 15 is to 8 as 75 is to 40.
5. 5 wins in 7 games is the same as 115 wins in 161 games.	6. 4 wins in 9 games is the same as 36 wins in 81 games.
7. 8 campers to 1 counsellor is the same as 48 campers to 6 counsellors.	8. 6 campers to 1 counselor is the same as 48 campers to 8 counselors.
9. $\$9.36$ for 18 ounces is the same as $\$2.60$ for 5 ounces.	10. $\$3.92$ for 8 ounces is the same as $\$1.47$ for 3 ounces.
11. $\$18.04$ for 11 pounds is the same as $\$4.92$ for 3 pounds.	12. $\$12.42$ for 27 pounds is the same as $\$5.52$ for 12 pounds.

In the following exercises, determine whether each equation is a proportion.

13. $\frac{7}{15} = \frac{56}{120}$	14. $\frac{5}{12} = \frac{45}{108}$
15. $\frac{11}{6} = \frac{21}{16}$	16. $\frac{9}{4} = \frac{39}{34}$
17. $\frac{12}{18} = \frac{4.99}{7.56}$	18. $\frac{9}{16} = \frac{2.16}{3.89}$
19. $\frac{13.5}{8.5} = \frac{31.05}{19.55}$	20. $\frac{10.1}{8.4} = \frac{3.03}{2.52}$

Solve Proportions

In the following exercises, solve each proportion.

21. $\frac{x}{56} = \frac{7}{8}$	22. $\frac{n}{91} = \frac{8}{13}$
23. $\frac{49}{63} = \frac{z}{9}$	24. $\frac{56}{72} = \frac{y}{9}$
25. $\frac{5}{a} = \frac{65}{117}$	26. $\frac{4}{b} = \frac{64}{144}$
27. $\frac{98}{154} = \frac{-7}{p}$	28. $\frac{72}{156} = \frac{-6}{q}$
29. $\frac{a}{-8} = \frac{-42}{48}$	30. $\frac{b}{-7} = \frac{-30}{42}$
31. $\frac{2.6}{3.9} = \frac{c}{3}$	32. $\frac{2.7}{3.6} = \frac{d}{4}$
33. $\frac{2.7}{j} = \frac{0.9}{0.2}$	34. $\frac{2.8}{k} = \frac{2.1}{1.5}$
35. $\frac{\frac{1}{2}}{1} = \frac{m}{8}$	36. $\frac{\frac{1}{3}}{3} = \frac{9}{n}$

Solve Applications Using Proportions

In the following exercises, solve the proportion problem.

37. Pediatricians prescribe 5 millilitre s (ml) of acetaminophen for every 25 pounds of a child’s weight. How many millilitres of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

38. Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child’s weight. How many milligrams will the doctor prescribe?

39. At the gym, Carol takes her pulse for 10 sec and counts 19 beats. How many beats per minute is this? Has Carol met her target heart rate of 140 beats per minute?

40. Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds. How many beats per minute is this? Has Kevin met his target heart rate?

41. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

42. One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?

43. Karen eats $\frac{1}{2}$ cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?

44. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

45. Janice is traveling to the US and will change $250 Canadian dollars into US dollars. At the current exchange rate, $1 Canadian is equal to $0.70 US. How many Canadian dollars will she get for her trip?

46. Todd is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 17.20 pesos, how many pesos will he get for his trip?

47. Steve changed $782 into 507.08 Euros. How many Euros did he receive per Canadian dollar?

48. Martha changed $350 Canadian into 392.28 Australian dollars. How many Australian dollars did she receive per US dollar?

49. At the laundromat, Lucy changed $12.00 into quarters. How many quarters did she get?

50. When she arrived at a casino, Gerty changed $20 into nickels. How many nickels did she get?

51. Jesse’s car gets 30 miles per gallon of gas. If Toronto is 285 miles away, how many gallons of gas are needed to get there and then home? If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

52. Danny wants to drive to Banff to see his grandfather. Banff is 370 miles from Danny’s home and his car gets 18.5 miles per gallon. How many gallons of gas will Danny need to get to and from Banff? If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?

53. Hugh leaves early one morning to drive from his home in White Rock to go to Edmonton, 702 miles away. After 3 hours, he has gone 190 miles. At that rate, how long will the whole drive take?

54. Kelly leaves her home in Seattle to drive to Spokane, a distance of 280 miles. After 2 hours, she has gone 152 miles. At that rate, how long will the whole drive take?

55. Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil’s lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

56. April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

Write Percent Equations as Proportions

In the following exercises, translate to a proportion.

57. What number is 35% of 250?	58. What number is 75% of 920?
59. What number is 110% of 47?	60. What number is 150% of 64?
61. 45 is 30% of what number?	62. 25 is 80% of what number?
63. 90 is 150% of what number?	64. 77 is 110% of what number?
64. 77 is 110% of what number?	65. What percent of 85 is 17?
66. What percent of 92 is 46?	67. What percent of 260 is 340?
68. What percent of 180 is 220?	

Translate and Solve Percent Proportions

In the following exercises, translate and solve using proportions.

69. What number is 65% of 180?	70. What number is 55% of 300?
71. 18% of 92 is what number?	72. 22% of 74 is what number?
73. 175% of 26 is what number?	74. 250% of 61 is what number?
75. What is 300% of 488?	76. What is 500% of 315?
77. 17% of what number is $\$7.65$?	78. 19% of what number is $\$6.46$?
79. $\$13.53$ is 8.25% of what number?	80. $\$18.12$ is 7.55% of what number?
81. What percent of 56 is 14?	82. What percent of 80 is 28?
83. What percent of 96 is 12?	84. What percent of 120 is 27?

Everyday Math

85. **Mixing a concentrate** Sam bought a large bottle of concentrated cleaning solution at the warehouse store. He must mix the concentrate with water to make a solution for washing his windows. The directions tell him to mix 3 ounces of concentrate with 5 ounces of water. If he puts 12 ounces of concentrate in a bucket, how many ounces of water should he add? How many ounces of the solution will he have altogether?	86. **Mixing a concentrate** Travis is going to wash his car. The directions on the bottle of car wash concentrate say to mix 2 ounces of concentrate with 15 ounces of water. If Travis puts 6 ounces of concentrate in a bucket, how much water must he mix with the concentrate?

Writing Exercises

87. To solve "what number is 45% of 350" do you prefer to use an equation like you did in the section on Decimal Operations or a proportion like you did in this section? Explain your reason.	88. To solve "what percent of 125 is 25" do you prefer to use an equation like you did in the section on Decimal Operations or a proportion like you did in this section? Explain your reason.

Answers

1. $\frac{4}{15}=\frac{36}{135}$	3. $\frac{12}{5}=\frac{96}{40}$	5. $\frac{5}{7}=\frac{115}{161}$
7. $\frac{8}{1}=\frac{48}{6}$	9. $\frac{9.36}{18}=\frac{2.60}{5}$	11. $\frac{18.04}{11}=\frac{4.92}{3}$
13. yes	15. no	17. no
19. yes	21. 49	23. 47
25. 9	27. -11	29. 7
31. 2	33. 0.6	35. 4
37. 9 ml	39. 114, no	41. 159 cal
43. $\frac{3}{4}$ cup	45. $175.00	47. 0.65
49. 48 quarters	51. 19, $58.71	53. 11.1 hours
55. 4 bags	57. $\frac{n}{250}=\frac{35}{100}$	59. $\frac{n}{47}=\frac{110}{100}$
61. $\frac{45}{n}=\frac{30}{100}$	63. $\frac{90}{n}=\frac{150}{100}$	65. $\frac{17}{85}=\frac{p}{100}$
67. $\frac{340}{260}=\frac{p}{100}$	69. 117	70. 165
71. 16.56	73. 45.5	75. 1464
77. $45	79. $164	81. 25%
83. 12.5%	85. 20, 32	87. Answers will vary.

Attributions

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4.4 Solve General Applications of Percent

Learning Objectives

By the end of this section, you will be able to:

- Translate and solve basic percent equations
- Solve applications of percent
- Find percent increase and percent decrease

Translate and Solve Basic Percent Equations

In the last section, we solved percent problems by setting them up as proportions. That is the best method available when you did not have the tools of algebra. Now, in this section we will translate word sentences into algebraic equations, and then solve the percent equations.

We'll look at a common application of percent—tips to a server at a restaurant—to see how to set up a basic percent application.

When Kim and her friends went on a road trip to Vancouver, they ate lunch at Marta's Cafe Tower. The bill came to $\$80$. They wanted to leave a 20% tip. What amount would the tip be?

To solve this, we want to find what *amount* is 20% of $\$80$. The $\$80$ is called the *base*. The amount of the tip would be $0.20\left(80\right)$, or $\$16$ See (Figure 1). To find the amount of the tip, we multiplied the percent by the base.

A 20% tip for an $\$80$ restaurant bill comes out to $\$16$.

Marta's Cafe Tower
14 Jan Blvd.
Vancouver, BC 672901
555-114-1313

DATE 07/17/2020
TIME 2:36 PM
ACCT# xxxxxxxxxxxxx8032
CARD TYPE VISA
SERVER ID 5
CARD TYPE SALE
AUTH CODE 445938
TABLE # R3
CHECK # 1
AMOUNT $80.00
TIP AMOUNT $16.00
TOTAL AMOUNT ________

SIGN X ____________________
I AGREE TO PAY THE ABOVE
TOTAL AMOUNT ACCORDING TO
CARD ISSUER AGREEMENT
THANK YOU!
PLEASE COME AGAIN
CUSTOMER COPY

Figure 1.(credit: Marta Oraniewicz)

In the next examples, we will find the amount. We must be sure to change the given percent to a decimal when we translate the words into an equation.

EXAMPLE 1

What number is 35% of 90?

Solution

Translate into algebra. Let n =the number.	$\underbrace{\text{What number}}_{n}\ \underbrace{\text{is}}_{=}\ \underbrace{35\%}_{0.35}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{90?}_{90}$
Multiply.	$n = 31.5$
	31.5 is 35% of 90

TRY IT 1.1

What number is 45% of 80?

Show answer
36

TRY IT 1.2

What number is 55% of $60?$

Show answer

33

EXAMPLE 2

125% of 28 is what number?

Solution

Translate into algebra. Let a $=$the number.	$\underbrace{125\%}_{1.25}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{28}_{28}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what number?}}_{a}$
Multiply.	$35 = a$
	125% of 28 is 35.

Remember that a percent over 100 is a number greater than 1. We found that 125% of 28 is 35, which is greater than 28.

TRY IT 2.1

150% of 78 is what number?

Show answer

117

TRY IT 2.2

175% of 72 is what number?

Show answer

126

In the next examples, we are asked to find the base.

EXAMPLE 3

Translate and solve: 36 is 75% of what number?

Solution

Translate. Let $b =$ the number.	$\underbrace{36}_{36}\ \underbrace{\text{is}}_{=}\ \underbrace{75\%}_{0.75}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\text{what number?}}_{b}$
Divide both sides by 0.75.	$\frac{36}{0.75} = \frac{0.75b}{0.75}$
Simplify.	$48 = b$ 36 is 75% of 48.

TRY IT 3.1

17 is 25% of what number?

Show answer

68

TRY IT 3.2

40 is 62.5% of what number?

Show answer

64

EXAMPLE 4

6.5% of what number is $\$1.17$?

Solution

Translate. Let $b =$ the number.	$\underbrace{6.5\%}_{0.065}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\text{what number}}_{b}\ \underbrace{\text{is}}_{=}\ \underbrace{\$1.17?}_{1.17}$
Divide both sides by 0.065.	$\frac{0.065n}{0.065} = \frac{1.17}{0.065}$
Simplify.	$n = 18$ 6.5% of \$18 is \$1.17.

TRY IT 4.1

7.5% of what number is $\$1.95?$

Show answer
$26

TRY IT 4.1

8.5% of what number is $\$3.06?$

Show answer
$36

In the next examples, we will solve for the percent.

EXAMPLE 5

What percent of 36 is $9?$

Solution

Translate into algebra. Let p = the percent.	$\underbrace{\text{What percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{36}_{36}\ \underbrace{\text{is}}_{=}\ \underbrace{9?}_{9}$
Divide by 36.	$\frac{36p}{36} = \frac{9}{36}$
Simplify.	$p = \frac{1}{4}$
Convert to decimal form.	$p = 0.25$
Convert to percent.	$p = 25\%$ 25% of 36 is 9.

TRY IT 5.1

What percent of 76 is 57?

Show answer
75%

TRY IT 5.2

What percent of 120 is 96?

Show answer
80%

EXAMPLE 6

144 is what percent of 96?

Solution

Translate into algebra. Let $p =$ the percent.	$\underbrace{144}_{144}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{96?}_{96}$
Divide by 96.	$\frac{144}{96} = \frac{96p}{96}$
Simplify.	$1.5 = p$
Convert to percent.	$150\% = p$ 144 is 150% of 96.

TRY IT 6.1

110 is what percent of $88?$

Show answer
125%

TRY IT 6.2

126 is what percent of $72?$

Show answer
175%

Solve Applications of Percent

Many applications of percent occur in our daily lives, such as tips, sales tax, discount, and interest. To solve these applications we'll translate to a basic percent equation, just like those we solved in the previous examples in this section. Once you translate the sentence into a percent equation, you know how to solve it.

We will update the strategy we used in our earlier applications to include equations now. Notice that we will translate a sentence into an equation.

HOW TO: Solve an Application

1. Identify what you are asked to find and choose a variable to represent it.
2. Write a sentence that gives the information to find it.
3. Translate the sentence into an equation.
4. Solve the equation using good algebra techniques.
5. Check the answer in the problem and make sure it makes sense.
6. Write a complete sentence that answers the question.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we'll solve involve everyday situations, you can rely on your own experience.

EXAMPLE 7

Dezohn and his girlfriend enjoyed a dinner at a restaurant, and the bill was $\$68.50$. They want to leave an 18% tip. If the tip will be 18% of the total bill, how much should the tip be?

Solution

What are you asked to find?	The amount of the tip
Choose a variable to represent it.	Let t = amount of tip.
Write a sentence that give the information to find it.	The tip is 18% of the total bill.
Translate the sentence into an equation.	$\underbrace{\text{The tip}}_{t}\ \underbrace{\text{is}}_{=}\ \underbrace{18\%}_{0.18}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\$68.50.}_{\$68.50}$
Multiply.	$t = 12.33$
Check. Is this answer reasonable?	
If we approximate the bill to \$70 and the percent to 20%, we would have a tip of \$14. So a tip of \$12.33 seems reasonable.	
Write a complete sentence that answers the question.	The couple should leave a tip of \$12.33.

TRY IT 7.1

Cierra and her sister enjoyed a special dinner in a restaurant, and the bill was $\$81.50$. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Show answer
$14.67

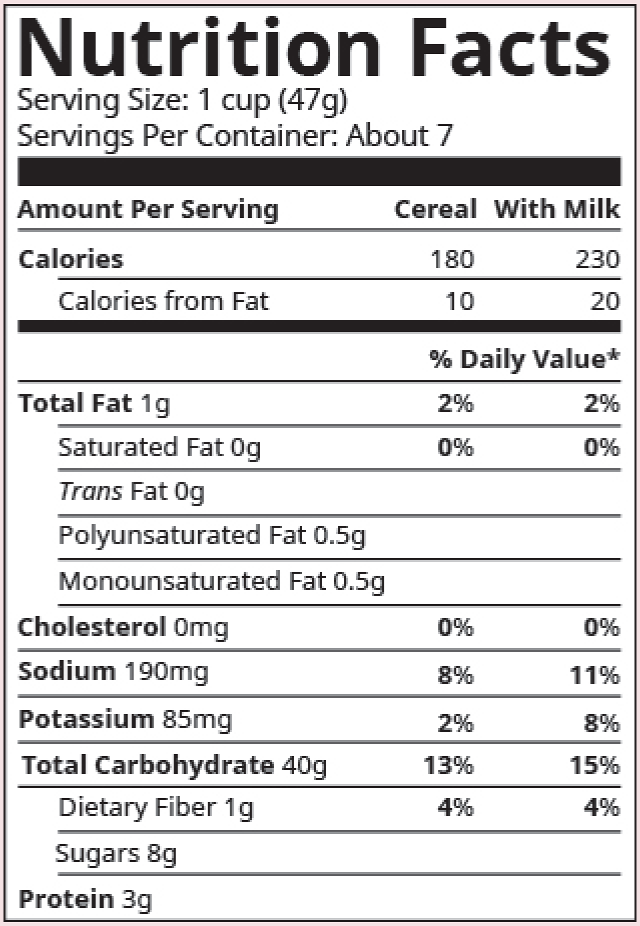
TRY IT 7.2

Kimngoc had lunch at her favorite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was $\$14.40$, how much will she leave for the tip?

Show answer
$2.16

EXAMPLE 8

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?



Nutrition Facts

Serving Size: 1 cup (47g)
Servings Per Container: About 7

Amount Per Serving	**Cereal**	**With Milk**
Calories	180	230
Calories from Fat	10	20
	% Daily Value*	
Total Fat 1g	**2%**	**2%**
Saturated Fat 0g	**0%**	**0%**
Trans Fat 0g		
Polyunsaturated Fat 0.5g		
Monounsaturated Fat 0.5g		
Cholesterol 0mg	**0%**	**0%**
Sodium 190mg	**8%**	**11%**
Potassium 85mg	**2%**	**8%**
Total Carbohydrate 40g	**13%**	**15%**
Dietary Fiber 1g	**4%**	**4%**
Sugars 8g		
Protein 3g		

Solution

What are you asked to find?	the total amount of potassium recommended
Choose a variable to represent it.	Let $a =$ total amount of potassium.
Write a sentence that gives the information to find it.	85 mg is 2% of the total amount.
Translate the sentence into an equation.	$\underbrace{85\text{ mg}}_{85}\ \underbrace{\text{is}}_{=}\ \underbrace{2\%}_{0.02}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{a}_{a}?$
Divide both sides by 0.02.	$\frac{85}{0.02} = \frac{0.02a}{0.02}$
Simplify.	$4{,}250 = a$
Check: Is this answer reasonable?	
Yes. 2% is a small percent and 85 is a small part of 4,250.	
Write a complete sentence that answers the question.	The amount of potassium that is recommended is 4250 mg.

TRY IT 8.1

One serving of wheat square cereal has 7 grams of fiber, which is 29% of the recommended daily amount. What is the total recommended daily amount of fiber?

Show answer
24.1 grams

TRY IT 8.2

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Show answer
2,375 mg

EXAMPLE 9

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution

What are you asked to find?	the percent of the total calories from fat
Choose a variable to represent it.	Let $p =$ percent from fat.
Write a sentence that gives the information to find it.	What percent of 480 is 240?
Translate the sentence into an equation.	$\underbrace{\text{What percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{480}_{480}\ \underbrace{\text{is}}_{=}\ \underbrace{240?}_{240}$
Divide both sides by 480.	$\frac{p \cdot 480}{480} = \frac{240}{480}$
Simplify.	$p = 0.5$
Convert to percent form.	$p = 50\%$
Check. Is this answer reasonable?	
Yes. 240 is half of 480, so 50% makes sense.	
Write a complete sentence that answers the question.	Of the total calories in each brownie, 50% is fat.

TRY IT 9.1

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat? (Round to the nearest whole percent.)

Show answer
26%

Exercises

The brownie mix Ricardo plans to use says that each brownie will be 190 calories, and 70 calories are from fat. What percent of the total calories are from fat?

Show answer
37%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the percent increase, first we find the amount of increase, which is the difference between the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

HOW TO: Find Percent Increase

Step 1. Find the amount of increase.

- $\text{increase} = \text{new amount} - \text{original amount}$

Step 2. Find the percent increase as a percent of the original amount.

EXAMPLE 10

In 2017, university tuition fees in Canada for domestic students increased from $\$26$ per school year to $\$36$ per school year. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution

What are you asked to find?	the percent increase
Choose a variable to represent it.	Let p = percent.
Find the amount of increase.	$\underbrace{10}_{10}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{26?}_{26}$
Find the percent increase.	The increase is what percent of the original amount?
Translate to an equation.	
Divide both sides by 26.	$\frac{10}{26} = \frac{26p}{26}$
Round to the nearest thousandth.	$0.384 = p$
Convert to percent form.	$38.4\% = p$
Write a complete sentence.	The new fees represent a 38.4\% increase over the old fees.

TRY IT 10.1

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents. Find the percent increase. (Round to the nearest tenth of a percent.)

Show answer
8.8%

TRY IT 10.2

In 1984, the standard bus fare in Vancouver was $\$1.25$. In 2008, the standard bus fare was $\$2.50$. Find the percent increase. (Round to the nearest tenth of a percent.)

Show answer
50%

Finding the percent decrease is very similar to finding the percent increase, but now the amount of decrease is the difference between the original amount and the final amount. Then we find what percent the amount of decrease is of the original amount.

HOW TO: Find Percent Decrease

1. Find the amount of decrease.
    - $\text{decrease} = \text{original amount} - \text{new amount}$
2. Find the percent decrease as a percent of the original amount.

EXAMPLE 11

The average price of a gallon of gas in one city in June 2014 was $\$3.71$. The average price in that city in July was $\$3.64$. Find the percent decrease.

Solution

What are you asked to find?	the percent decrease
Choose a variable to represent it.	Let p = percent.
Find the amount of decrease.	$\underbrace{3.71}_{\text{original amount}} - \underbrace{3.64}_{\text{new amount}} = \underbrace{0.07}_{\text{increase}}$
Find the percent of decrease.	The decrease is what percent of the original amount?
Translate to an equation.	$\underbrace{0.07}_{0.07}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{3.71}_{3.71}$
Divide both sides by 3.71.	$\frac{0.07}{3.71} = \frac{3.71p}{3.71}$
Round to the nearest thousandth.	$0.019 = p$
Convert to percent form.	$1.9\% = p$
Write a complete sentence.	The price of gas decreased 1.9%.

TRY IT 11.1

The population of one city was about $672,000$ in 2010. The population of the city is projected to be about $630,000$ in 2020. Find the percent decrease. (Round to the nearest tenth of a percent.)

Show answer

6.3%

TRY IT 11.2

Last year Sheila's salary was $\$42{,}000$. Because of furlough days, this year her salary was $\$37{,}800$. Find the percent decrease. (Round to the nearest tenth of a percent.)

Show answer
10%

Access Additional Online Resources

- Percent Increase and Percent Decrease Visualization

Key Concepts

- **Solve an application.**
    1. Identify what you are asked to find and choose a variable to represent it.
    2. Write a sentence that gives the information to find it.
    3. Translate the sentence into an equation.
    4. Solve the equation using good algebra techniques.
    5. Write a complete sentence that answers the question.
    6. Check the answer in the problem and make sure it makes sense.
- **Find percent increase.**
    1. Find the amount of increase:
       $\text{increase} = \text{new amount} - \text{original amount}$
    2. Find the percent increase as a percent of the original amount.
- **Find percent decrease.**
    1. Find the amount of decrease.
       $\text{decrease} = \text{original amount} - \text{new amount}$
    2. Find the percent decrease as a percent of the original amount.

Glossary

percent increase

The percent increase is the percent the amount of increase is of the original amount.

percent decrease

The percent decrease is the percent the amount of decrease is of the original amount.

Practice Makes Perfect

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

1. What number is 45% of 120?
2. What number is 65% of 100?
3. What number is 24% of 112?
4. What number is 36% of 124?
5. 250% of 65 is what number?
6. 150% of 90 is what number?
7. 800% of $2,250$ is what number?
8. 600% of $1,740$ is what number?
9. 28 is 25% of what number?
10. 36 is 25% of what number?
11. 81 is 75% of what number?
12. 93 is 75% of what number?
13. 8.2% of what number is $\$2.87$?
14. 6.4% of what number is $\$2.88$?
15. 11.5% of what number is $\$108.10$?
16. 12.3% of what number is $\$92.25$?
17. What percent of 260 is 78?
18. What percent of 215 is 86?
19. What percent of $1,500$ is 540?
20. What percent of $1,800$ is 846?
21. 30 is what percent of 20?
22. 50 is what percent of 40?
23. 840 is what percent of 480?
24. 790 is what percent of 395?

Solve Applications of Percents

In the following exercises, solve the applications of percents.

25. Geneva treated her parents to dinner at their favorite restaurant. The bill was $\$74.25$. She wants to leave 16% of the total bill as a tip. How much should the tip be?

26. When Hiro and his co-workers had lunch at a restaurant the bill was $\$90.50$. They want to leave 18% of the total bill as a tip. How much should the tip be?

27. Trong has 12% of each paycheck automatically deposited to his savings account. His last paycheck was $\$2{,}165$. How much money was deposited to Trong's savings account?

28. Cherise deposits 8% of each paycheck into her retirement account. Her last paycheck was $\$1{,}485$. How much did Cherise deposit into her retirement account?

29. One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

30. One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

31. A bacon cheeseburger at a popular fast food restaurant contains $2,070$ milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

32. A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

33. The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

34. The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

35. Emma gets paid $\$3{,}000$ per month. She pays $\$750$ a month for rent. What percent of her monthly pay goes to rent?

36. Dimple gets paid $\$3{,}200$ per month. She pays $\$960$ a month for rent. What percent of her monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, find the percent increase or percent decrease.

37. Tamanika got a raise in her hourly pay, from $\$15.50$ to $\$17.55$. Find the percent increase.

38. Ayodele got a raise in her hourly pay, from $\$24.50$ to $\$25.48$. Find the percent increase.

39. According to Statistics Canada, annual international graduate student fees in Canada rose from about $\$13{,}000$ in 2015 to about $\$15{,}000$ in 2019. Find the percent increase.

40. The price of a share of one stock rose from $\$12.50$ to $\$50$. Find the percent increase.

41. According to Time magazine $(7/19/2011)$ annual global seafood consumption rose from 22 pounds per person in 1960 to 38 pounds per person today. Find the percent increase. (Round to the nearest tenth of a percent.)

42. In one month, the median home price in the Northeast rose from $\$225{,}400$ to $\$241{,}500$. Find the percent increase. (Round to the nearest tenth of a percent.)

43. A grocery store reduced the price of a loaf of bread from $\$2.80$ to $\$2.73$. Find the percent decrease.

44. The price of a share of one stock fell from $\$8.75$ to $\$8.54$. Find the percent decrease.

45. Hernando’s salary was $\$49{,}500$ last year. This year his salary was cut to $\$44{,}055$. Find the percent decrease.

46. From 2000 to 2010, the population of Detroit fell from about $951,\ 000$ to about $714,\ 000$. Find the percent decrease. (Round to the nearest tenth of a percent.)

47. In one month, the median home price in the West fell from $\$203{,}400$ to $\$192{,}300$. Find the percent decrease. (Round to the nearest tenth of a percent.)

48. Sales of video games and consoles fell from $\$1{,}150$ million to $\$1{,}030$ million in one year. Find the percent decrease. (Round to the nearest tenth of a percent.)

Everyday Math

49. **Tipping** At the campus coffee cart, a medium coffee costs $\$1.65$. MaryAnne brings $\$2.00$ with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

50. **Late Fees** Alison was late paying her credit card bill of $\$249$. She was charged a 5% late fee. What was the amount of the late fee?

Writing Exercises

51. Without solving the problem “44 is 80% of what number”, think about what the solution might be. Should it be a number that is greater than 44 or less than $44?$ Explain your reasoning.

52. Without solving the problem “What is 20% of $300?''$ think about what the solution might be. Should it be a number that is greater than 300 or less than $300?$ Explain your reasoning.

53. After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

54. Because of road construction in one city, commuters were advised to plan their Monday morning commute to take 150% of their usual commuting time. Explain what this means.

Answers

1. 54	3. 26.88	5. 162.5
7. 18,000	9. 112	11. 108
13. $35	15. $940	17. 30%
19. 36%	21. 150%	23. 175%
25. $11.88	27. $259.80	29. 24.2 grams
31. 2,407 grams	33. 45%	35. 25%
37. 13.2%	39. 15%	41. 72.7%
43. 2.5%	45. 11%	47. 5.5%
49. 21.2%	51. The original number should be greater than 44.80% is less than 100%, so when 80% is converted to a decimal and multiplied to the base in the percent equation, the resulting amount of 44 is less. 44 is only the larger number in cases where the percent is greater than 100%.	53. Alex should have packed half as many shorts and twice as many shirts.

Attributions

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4.5 Chapter Review

Review Exercises

Write a Ratio as a Fraction

In the following exercises, write each ratio as a fraction. Simplify the answer if possible.

1. 56 to 32	2. 28 to 40
3. 1.2 to 1.8	4. 3.5 to 0.5
5. $2\frac{1}{3}$ to $5\frac{1}{4}$	6. $1\frac{3}{4}$ to $1\frac{5}{8}$
7. 28 inches to 3 feet	8. 64 ounces to 30 ounces

Write a Rate as a Fraction

In the following exercises, write each rate as a fraction. Simplify the answer if possible.

9. 90 pounds per 7.5 square inches	10. 180 calories per 8 ounces
11.$\$612.50$ for 35 hours	12. 126 miles in 4 hours

Find Unit Rates

In the following exercises, find the unit rate.

13. 90 pounds per 7.5 square inches	14. 180 calories per 8 ounces
15. $\$612.50$ for 35 hours	16. 126 miles in 4 hours

Find Unit Price

In the following exercises, find the unit price.

17.Highlighters: 6 for $\$2.52$	18. T-shirts: 3 for $\$8.97$
19. Anna bought a pack of 8 kitchen towels for $\$13.20$. How much did each towel cost? Round to the nearest cent if necessary.	20. An office supply store sells a box of pens for $\$11$. The box contains 12 pens. How much does each pen cost?

In the following exercises, find each unit price and then determine the better buy.

21.Vitamins: 60 tablets for $\$6.49$ or 100 for $\$11.99$?	22. Shampoo: 12 ounces for $\$4.29$ or 22 ounces for $\$7.29$?

Translate Phrases to Expressions with Fractions

In the following exercises, translate the English phrase into an algebraic expression.

23. a adults to 45 children	24. 535 miles per h hours
25. the ratio of 19 and the sum of 3 and n	26. the ratio of $4y$ and the difference of x and 10

In the following exercises, write each percent as a ratio.

27. 32% admission rate for the university	28. 53.3% rate of college students with student loans

In the following exercises, write as a ratio and then as a percent.

29. 13 out of 100 architects are women.	30. 9 out of every 100 nurses are men.

In the following exercises, convert each percent to a fraction.

31. 48%	32. 175%
33. 64.1%	34. $8\frac{1}{4}\%$

In the following exercises, convert each percent to a decimal.

35. 6%	36. 23%
37. 128%	38. 4.9%

In the following exercises, convert each percent to a) a simplified fraction and b) a decimal.

39. In 2016, 17% of the Canadian population was age 65 or over. (Source: www12.statcan.gc.ca)	40. In 2016, 16.6% of the Canadian population was under 15 years old. (Source: www12.statcan.gc.ca)
41. When a die is tossed, the probability it will land with an even number of dots on the top side is 50%.	42. A couple plans to have three children. The probability they will all be girls is 12.5%.

In the following exercises, convert each decimal to a percent.

43. 0.04	44. 0.15
45. 2.82	46. 3
47. 0.003	48. 1.395

In the following exercises, convert each fraction to a percent.

49. $\frac{3}{4}$	50. $\frac{11}{5}$
51. $3\frac{5}{8}$	52. $\frac{2}{9}$
53. According to the Centers for Disease Control, $\frac{2}{5}$ of adults do not take a vitamin or supplement.	54. According to the Centers for Disease Control, among adults who do take a vitamin or supplement, $\frac{3}{4}$ take a multivitamin.

In the following exercises, translate and solve.

55. What number is 46% of 350?	56. 120% of 55 is what number?
57. 84 is 35% of what number?	58. 15 is 8% of what number?
59. 200% of what number is 50?	60. 7.9% of what number is $\$4.74$?
61. What percent of 120 is 81.6?	62. What percent of 340 is 595?

Solve General Applications of Percents

In the following exercises, solve.

63. When Aurelio and his family ate dinner at a restaurant, the bill was $\$83.50$. Aurelio wants to leave 20% of the total bill as a tip. How much should the tip be?	64. One granola bar has 2 grams of fiber, which is 8% of the recommended daily amount. What is the total recommended daily amount of fiber?
65. The nutrition label on a package of granola bars says that each granola bar has 190 calories, and 54 calories are from fat. What percent of the total calories is from fat?	66. Elsa gets paid $\$4,600$ per month. Her car payment is $\$253$. What percent of her monthly pay goes to her car payment?

67. Marta got a gift of $\$1900$ from her uncle. She spent 35% of that money for her trip to Victoria. How much money she has left?.	68. Last year Bernard bought a new car for $\$30,000$. If the value of the car depreciated 20% every year , find the value of the car this year.

Solve Proportions and their Applications

In the following exercises, write each sentence as a proportion.

69. 3 is to 8 as 12 is to 32.	70. 95 miles to 3 gallons is the same as 475 miles to 15 gallons.
71. 1 teacher to 18 students is the same as 23 teachers to 414 students.	72. $\$7.35$ for 15 ounces is the same as $\$2.94$ for 6 ounces.

In the following exercises, determine whether each equation is a proportion.

73. $\frac{5}{13} = \frac{30}{78}$	74. $\frac{16}{7} = \frac{48}{23}$
75. $\frac{12}{18} = \frac{6.99}{10.99}$	76. $\frac{11.6}{9.2} = \frac{37.12}{29.44}$

In the following exercises, solve each proportion.

77. $\frac{x}{36} = \frac{5}{9}$	78. $\frac{7}{a} = \frac{-6}{84}$
79. $\frac{1.2}{1.8} = \frac{d}{6}$	80. $\frac{\frac{1}{2}}{2} = \frac{m}{20}$

In the following exercises, solve the proportion problem.

81. The children's dosage of acetaminophen is 5 millilitre s (ml) for every 25 pounds of a child's weight. How many millilitre s of acetaminophen will be prescribed for a 60 pound child?	82. After a workout, Dennis takes his pulse for 10 sec and counts 21 beats. How many beats per minute is this?
83. An 8 ounce serving of ice cream has 272 calories. If Lavonne eats 10 ounces of ice cream, how many calories does she get?	84. Alma is going to Europe and wants to exchange $\$1{,}200$ into Euros. If each dollar is 0.65 Euros, how many Euros will Alma get?
85. Zack wants to drive from Abbotsford to Banff, a distance of 494 miles. If his car gets 38 miles to the gallon, how many gallons of gas will Zack need to get to Banff?	86. Teresa is planning a party for 100 people. Each gallon of punch will serve 18 people. How many gallons of punch will she need?

In the following exercises, translate to a proportion.

87. What number is 62% of $395?$	88. 42 is 70% of what number?
89. What percent of $1,000$ is $15?$	90. What percent of 140 is $210?$

In the following exercises, translate and solve using proportions.

91. What number is 85% of 900?	92. 6% of what number is $\$24$?
93. $\$3.51$ is 4.5% of what number?	94. What percent of $3,100$ is 930?

In the following exercises, convert each percent to a) a decimal b) a simplified fraction.

95. 24%	96. 5%
97. 350%	

In the following exercises, convert each fraction to a percent. (Round to 3 decimal places if needed.)

98. $\frac{7}{8}$	99. $\frac{1}{3}$
100. $\frac{11}{12}$	

In the following exercises, solve the percent problem.

101. 65 is what percent of 260?	102. What number is 27% of $3,000$?
103. 150% of what number is 60?	104. Write as a proportion: 4 gallons to 144 miles is the same as 10 gallons to 360 miles.
105. Vin read 10 pages of a book in 12 minutes. At that rate, how long will it take him to read 35 pages?	

Review Answers

1. $\frac{7}{4}$	3. $\frac{2}{3}$	5. $\frac{4}{9}$
7. $\frac{7}{9}$	9. $\frac{\text{90 pounds}}{\text{7.5 square inches}}$	11. $\frac{\$612.50}{\text{35 hours}}$
13. 12 pounds/sq.in.	15. $17.50/hour	17. $0.42
19. $1.65	21. $0.11, $0.12; 60 tablets for $6.49	23. $\frac{a\text{ adults}}{\text{45 children}}$
25. $\frac{19}{3+n}$	27. $\frac{32}{100}$	29. $\frac{13}{100}$, 13%
31. $\frac{12}{25}$	33. $\frac{641}{1000}$	35. 0.06
37. 1.28	39. a) $\frac{17}{100}$ b) 0.17	41. a) $\frac{1}{2}$ b) 0.5
43. 4%	45. 282%	47. 0.3%
49. 75%	51. 362.5%	53. 40%
55. 161	57. 240	59. 25
61. 68%	63. $16.70	65. 28.4%
67. 1235	69. $\frac{3}{8} = \frac{12}{32}$	71. $\frac{1}{18} = \frac{23}{414}$
73. yes	75. no	77. 20
79. 4	81. 12	83. 340
87. $\frac{x}{395} = \frac{62}{100}$	89. $\frac{x}{100} = \frac{15}{1000}$	91.765
93. $78	95.0.24, $\frac{6}{25}$	97. $3.5, 3\frac{1}{2}$
99. 33.333%	101.25%	103. 40
105. 42		

Chapter Test

1. Write a ratio as a fraction. Simplify the answer if possible. 42 to 28	2. Write a rate as a fraction. Simplify the answer if possible. 80 pounds per 6.5 square inches
3. Find the unit rate. $\$868.80$ for 24 hours	4. Marta bought a pack of 6 paint brushes for $\$32.20$. How much did each brush cost? Round to the nearest cent if necessary
5. Find each unit price and then the better buy. Laundry detergent: 64 ounces for $\$10.99$ or 48 ounces for $\$8.49$	6. Convert a percent to a fraction: 245%
7. Convert a decimal to a percent: 0.07	8. Convert a fraction to a percent. (Round to 3 decimal places if needed.) $\frac{11}{8}$
9. What number is 36% of $450?$	10. 8% of what number is $\$34?$
11. 57.6 is what percent of $360?$	12. One granola bar has 3 grams of fiber, which is 12% of the recommended daily amount. What is the total recommended daily amount of fiber?
13. Klaudia is going to Poland and wants to exchange $\$1{,}400$ into Polish zlotych. If each dollar is 2.91 zlotych, how many zlotych will Klaudia get?	14. Solve a proportion:$\frac{24}{x}=\frac{3}{7}$
15. Solve a proportion:$\frac{x}{6}=\frac{9}{24}$	16.Solve a proportion:$\frac{2.4}{1.6}=\frac{t}{6.2}$

Test Answers

1. $\frac{3}{2}$	2. $\frac{160}{13}$	3. $36.20
4. $5.37	5. 64 ounces for $10.99 is the better buy	6. $\frac{49}{20}$
7. 7%	8. 137.5%	9. 162
10. 425	11. 16%	12. 25 grams
13. 4074 zlotych	14. 56	15. 2.25
16. 9.3		

CHAPTER 5 Solving First Degree Equations in One Variable

The rocks in this formation must remain perfectly balanced around the centre for the formation to hold its shape.

If we carefully placed more rocks of equal weight on both sides of this formation, it would still balance. Similarly, the expressions in an equation remain balanced when we add the same quantity to both sides of the equation. In this chapter, we will solve equations, remembering that what we do to one side of the equation, we must also do to the other side.

Attributions

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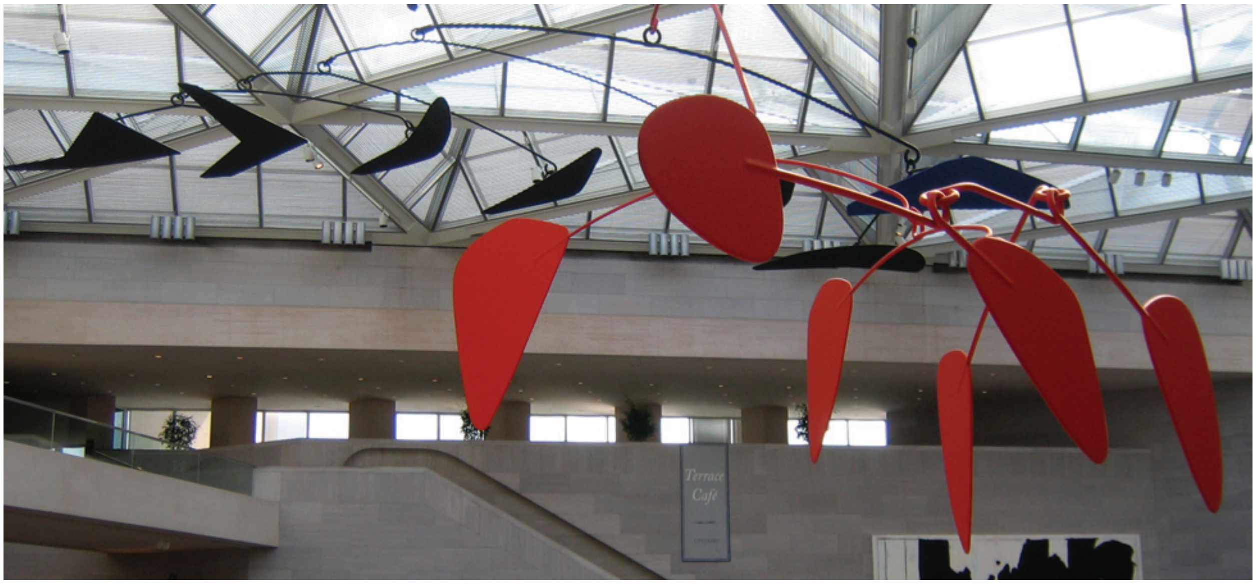
Introduction

A Calder mobile is balanced and has several elements on each side. (credit: paurian, Flickr)



Teetering high above the floor, this amazing mobile remains aloft thanks to its carefully balanced mass. Any shift in either direction could cause the mobile to become lopsided, or even crash downward. In this chapter, we will solve equations by keeping quantities on both sides of an equal sign in perfect balance.

5.1 Solve Equations Using the Subtraction and Addition Properties of Equality

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that need to be simplified
- Translate an equation and solve
- Translate and solve applications

We are now ready to “get to the good stuff.” You have the basics down and are ready to begin one of the most important topics in algebra: solving equations. The applications are limitless and extend to all careers and fields. Also, the skills and techniques you learn here will help improve your critical thinking and problem-solving skills. This is a great benefit of studying mathematics and will be useful in your life in ways you may not see right now.

Solve Equations Using the Subtraction and Addition Properties of Equality

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle.

Solution of an Equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

The steps to determine if a value is a solution to an equation are listed here.

HOW TO: Determine whether a number is a solution to an equation.

1. Substitute the number for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true.
    - If it is true, the number is a solution.
    - If it is not true, the number is not a solution.

EXAMPLE 1

Determine whether $y = \frac{3}{4}$ is a solution for $4y + 3 = 8y$.

Solution

	$4y + 3 = 8y$
Substitute $\frac{3}{4}$ for y.	$4\left(\frac{3}{4}\right) + 3 \overset{?}{=} 8\left(\frac{3}{4}\right)$
Multiply.	$3 + 3 \overset{?}{=} 6$
Add.	$6 = 6$ ✓

Since $y = \frac{3}{4}$ results in a true equation, $\frac{3}{4}$ is a solution to the equation $4y + 3 = 8y$.

TRY IT 1.1

Is $y = \frac{2}{3}$ a solution for $9y + 2 = 6y$?

Show answer
no

TRY IT 1.2

Is $y = \frac{2}{5}$ a solution for $5y - 3 = 10y$?

Show answer
no

In that section,we will model how the Subtraction and Addition Properties work and then we will apply them to solve equations.

Subtraction Property of Equality

For all real numbers a, b, and c, if $a = b$, then $a - c = b - c$.

Addition Property of Equality

For all real numbers a, b, and c, if $a = b$, then $a + c = b + c$.

When you add or subtract the same quantity from both sides of an equation, you still have equality.

We will introduce the Subtraction Property of Equality by modeling equations with envelopes and counters. (Figure .1) models the equation $x + 3 = 8$.

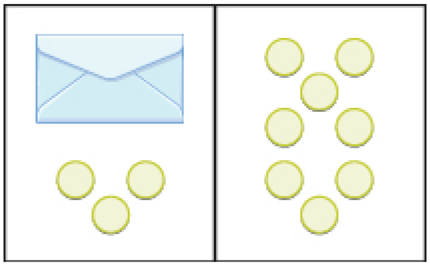


Figure .1

The goal is to isolate the variable on one side of the equation. So we ‘took away’ 3 from both sides of the equation and found the solution $x = 5$.

Some people picture a balance scale, as in (Figure .2), when they solve equations.

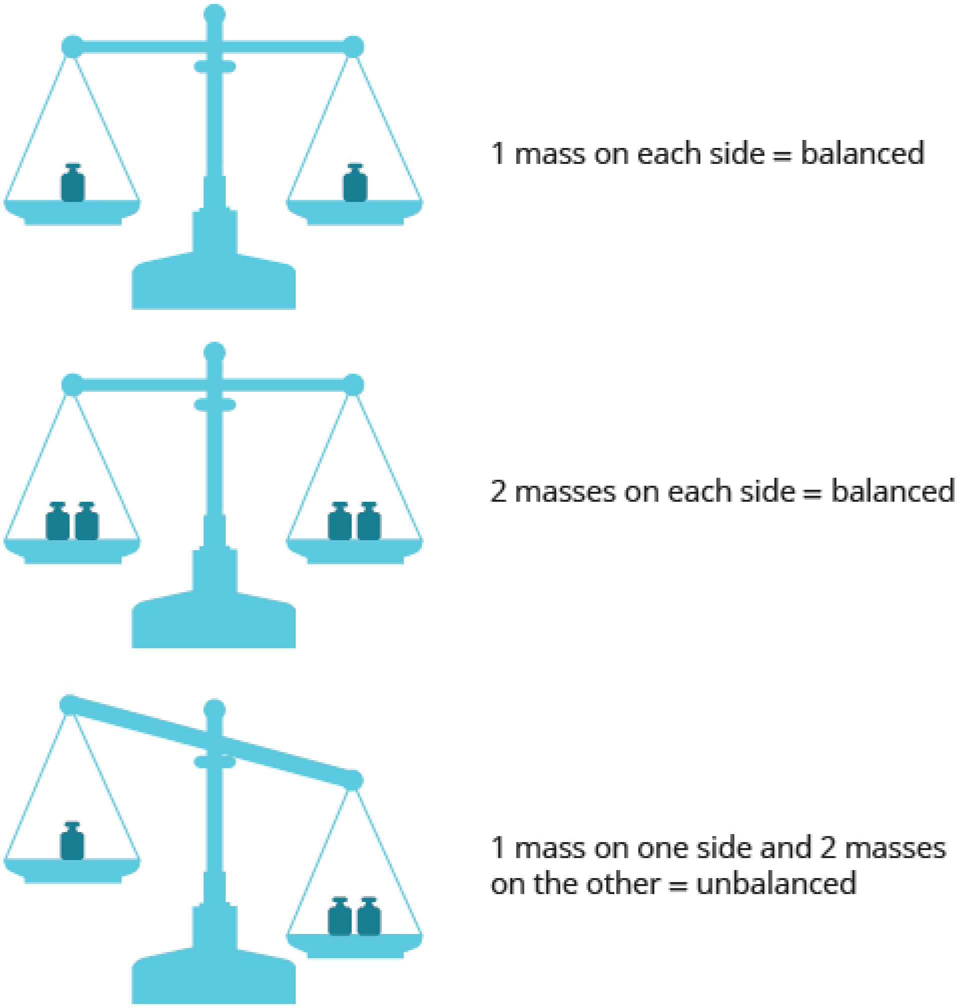


Figure .2

The quantities on both sides of the equal sign in an equation are equal, or balanced. Just as with the balance scale, whatever you do to one side of the equation you must also do to the other to keep it balanced.

Let's see how to use Subtraction and Addition Properties of Equality to solve equations. We need to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

EXAMPLE 2

Solve: $x + 11 = -3$.

Solution

To isolate x, we undo the addition of 11 by using the Subtraction Property of Equality.

		$x - 11 = -3$
Subtract 11 from each side to “undo” the addition.		$x + 11 - 11 = -3 - 11$
Simplify.		$x = -14$
Check:	$x - 11 = -3$	
Substitute $x = -14$.	$-14 + 11 \overset{?}{=} -3$	
	$-3 = -3$ ✓	

Since $x = -14$ makes $x + 11 = -3$ a true statement, we know that it is a solution to the equation.

TRY IT 2.1

Solve: $x + 9 = -7$.

Show answer
$x = -16$

TRY IT 2.2

Solve: $x + 16 = -4$.

Show answer
$x = -20$

In the original equation in the previous example, 11 was added to the x, so we subtracted 11 to ‘undo’ the addition. In the next example, we will need to ‘undo’ subtraction by using the Addition Property of Equality.

EXAMPLE 3

Solve: $m - 4 = -5$.

Solution

		$m + 4 = -5$
Add 4 to each side to “undo” the subtraction.		$m + 4 - 4 = -5 - 4$
Simplify.		$m = -1$
Check:	$m + 4 = -5$	
Substitute $m = -1$.	$-1 + 4 \overset{?}{=} -5$	
	$-5 = -5$ ✓	
		The solution to $m - 4 = -5$ is $m = -1$.

TRY IT 3.1

Solve: $n - 6 = -7$.

Show answer
−1

TRY IT 3.2

Solve: $x - 5 = -9$.

Show answer
−4

Now let’s solve equations with fractions.

EXAMPLE 4

Solve: $n - \frac{3}{8} = \frac{1}{2}$.

Solution

		$n-\frac{3}{8}=\frac{1}{2}$
Use the Addition Property of Equality.		$n-\frac{3}{8}+\frac{3}{8}=\frac{1}{2}+\frac{3}{8}$
Find the LCD to add the fractions on the right.		$n-\frac{3}{8}+\frac{3}{8}=\frac{4}{8}+\frac{3}{8}$
Simplify		$n=\frac{7}{8}$
Check:	$n-\frac{3}{8}=\frac{1}{2}$	
Substitute $n=\frac{7}{8}$.	$\frac{7}{8}-\frac{3}{8}\overset{?}{=}\frac{1}{2}$	
Subtract.	$\frac{4}{8}\overset{?}{=}\frac{1}{2}$	
Simplify.	$\frac{1}{2}=\frac{1}{2}$ ✓	
The solution checks.		

TRY IT 4.1

Solve: $p-\frac{1}{3}=\frac{5}{6}$.

Show answer

$p=\frac{7}{6}$

TRY IT 4.2

Solve: $q-\frac{1}{2}=\frac{1}{6}$.

Show answer

$q=\frac{2}{3}$

Let’s solve equations that contained decimals.

EXAMPLE 5

Solve $a - 3.7 = 4.3$.

Solution

		$a - 3.7 = 4.3$
Use the Addition Property of Equality.		$a - 3.7 + 3.7 = 4.3 + 3.7$
Add.		$a = 8$
Check:	$a - 3.7 = 4.3$	
Substitute $a = 8$.	$8 - 3.7 \overset{?}{=} 4.3$	
Simplify.	$4.3 = 4.3$ ✓	
The solution checks.		

TRY IT 5.1

Solve: $b - 2.8 = 3.6$.

Show answer
$b = 6.4$

TRY IT 5.2

Solve: $c - 6.9 = 7.1$.

Show answer
$c = 14$

Solve Equations That Need to Be Simplified

In the examples up to this point, we have been able to isolate the variable with just one operation. Many of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality. You should always simplify as much as possible before trying to isolate the variable.

EXAMPLE 6

Solve: $3x - 7 - 2x - 4 = 1$.

Solution

The left side of the equation has an expression that we should simplify before trying to isolate the variable.

	$3x - 7 - 2x - 4 = 1$
Rearrange the terms, using the Commutative Property of Addition.	$3x - 2x - 7 - 4 = 1$
Combine like terms.	$x - 11 = 1$
Add 11 to both sides to isolate x.	$x - 11 + 11 = 1 + 11$
Simplify.	$x = 12$
Check. Substitute $x = 12$ into the original equation. $3x - 7 - 2x - 4 = 1$ $3(12) - 7 - 2(12) - 4 = 1$ $36 - 7 - 24 - 4 = 1$ $29 - 24 - 4 = 1$ $5 - 4 = 1$ $1 = 1$ ✓	

The solution checks.

TRY IT 6.1

Solve: $8y - 4 - 7y - 7 = 4$.

Show answer

$y = 15$

TRY IT 6.2

Solve: $6z + 5 - 5z - 4 = 3$.

Show answer

$z = 2$

EXAMPLE 7

Solve: $3\left(n-4\right)-2n=-3.$

Solution

The left side of the equation has an expression that we should simplify.

	$3(n-4)-2n=-3$
Distribute on the left.	$3n-12-2n=-3$
Use the Commutative Property to rearrange terms.	$3n-2n-12=-3$
Combine like terms.	$n-12=-3$
Isolate *n* using the Addition Property of Equality.	$n-12+12=-3+12$
Simplify.	$n=9$
Check. Substitute $n=9$ into the original equation. $3(n-4)-2n=-3$ $3(9-4)-2\cdot 9=-3$ $3(5)-18=-3$ $15-18=-3$ $-3=-3\ \checkmark$ The solution checks.	

TRY IT 7.1

Solve: $5\left(p-3\right)-4p=-10.$

Show answer

$p=5$

TRY IT 7.2

Solve: $4\left(q+2\right)-3q=-8.$

Show answer

$q=-16$

EXAMPLE 8

Solve: $2\left(3k-1\right)-5k=-2-7.$

Solution

Both sides of the equation have expressions that we should simplify before we isolate the variable.

	$2(3k-1)-5k=-2-7$
Distribute on the left, subtract on the right.	$6k-2-5k=-9$
Use the Commutative Property of Addition.	$6k-5k-2=-9$
Combine like terms.	$k-2=-9$
Undo subtraction by using the Addition Property of Equality.	$k-2+2=-9+2$
Simplify.	$k=-7$
Check. Let $k=-7.$	$2(3k-1)-5k=-2-7$ $2(3(-7)-1)-5(-7)=-2-7$ $2(-21-1)-5(-7)=-9$ $2(-22)+35=-9$ $-44+35=-9$ $-9=-9$ ✓
The solution checks.	

TRY IT 8.1

Solve: $4\left(2h-3\right)-7h=-6-7.$

Show answer

$h=-1$

TRY IT 8.2

Solve: $2\left(5x+2\right)-9x=-2+7.$

Show answer

$x = 1$

Translate an Equation and Solve

Previously, we translated word sentences into equations. The first step is to look for the word (or words) that translate(s) to the equal sign. The list below reminds us of some of the words that translate to the equal sign (=):

- is
- is equal to
- is the same as
- the result is
- gives
- was
- will be

Let's review the steps we used to translate a sentence into an equation.

HOW TO: Translate a word sentence to an algebraic equation.

1. Locate the "equals" word(s). Translate to an equal sign.
2. Translate the words to the left of the "equals" word(s) into an algebraic expression.
3. Translate the words to the right of the "equals" word(s) into an algebraic expression.

Now we are ready to try an example.

EXAMPLE 9

Translate and solve: five more than x is equal to 26.

Solution

Translate.	$\underbrace{\text{Five more than } x}_{x+5}\quad\underbrace{\text{is equal to}}_{=}\quad\underbrace{26}_{26}$
Subtract 5 from both sides.	$x+5-5=26-5$
Simplify.	$x=21$
Check: Is 26 five more than 21?	$21+5\overset{?}{=}26$ $26=26\checkmark$ The solution checks.

TRY IT 9.1

Translate and solve: Eleven more than x is equal to 41.

Show answer
$x + 11 = 41$; $x = 30$

TRY IT 9.2

Translate and solve: Twelve less than y is equal to 51.

Show answer
$y - 12 = 51$; $y = 63$

EXAMPLE 10

Translate and solve: The difference of $5p$ and $4p$ is 23.

Solution

Translate.	$\underbrace{\text{The difference of } 5p \text{ and } 4p}_{5p-4p}\quad\underbrace{\text{is}}_{=}\quad\underbrace{23}_{23}$
Simplify.	$p=23$
Check.	$5p-4p=23$ $5(23)-4(23)\overset{?}{=}23$ $115-22\overset{?}{=}23$ $23=23$ ✓
The solution checks.	

TRY IT 10.1

Translate and solve: The difference of $4x$ and $3x$ is 14.

Show answer
$4x-3x=14$; $x=14$

TRY IT 10.2

Translate and solve: The difference of $7a$ and $6a$ is -8.

Show answer
$7a-6a=-8$; $a=-8$

Translate and Solve Applications

In most of the application problems we solved earlier, we were able to find the quantity we were looking for by simplifying an algebraic expression. Now we will be using equations to solve application problems. We’ll start by restating the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for.

EXAMPLE 11

The Robles family has two dogs, Buster and Chandler. Together, they weigh 71 pounds. Chandler weighs 28 pounds. How much does Buster weigh?

Solution

Read the problem carefully.	
Identify what you are asked to find, and choose a variable to represent it.	How much does Buster weigh? Let $b =$ Buster’s weight
Write a sentence that gives the information to find it.	Buster’s weight plus Chandler’s weight equals 71 pounds.
We will restate the problem, and then include the given information.	Buster’s weight plus 28 equals 71.
Translate the sentence into an equation, using the variable b.	$b + 28 = 71$
Solve the equation using good algebraic techniques.	$b + 28 - 28 = 71 - 28$ $b = 43$
Check the answer in the problem and make sure it makes sense.	Is 43 pounds a reasonable weight for a dog? Yes. Does Buster’s weight plus Chandler’s weight equal 71 pounds?
	$43 + 28 \overset{?}{=} 71$
	$71 = 71$?
Write a complete sentence that answers the question, “How much does Buster weigh?”	Buster weighs 43 pounds

TRY IT 11.1

Translate into an algebraic equation and solve: The Pappas family has two cats, Zeus and Athena. Together, they weigh 13 pounds. Zeus weighs 6 pounds. How much does Athena weigh?

Show answer

$a + 6 = 13$; Athena weighs 7 pounds.

TRY IT 11.2

Translate into an algebraic equation and solve: Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Show answer

$26 + h = 68$; Henry has 42 books.

Devise a Problem-Solving Strategy

1. Read the problem. Make sure you understand all the words and ideas.
2. Identify what you are looking for.
3. Name what you are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

EXAMPLE 12

Shayla paid $\$24,575$ for her new car. This was $\$875$ less than the sticker price. What was the sticker price of the car?

Solution

What are you asked to find?	“What was the sticker price of the car?”
Assign a variable.	Let s = the sticker price of the car.
Write a sentence that gives the information to find it.	$24,575 is $875 less than the sticker price $24,575 is $875 less than s
Translate into an equation.	$24{,}575 = s - 875$
Solve.	$24{,}575 + 875 = s - 875 + 875$ $24{,}575 = s$
Check:	Is $875 less than $25,450 equal to $24,575? $25,450 - 875 \overset{?}{=} 24,575$ $24,575 = 24,575$?
Write a sentence that answers the question.	The sticker price was $25,450.

TRY IT 12.1

Translate into an algebraic equation and solve: Eddie paid $\$19{,}875$ for his new car. This was $\$1{,}025$ less than the sticker price. What was the sticker price of the car?

Show answer

19,875 = s − 1025; the sticker price is $20,900.

TRY IT 12.2

Translate into an algebraic equation and solve: The admission price for the movies during the day is $\$7.75$. This is $\$3.25$ less than the price at night. How much does the movie cost at night?

Show answer

7.75 = n − 3.25; the price at night is $11.00.

Key Concepts

- **Determine whether a number is a solution to an equation.**
    1. Substitute the number for the variable in the equation.

2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true.

If it is true, the number is a solution.
If it is not true, the number is not a solution.

- **Subtraction and Addition Properties of Equality**
  - **Subtraction Property of Equality**
    For all real numbers a, b, and c,
    if $a = b$ then $a - c = b - c$.
  - **Addition Property of Equality**
    For all real numbers a, b, and c,
    if $a = b$ then $a + c = b + c$.
- **Translate a word sentence to an algebraic equation.**
  1. Locate the “equals” word(s). Translate to an equal sign.
  2. Translate the words to the left of the “equals” word(s) into an algebraic expression.
  3. Translate the words to the right of the “equals” word(s) into an algebraic expression.
- **Problem-solving strategy**
  1. Read the problem. Make sure you understand all the words and ideas.
  2. Identify what you are looking for.
  3. Name what you are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.

Glossary

solution of an equation
A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

Practice Makes Perfect

Solve Equations Using the Subtraction and Addition Properties of Equality

In the following exercises, determine whether the given value is a solution to the equation.

1. Is $y=\frac{1}{3}$ a solution of $4y+2=10y?$	2. Is $x=\frac{3}{4}$ a solution of $5x+3=9x?$
3. Is $u=-\frac{1}{2}$ a solution of $8u-1=6u?$	4. Is $v=-\frac{1}{3}$ a solution of $9v-2=3v?$

In the following exercises, solve each equation.

5. $x+7=12$	6. $y+5=-6$
7. $b+\frac{1}{4}=\frac{3}{4}$	8. $a+\frac{2}{5}=\frac{4}{5}$
9. $p+2.4=-9.3$	10. $m+7.9=11.6$
11. $a-3=7$	12. $m-8=-20$
13. $x-\frac{1}{3}=2$	14. $x-\frac{1}{5}=4$
15. $y-3.8=10$	16. $y-7.2=5$
17. $x-15=-42$	18. $z+5.2=-8.5$
19. $q+\frac{3}{4}=\frac{1}{2}$	20. $p-\frac{2}{5}=\frac{2}{3}$

Solve Equations that Need to be Simplified

In the following exercises, solve each equation.

21. $m+6-8=15$	22. $c+3-10=18$
23. $6x+8-5x+16=32$	24. $9x+5-8x+14=20$
25. $-8n-17+9n-4=-41$	26. $-6x-11+7x-5=-16$
27. $4\left(y-2\right)-3y=-6$	28. $3\left(y-5\right)-2y=-7$
29. $5\left(w+2.2\right)-4w=9.3$	30. $8\left(u+1.5\right)-7u=4.9$
31. $-8\left(x-1\right)+9x=-3+9$	32. $-5\left(y-2\right)+6y=-7+4$
33. $2\left(8m+3\right)-15m-4=3-5$	34. $3\left(5n-1\right)-14n+9=1-2$
35. $-\left(k+7\right)+2k+8=7$	36. $-\left(j+2\right)+2j-1=5$
37. $8c-7\left(c-3\right)+4=-16$	38. $6a-5\left(a-2\right)+9=-11$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

39. The sum of x and -5 is 33.	40.Five more than x is equal to 21.
41.Three less than y is -19.	42. Ten less than m is -14.
43. Eight more than p is equal to 52.	44. The sum of y and -3 is 40.
45. The difference of $5c$ and $4c$ is 60.	46. The difference of $9x$ and $8x$ is 17.
47. The difference of f and $\frac{1}{3}$ is $\frac{1}{12}$.	48. The difference of n and $\frac{1}{6}$ is $\frac{1}{2}$.
49. The sum of $-9m$ and $10m$ is -25.	50. The sum of $-4n$ and $5n$ is -32.

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

51.Jeff read a total of 54 pages in his English and Psychology textbooks. He read 41 pages in his English textbook. How many pages did he read in his Psychology textbook?

52. Pilar drove from home to school and then to her aunt’s house, a total of 18 miles. The distance from Pilar’s house to school is 7 miles. What is the distance from school to her aunt’s house?

53. Eva’s daughter is 5 years younger than her son. Eva’s son is 12 years old. How old is her daughter?

54. Pablo’s father is 3 years older than his mother. Pablo’s mother is 42 years old. How old is his father?

55. For a family birthday dinner, Celeste bought a turkey that weighed 5 pounds less than the one she bought for Thanksgiving. The birthday dinner turkey weighed 16 pounds. How much did the Thanksgiving turkey weigh?

56. Allie weighs 8 pounds less than her twin sister Lorrie. Allie weighs 124 pounds. How much does Lorrie weigh?

57. Connor’s temperature was 0.7 degrees higher this morning than it had been last night. His temperature this morning was 101.2 degrees. What was his temperature last night?

58. The nurse reported that Tricia’s daughter had gained 4.2 pounds since her last checkup and now weighs 31.6 pounds. How much did Tricia’s daughter weigh at her last checkup?

59. Ron’s paycheck this week was $\$17.43$ less than his paycheck last week. His paycheck this week was $\$103.76$. How much was Ron’s paycheck last week?

60. Melissa’s math book cost $\$22.85$ less than her art book cost. Her math book cost $\$93.75$. How much did her art book cost?

Everyday Math

61.Construction Miguel wants to drill a hole for a $\frac{5}{8}\text{-inch}$ screw. The screw should be $\frac{1}{12}$ inch larger than the hole. Let d equal the size of the hole he should drill. Solve the equation $d+\frac{1}{12}=\frac{5}{8}$ to see what size the hole should be.

Baking 62. Kelsey needs $\frac{2}{3}$ cup of sugar for the cookie recipe she wants to make. She only has $\frac{1}{4}$ cup of sugar and will borrow the rest from her neighbour. Let s equal the amount of sugar she will borrow. Solve the equation $\frac{1}{4}+s=\frac{2}{3}$ to find the amount of sugar she should ask to borrow.

Writing Exercises

63. Write a word sentence that translates the equation $y-18=41$ and then make up an application that uses this equation in its solution.

64. Is -18 a solution to the equation $3x=16-5x?$ How do you know?

Answers

1. yes	3. no	5. $x = 5$
7. $b = \frac{1}{2}$	9. $p = -11.7$	11. $a = 10$
13. $x = \frac{7}{3}$	15. $y = 13.8$	17. $x = -27$
19. $q = -\frac{1}{4}$	21. 17	23. 8
25. −20	27. 2	29. −1.7
31. −2	33. −4	35. 6
37. −41	39. $x + (-5) = 33$; $x = 38$	41.$y - 3 = -19$; $y = -16$
43. $p + 8 = 52$; $p = 44$	45. $5c - 4c = 60$; 60	47. $f - \frac{1}{3} = \frac{1}{12}$; $\frac{5}{12}$
49. $-9m + 10m = -25$; $m = -25$	51. Let p equal the number of pages read in the Psychology book 41 + p = 54. Jeff read pages in his Psychology book.	53. Let d equal the daughter's age. d = 12 − 5. Eva's daughter's age is 7 years old.
55. 21 pounds	57. 100.5 degrees	59. $121.19
61. $d = \frac{13}{24}$	63. Answers will vary.	

Attributions

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5.2 Solve Equations Using the Division and Multiplication Properties of Equality

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using the Division and Multiplication Properties of Equality
- Solve equations that need to be simplified

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

Let's look at our puzzle again with the envelopes and counters in (Figure 1).

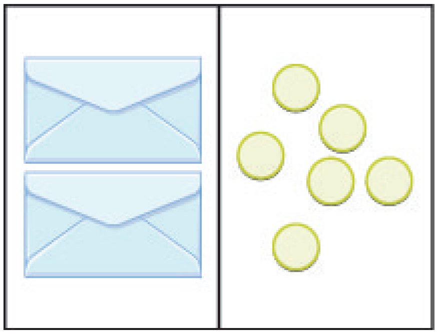

Figure .1

In the illustration there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are "hidden" in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in (Figure 2)? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters.

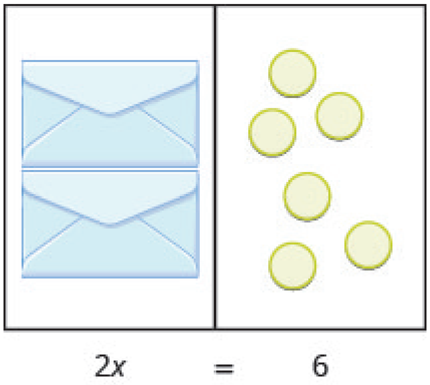

Figure .2

	$2x = 6$
If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2} = \frac{6}{2}$
we get:	$x = 3$

We found that each envelope contains 3 counters. Does this check? We know $2 \times 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the Division Property of Equality.

Division and Multiplication Properties of Equality

Division Property of Equality: For all real numbers a, b, c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.

Multiplication Property of Equality: For all real numbers a, b, c, if $a = b$, then $ac = bc$.

When you divide or multiply both sides of an equation by the same quantity, you still have equality.

Let's review how these properties of equality can be applied in order to solve equations. Remember, the goal is to 'undo' the operation on the variable. In the example below the variable is multiplied by 4, so we will divide both sides by 4 to 'undo' the multiplication.

EXAMPLE 1

Solve: $4x = -28$.

Solution

We use the Division Property of Equality to divide both sides by 4.

	$4x = -28$
Divide both sides by 4 to undo the multiplication.	$\frac{4x}{4} = \frac{-28}{4}$
Simplify.	$x = -7$
Check your answer. Let $x = -7$.	$4x = -28$ $4(-7) \overset{?}{=} -28$ $-28 = -28$ ✓

Since this is a true statement, $x = -7$ is a solution to $4x = -28$.

TRY IT 1.1

Solve: $3y = -48$.

Show answer

$y = -16$

TRY IT 1.2

Solve: $4z = -52$.

Show answer

$z = -13$

In the previous example, to 'undo' multiplication, we divided. How do you think we 'undo' division?

EXAMPLE 2

Solve: $\frac{a}{-7} = -42$.

Solution

Here a is divided by -7. We can multiply both sides by -7 to isolate a.

	$\frac{a}{-7} = -42$
Multiply both sides by -7.	$-7\left(\frac{a}{-7}\right) = -7(-42)$ $\frac{-7a}{-7} = 294$
Simplify.	$a = 294$
Check your answer. Let $a = 294$.	
$\frac{a}{-7} = -42$	
$\frac{294}{-7} \stackrel{?}{=} -42$	
$-42 = -42$ ✓	

TRY IT 2.1

Solve: $\frac{b}{-6} = -24$.

Show answer
$b = 144$

TRY IT 2.2

Solve: $\frac{c}{-8} = -16$.

Show answer
$c = 128$

EXAMPLE 3

Solve: $-r = 2$.

Solution

Remember $-r$ is equivalent to $-1r$.

		$-r=2$
Rewrite $-r$ as $-1r$.		$-1r=2$
Divide both sides by -1.		$\frac{-1r}{-1}=\frac{2}{-1}$
		$r=-2$
Check.	$-r=2$	
Substitute $r=-2$	$-(-2)\overset{?}{=}2$	
Simplify.	$2=2\checkmark$	

We see that there are two other ways to solve $-r=2$.

We could multiply both sides by -1.

We could take the opposite of both sides.

TRY IT 3.1

Solve: $-k=8$.

Show answer

$k=-8$

TRY IT 3.2

Solve: $-g=3$.

Show answer

$g=-3$

EXAMPLE 4

Solve: $\frac{2}{3}x=18$.

Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{2}{3}$.

	$\frac{2}{3}x = 18$
Multiply by the reciprocal of $\frac{2}{3}$.	$\frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot 18$
Reciprocals multiply to one.	$1x = \frac{3}{2} \cdot \frac{18}{1}$
Multiply.	$x = 27$
Check your answer. Let $x = 27$	$\frac{2}{3}x = 18$ $\frac{2}{3} \cdot 27 \overset{?}{=} 18$
	$18 = 18$ ✓

Notice that we could have divided both sides of the equation $\frac{2}{3}x = 18$ by $\frac{2}{3}$ to isolate x. While this would work, multiplying by the reciprocal requires fewer steps.

TRY IT 4.1

Solve: $\frac{2}{5}n = 14$.

Show answer
$n = 35$

TRY IT 4.2

Solve: $\frac{5}{6}y = 15$.

Show answer
$y = 18$

Solve Equations That Need to be Simplified

Many equations start out more complicated than the ones we’ve just solved. First, we need to simplify both sides of the equation as much as possible

EXAMPLE 5

Solve: $8x + 9x - 5x = -3 + 15$.

Solution

Start by combining like terms to simplify each side.

	$8x + 9x - 5x = -3 + 15$
Combine like terms.	$12x = 12$
Divide both sides by 12 to isolate x.	$\frac{12x}{12} = \frac{12}{12}$
Simplify.	$x = 1$
Check your answer. Let $x = 1$	$8x + 9x - 5x = -3 + 15$ $8 \cdot 1 + 9 \cdot 1 - 5 \cdot 1 \overset{?}{=} -3 + 15$ $8 + 9 - 5 \overset{?}{=} -3 + 15$ $12 = 12 ✓$

TRY IT 5.1

Solve: $7x + 6x - 4x = -8 + 26$.

Show answer

$x = 2$

TRY IT 5.2

Solve: $11n - 3n - 6n = 7 - 17$.

Show answer

$n = -5$

EXAMPLE 6

Solve: $11 - 20 = 17y - 8y - 6y$.

Solution

Simplify each side by combining like terms.

	$11 - 20 = 17y - 8y - 6y$
Simplify each side.	$-9 = 3y$
Divide both sides by 3 to isolate y.	$\frac{-9}{3} = \frac{3y}{3}$
Simplify.	$-3 = y$
Check your answer. Let $y = -3$	
$11 - 20 = 17y - 8y - 6y$	
$11 - 20 \overset{?}{=} 17(-3) - 8(-3) - 6(-3)$	
$11 - 20 \overset{?}{=} -51 + 24 + 18$	
$-9 = -9$ ✓	

Notice that the variable ended up on the right side of the equal sign when we solved the equation. You may prefer to take one more step to write the solution with the variable on the left side of the equal sign.

TRY IT 6.1

Solve: $18 - 27 = 15c - 9c - 3c$.

Show answer
$c = -3$

TRY IT 6.2

Solve: $18 - 22 = 12x - x - 4x$.

Show answer
$x = -\frac{4}{7}$

EXAMPLE 7

Solve: $-3\left(n-2\right)-6=21$.

Solution

Remember—always simplify each side first.

	$-3(n-2)-6=21$
Distribute.	$-3n+6-6=21$
Simplify.	$-3n=21$
Divide both sides by -3 to isolate n.	$\frac{-3n}{-3}=\frac{21}{-3}$ $n=-7$
Check your answer. Let $n=-7$.	$-3(n-2)-6=21$ $-3(-7-2)-6\overset{?}{=}21$ $-3(-9)-6\overset{?}{=}21$ $27-6\overset{?}{=}21$ $21=21$ ✓

TRY IT 7.1

Solve: $-4\left(n-2\right)-8=24$.

Show answer
n = −6

TRY IT 7.2

Solve: $-6\left(n-2\right)-12=30$.

Show answer
n = −5

Key Concepts

- **Division and Multiplication Properties of Equality**
    - **Division Property of Equality:** For all real numbers *a*, *b*, *c*, and $c\ne 0$, if $a=b$, then $ac=bc$.
    - **Multiplication Property of Equality:** For all real numbers *a*, *b*, *c*, if $a=b$, then $ac=bc$.

Practice Makes Perfect

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation for the variable using the Division Property of Equality and check the solution.

1. $7p=63$	2. $8x=32$
3. $-9x=-27$	4. $-5c=55$
5. $-72=12y$	6. $-90=6y$
7. $-8m=-56$	8. $-16p=-64$
9. $0.75a=11.25$	10. $0.25z=3.25$
11. $4x=0$	12. $-3x=0$

In the following exercises, solve each equation for the variable using the Multiplication Property of Equality and check the solution.

13. $\frac{z}{2} = 14$	14. $\frac{x}{4} = 15$
15. $\frac{c}{-3} = -12$	16. $-20 = \frac{q}{-5}$
17. $\frac{q}{6} = -8$	18. $\frac{y}{9} = -6$
19. $-4 = \frac{p}{-20}$	20. $\frac{m}{-12} = 5$
21. $\frac{3}{5}r = 15$	22. $\frac{2}{3}y = 18$
23. $24 = -\frac{3}{4}x$	24. $-\frac{5}{8}w = 40$
25. $-\frac{1}{3}q = -\frac{5}{6}$	26. $-\frac{2}{5} = \frac{1}{10}a$

Solve Equations That Need to be Simplified

In the following exercises, solve the equation.

27. $6y - 3y + 12y = -43 + 28$	28. $8a + 3a - 6a = -17 + 27$
29. $-5m + 7m - 8m = -6 + 36$	30. $-9x - 9x + 2x = 50 - 2$
31. $-18 - 7 = 5t - 9t - 6t$	32. $100 - 16 = 4p - 10p - p$
33. $\frac{5}{12}q + \frac{1}{2}q = 25 - 3$	34. $\frac{7}{8}n - \frac{3}{4}n = 9 + 2$
35. $0.05p - 0.01p = 2 + 0.24$	36. $0.25d + 0.10d = 6 - 0.75$

Everyday Math

37. Teaching Connie's kindergarten class has 24 children. She wants them to get into 4 equal groups. Find the number of children in each group, g, by solving the equation $4g = 24$.	**38. Balloons** Ramona bought 18 balloons for a party. She wants to make 3 equal bunches. Find the number of balloons in each bunch, b, by solving the equation $3b = 18$.
39. Unit price Nishant paid $\$12.96$ for a pack of 12 juice bottles. Find the price of each bottle, b, by solving the equation $12b = 12.96$.	**40. Ticket price** Daria paid $\$36.25$ for 5 children's tickets at the ice skating rink. Find the price of each ticket, p, by solving the equation $5p = 36.25$.
41. Fabric The drill team used 14 yards of fabric to make flags for one-third of the members. Find how much fabric, f, they would need to make flags for the whole team by solving the equation $\frac{1}{3}f = 14$.	**42. Fuel economy** Tania's SUV gets half as many miles per gallon (mpg) as her husband's hybrid car. The SUV gets $18\ \text{mpg}$. Find the miles per gallons, m, of the hybrid car, by solving the equation $\frac{1}{2}m = 18$.

Writing Exercises

43. Emiliano thinks $x = 40$ is the solution to the equation $\frac{1}{2}x = 80$. Explain why he is wrong.	44. Frida started to solve the equation $-3x = 36$ by adding 3 to both sides. Explain why Frida's method will result in the correct solution.

Answers

1. 9	3. 3	5. −6
7. 7	9. 15	11. 0
13. 28	15. 36	17. −48
19. 80	21. 25	23. −32
25. 5/2	27. y = −1	29. m = −5
31. $t = \frac{5}{2}$	33. q = 24	35. p = 56
37. 6 children	39. $1.08	41. 42 yards
43. Answer will vary.		

Attributions

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5.3 Solve Equations with Variables and Constants on Both Sides

Learning Objectives

By the end of this section, you will be able to:

- Solve an equation with constants on both sides
- Solve an equation with variables on both sides
- Solve an equation with variables and constants on both sides
- Solve equations using a general strategy

Solve an Equation with Constants on Both Sides

You may have noticed that in all the equations we have solved so far, all the variable terms were on only one side of the equation with the constants on the other side. This does not happen all the time—so now we'll see how to solve equations where the variable terms and/or constant terms are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the variable side, and the other side of the equation to be the constant side. Then, we will use the Subtraction and Addition Properties of Equality, step by step, to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that started with variables and constants on both sides into the form $ax = b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

EXAMPLE 1

Solve: $4x + 6 = -14.$

Solution

In this equation, the variable is only on the left side. It makes sense to call the left side the variable side. Therefore, the right side will be the constant side. We'll write the labels above the equation to help us remember what goes where.

		$\overset{\text{variable}}{4x + 6} \overset{\text{constant}}{= -14}$
Since the left side is the variable side, the 6 is out of place. We must “undo” adding 6 by subtracting 6, and to keep the equality we must subtract 6 from both sides. Use the Subtraction Property of Equality.		$4x + 6 - 6 = -14 - 6$
Simplify.		$4x = -20$
Now all the xs are on the left and the constant on the right.		
Use the Division Property of Equality.		$\frac{4x}{4} = \frac{-20}{4}$
Simplify.		$x = -5$
Check:	$4x + 6 = -14$	
Let $x = -5$.	$4(-5) + 6 = -14$	
	$-20 + 6 = -14$	
	$-14 = -14$ ✓	

TRY IT 1.1

Solve: $3x + 4 = -8$.

Show answer
$x = -4$

TRY IT 1.2

Solve: $5a + 3 = -37$.

Show answer
$a = -8$

EXAMPLE 1.2

Solve: $2y - 7 = 15$.

Solution

Notice that the variable is only on the left side of the equation, so this will be the variable side and the right side will be the constant side. Since the left side is the variable side, the 7 is out of place. It is subtracted from the $2y$, so to ‘undo’ subtraction, add 7 to both sides.

		$\overset{\text{variable}}{2y-7}=\overset{\text{constant}}{15}$
Add 7 to both sides.		$2y-7+7=15+7$
Simplify.		$2y=22$
The variables are now on one side and the constants on the other.		
Divide both sides by 2.		$\frac{2y}{2}=\frac{22}{2}$
Simplify.		$y=11$
Check:	$2y-7=15$	
Substitute: $y=11$.	$2\cdot 11-7\stackrel{?}{=}15$	
	$22-7\stackrel{?}{=}15$	
	$15=15\checkmark$	

TRY IT 2.1

Solve: $5y-9=16$.

Show answer
y = 5

TRY IT 2.2

Solve: $3m-8=19$.

Show answer
m = 9

Solve an Equation with Variables on Both Sides

What if there are variables on both sides of the equation? We will start like we did above—choosing a variable side and a constant side, and then use the Subtraction and Addition Properties of Equality to collect all variables on one side and all constants on the other side. Remember, what you do to the left side of the equation, you must do to the right side too.

EXAMPLE 3

Solve: $5x = 4x + 7$.

Solution

Here the variable, x, is on both sides, but the constants appear only on the right side, so let's make the right side the "constant" side. Then the left side will be the "variable" side.

		$\overset{\text{variable}}{5x} = \overset{\text{constant}}{4x + 7}$
We don't want any variables on the right, so subtract the $4x$.		$5x - 4x = 4x - 4x + 7$
Simplify.		$x = 7$
We have all the variables on one side and the constants on the other. We have solved the equation.		
Check:	$5x = 4x + 7$	
Substitute 7 for x.	$5(7) \overset{?}{=} 4(7) + 7$	
	$35 \overset{?}{=} 28 + 7$	
	$35 = 35\ ✓$	

TRY IT 3.1

Solve: $6n = 5n + 10$.

Show answer

$n = 10$

TRY IT 3.2

Solve: $-6c = -7c + 1$.

Show answer
$c = 1$

EXAMPLE 4

Solve: $5y - 8 = 7y$.

Solution

The only constant, -8, is on the left side of the equation and variable, y, is on both sides. Let's leave the constant on the left and collect the variables to the right.

	$\overset{\text{constant}}{5y - 8} = \overset{\text{variable}}{7y}$
Subtract $5y$ from both sides.	$5y - 5y - 8 = 7y - 5y$
Simplify.	$-8 = 2y$
We have the variables on the right and the constants on the left. Divide both sides by 2.	$\frac{-8}{2} = \frac{2y}{2}$
Simplify.	$-4 = y$
Rewrite with the variable on the left.	$y = -4$
Check: Let $y = -4$.	
$5y - 8 = 7y$	
$5(-4) - 8 \overset{?}{=} 7(-4)$	
$-20 - 8 \overset{?}{=} -28$	
$-28 = -28$ ✓	

TRY IT 4.1

Solve: $3p - 14 = 5p$.

Show answer
$p = -7$

TRY IT 4.2

Solve: $8m + 9 = 5m$.

Show answer
$m = -3$

EXAMPLE 5

Solve: $7x = -x + 24$.

Solution

The only constant, 24, is on the right, so let the left side be the variable side.

	$\overset{\text{variable side}}{7x} = \overset{\text{constant side}}{-x + 24}$
Remove the $-x$ from the right side by adding x to both sides.	$7x + x = -x + x + 24$
Simplify.	$8x = 24$
All the variables are on the left and the constants are on the right. Divide both sides by 8.	$\frac{8x}{8} = \frac{24}{8}$
Simplify.	$x = 3$
Check: Substitute $x = 3$.	
$7x = -x + 24$ $7(3) \overset{?}{=} -(3) + 24$ $21 = 21$ ✓	

TRY IT 5.1

Solve: $12j = -4j + 32$.

Show answer
$j = 2$

TRY IT 5.2

Solve: $8h = -4h + 12.$

Show answer

$h = 1$

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables *and* constants on both sides of the equation. As we did before, we'll collect the variable terms to one side and the constants to the other side.

EXAMPLE 6

Solve: $7x + 5 = 6x + 2.$

Solution

Start by choosing which side will be the variable side and which side will be the constant side. The variable terms are $7x$ and $6x$. Since 7 is greater than 6, make the left side the variable side and so the right side will be the constant side.

	$7x + 5 = 6x + 2$
Collect the variable terms to the left side by subtracting $6x$ from both sides.	$7x - 6x + 5 = 6x - 6x + 2$
Simplify.	$x + 5 = 2$
Now, collect the constants to the right side by subtracting 5 from both sides.	$x + 5 - 5 = 2 - 5$
Simplify.	$x = -3$
The solution is $x = -3$.	
Check: Let $x = -3$.	
$7x + 5 = 6x + 2$ $7(-3) + 5 \stackrel{?}{=} 6(-3) + 2$ $-21 + 5 \stackrel{?}{=} -18 + 2$ $-16 = -16$ ✓	

TRY IT 6.1

Solve: $12x+8=6x+2$.

Show answer
$x=-1$

TRY IT 6.2

Solve: $9y+4=7y+12$.

Show answer
$y=4$

We’ll summarize the steps we took so you can easily refer to them.

HOW TO: Solve an Equation with Variables and Constants on Both Sides

1. Choose one side to be the variable side and then the other will be the constant side.
2. Collect the variable terms to the variable side, using the Addition or Subtraction Property of Equality.
3. Collect the constants to the other side, using the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

It is a good idea to make the variable side the one in which the variable has the larger coefficient. This usually makes the arithmetic easier.

EXAMPLE 7

Solve: $6n-2=-3n+7$.

Solution

We have $6n$ on the left and $-3n$ on the right. Since $6 > -3$, make the left side the “variable” side.

	$6n - 2 = -3n + 7$
We don’t want variables on the right side—add $3n$ to both sides to leave only constants on the right.	$6n + 3n - 2 = -3n + 3n + 7$
Combine like terms.	$9n - 2 = 7$
We don’t want any constants on the left side, so add 2 to both sides.	$9n - 2 + 2 = 7 + 2$
Simplify.	$9n = 9$
The variable term is on the left and the constant term is on the right. To get the coefficient of n to be one, divide both sides by 9.	$\frac{9n}{9} = \frac{9}{9}$
Simplify.	$\text{n} = 1$
Check: Substitute 1 for n.	$6n - 2 = -3n + 7$ $6(1) - 2 \overset{?}{=} -3(1) + 7$ $4 = 4 \checkmark$

TRY IT 7.1

Solve: $8q - 5 = -4q + 7$.

Show answer

$q = 1$

TRY IT 7.2

Solve: $7n - 3 = n + 3$.

Show answer

$n = 1$

EXAMPLE 8

Solve: $2a - 7 = 5a + 8$.

Solution

This equation has $2a$ on the left and $5a$ on the right. Since $5 > 2$, make the right side the variable side and the left side the constant side.

	$2a - 7 = 5a + 8$
Subtract $2a$ from both sides to remove the variable term from the left.	$2a - 2a - 7 = 5a - 2a + 8$
Combine like terms.	$-7 = 3a + 8$
Subtract 8 from both sides to remove the constant from the right.	$-7 - 8 = 3a + 8 - 8$
Simplify.	$-15 = 3a$
Divide both sides by 3 to make 1 the coefficient of a.	$\frac{-15}{3} = \frac{3a}{3}$
Simplify.	$-5 = a$
Check: Let $a = -5$.	$2a - 7 = 5a + 8$ $2(-5) - 7 \overset{?}{=} 5(-5) + 8$ $-10 - 7 \overset{?}{=} -25 + 8$ $-17 = -17$ ✓

Note that we could have made the left side the variable side instead of the right side, but it would have led to a negative coefficient on the variable term. While we could work with the negative, there is less chance of error when working with positives. The strategy outlined above helps avoid the negatives!

TRY IT 8.1

Solve: $2a - 2 = 6a + 18$.

Show answer

$a = -5$

TRY IT 8.2

Solve: $4k - 1 = 7k + 17$.

Show answer

$k = -6$

To solve an equation with fractions, we still follow the same steps to get the solution.

EXAMPLE 9

Solve: $\frac{3}{2}x + 5 = \frac{1}{2}x - 3$.

Solution

Since $\frac{3}{2} > \frac{1}{2}$, make the left side the variable side and the right side the constant side.

	$\frac{3}{2}x + 5 = \frac{1}{2}x - 3$
Subtract $\frac{1}{2}x$ from both sides.	$\frac{3}{2}x - \frac{1}{2}x + 5 = \frac{1}{2}x - \frac{1}{2}x - 3$
Combine like terms.	$x + 5 = -3$
Subtract 5 from both sides.	$x + 5 - 5 = -3 - 5$
Simplify.	$x = -8$
Check: Let $x = -8$.	$\frac{3}{2}x + 5 = \frac{1}{2}x - 3$ $\frac{3}{2}(-8) + 5 \overset{?}{=} \frac{1}{2}(-8) - 3$ $-12 + 5 \overset{?}{=} -4 - 3$ $-7 = -7$ ✓

TRY IT 9.1

Solve: $\frac{7}{8}x - 12 = -\frac{1}{8}x - 2$.

Show answer

$x = 10$

TRY IT 9.2

Solve: $\frac{7}{6}y + 11 = \frac{1}{6}y + 8$.

Show answer

$y = -3$

We follow the same steps when the equation has decimals, too.

EXAMPLE 10

Solve: $3.4x + 4 = 1.6x - 5$.

Solution

Since $3.4 > 1.6$, make the left side the variable side and the right side the constant side.

	$3.4x + 4 = 1.6x - 5$
Subtract $1.6x$ from both sides.	$3.4x - 1.6x + 4 = 1.6x - 1.6x - 5$
Combine like terms.	$1.8x + 4 = -5$
Subtract 4 from both sides.	$1.8x + 4 - 4 = -5 - 4$
Simplify.	$1.8x = -9$
Use the Division Property of Equality.	$\frac{1.8x}{1.8} = \frac{-9}{1.8}$
Simplify.	$x = -5$
Check: Let $x = -5$.	$3.4x + 4 = 1.6x - 5$ $3.4(-5) + 4 \overset{?}{=} 1.6(-5) - 5$ $-17 + 4 \overset{?}{=} -8 - 5$ $-13 = -13$ ✓

TRY IT 10.1

Solve: $2.8x + 12 = -1.4x - 9$.

Show answer

$x = -5$

TRY IT 10.2

Solve: $3.6y + 8 = 1.2y - 4$.

Show answer

$y = -5$

Solve Equations Using a General Strategy

Each of the first few sections of this chapter has dealt with solving one specific form of a linear equation. It’s time now to lay out an overall strategy that can be used to solve *any* linear equation. We call this the *general strategy*. Some equations won’t require all the steps to solve, but many will. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

HOW TO: Use a General Strategy for Solving Linear Equations

1. Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
2. Collect all the variable terms to one side of the equation. Use the Addition or Subtraction Property of Equality.
3. Collect all the constant terms to the other side of the equation. Use the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable term to equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
5. Check the solution. Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 11

Solve: $3\left(x+2\right)=18.$

Solution

	$3(x+2)=18$
Simplify each side of the equation as much as possible. Use the Distributive Property.	$3x+6=18$
Collect all variable terms on one side of the equation—all xs are already on the left side.	
Collect constant terms on the other side of the equation. Subtract 6 from each side	$3x+6-6=18-6$
Simplify.	$3x=12$
Make the coefficient of the variable term equal to 1. Divide each side by 3.	$\frac{3x}{3}=\frac{12}{3}$
Simplify.	$x=4$
Check: Let $x=4$.	$3(x+2)=18$ $3(4+2)\overset{?}{=}18$ $3(6)\overset{?}{=}18$ $18=18$ ✓

TRY IT 11.1

Solve: $5(x+3)=35$.

Show answer

$x = 4$

TRY IT 11.2

Solve: $6(y-4)=-18$.

Show answer

$y = 1$

EXAMPLE 12

Solve: $-(x+5)=7$.

Solution

	$-(x+5)=7$
Simplify each side of the equation as much as possible by distributing. The only x term is on the left side, so all variable terms are on the left side of the equation.	$-x-5=7$
Add 5 to both sides to get all constant terms on the right side of the equation.	$-x-5+5=7+5$
Simplify.	$-x=12$
Make the coefficient of the variable term equal to 1 by multiplying both sides by -1.	$-1(-x)=-1(12)$
Simplify.	$x=-12$
Check: Let $x=-12$.	$-(x+5)=7$ $-(-12+5)\overset{?}{=}7$ $-(-7)\overset{?}{=}7$ $7=7$ ✓

TRY IT 12.1

Solve: $-(y+8)=-2.$

Show answer

$y=-6$

TRY IT 12.2

Solve: $-(z+4)=-12.$

Show answer

$z=8$

EXAMPLE 13

Solve: $4\left(x-2\right)+5=-3$.

Solution

	$4(x-2)+5=-3$
Simplify each side of the equation as much as possible. Distribute.	$4x-8+5=-3$
Combine like terms	$4x-3=-3$
The only x is on the left side, so all variable terms are on one side of the equation.	
Add 3 to both sides to get all constant terms on the other side of the equation.	$4x-3+3=-3+3$
Simplify.	$4x=0$
Make the coefficient of the variable term equal to 1 by dividing both sides by 4.	$\frac{4x}{4}=\frac{0}{4}$
Simplify.	$x=0$
Check: Let $x=0$.	$4(x-2)+5=-3$ $4(0-2)+5\overset{?}{=}-3$ $4(-2)+5\overset{?}{=}-3$ $-8+5\overset{?}{=}-3$ $-3=-3$ ✓

TRY IT 13.1

Solve: $2\left(a-4\right)+3=-1$.

Show answer

$a=2$

TRY IT 13.2

Solve: $7\left(n-3\right)-8=-15$.

Show answer

$n=2$

EXAMPLE 14

Solve: $8 - 2\left(3y + 5\right) = 0.$

Solution

Be careful when distributing the negative.

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$
Divide both sides by −6.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$
Check: Let $y = -\frac{1}{3}$.	$8 - 2(3y + 5) = 0$ $8 - 2\left[3\left(-\frac{1}{3}\right) + 5\right] = 0$ $8 - 2(-1 + 5) \overset{?}{=} 0$ $8 - 2(4) \overset{?}{=} 0$ $8 - 8 \overset{?}{=} 0$ $0 = 0$ ✓

TRY IT 14.1

Solve: $12 - 3\left(4j + 3\right) = -17.$

Show answer

$j = \frac{5}{3}$

TRY IT 14.2

Solve: $-6 - 8\left(k - 2\right) = -10.$

Show answer

$k = \frac{5}{2}$

EXAMPLE 15

Solve: $3\left(x-2\right)-5=4\left(2x+1\right)+5.$

Solution

	$3(x-2)-5=4(2x+1)+5$
Distribute.	$3x-6-5=8x+4+5$
Combine like terms.	$3x-11=8x+9$
Subtract $3x$ to get all the variables on the right since $8>3$.	$3x-3x-11=8x-3x+9$
Simplify.	$-11=5x+9$
Subtract 9 to get the constants on the left.	$-11-9=5x+9-9$
Simplify.	$-20=5x$
Divide by 5.	$\frac{-20}{5}=\frac{5x}{5}$
Simplify.	$-4=x$
Check: Substitute: $-4=x$.	$3(x-2)-5=4(2x+1)+5$ $3(-4-2)-5\overset{?}{=}4[2(-4)+1]+5$ $3(-6)-5\overset{?}{=}4(-8+1)+5$ $-18-5\overset{?}{=}4(-7)+5$ $-23\overset{?}{=}-28+5$ $-23=-23$ ✓

TRY IT 14.1

Solve: $6\left(p-3\right)-7=5\left(4p+3\right)-12.$

Show answer

$p=-2$

TRY IT 14.2

Solve: $8(q+1)-5=3(2q-4)-1$.

Show answer

$q=-8$

EXAMPLE 15

Solve: $\frac{1}{2}(6x-2)=5-x$.

Solution

	$\frac{1}{2}(6x-2)=5-x$
Distribute.	$3x-1=5-x$
Add x to get all the variables on the left.	$3x-1+x=5-x+x$
Simplify.	$4x-1=5$
Add 1 to get constants on the right.	$4x-1+1=5+1$
Simplify.	$4x=6$
Divide by 4.	$\frac{4x}{4}=\frac{6}{4}$
Simplify.	$x=\frac{3}{2}$
Check: Let $x=\frac{3}{2}$.	$\frac{1}{2}(6x-2)=5-x$ $\frac{1}{2}\left(6\cdot\frac{3}{2}-2\right)\overset{?}{=}5-\frac{3}{2}$ $\frac{1}{2}(9-2)\overset{?}{=}\frac{10}{2}-\frac{3}{2}$ $\frac{1}{2}(7)\overset{?}{=}\frac{7}{2}$ $\frac{7}{2}=\frac{7}{2}$ ✓

TRY IT 15.1

Solve: $\frac{1}{3}\left(6u+3\right)=7-u$.

Show answer
$u = 2$

TRY IT 15.2

Solve: $\frac{2}{3}\left(9x-12\right)=8+2x$.

Show answer
$x = 4$

In many applications, we will have to solve equations with decimals. The same general strategy will work for these equations.

EXAMPLE 16

Solve: $0.24\left(100x+5\right)=0.4\left(30x+15\right)$.

Solution

	$0.24(100x+5)=0.4(30x+15)$
Distribute.	$24x+1.2=12x+6$
Subtract $12x$ to get all the xs to the left.	$24x+1.2-12x=12x+6-12x$
Simplify.	$12x+1.2=6$
Subtract 1.2 to get the constants to the right.	$12x+1.2-1.2=6-1.2$
Simplify.	$12x=4.8$
Divide.	$\frac{12x}{12}=\frac{4.8}{12}$
Simplify.	$x=0.4$
Check: Let $x=0.4$.	$0.24(100x+5)=0.4(30x+15)$ $0.24(100(0.4)+5)\overset{?}{=}0.4(30(0.4)+15)$ $0.24(40+5)\overset{?}{=}0.4(12+15)$ $0.24(45)\overset{?}{=}0.4(27)$ $10.8=10.8$ ✓

TRY IT 16.1

Solve: $0.55\,(100n+8)=0.6\,(85n+14)$.

Show answer
1

TRY IT 16.2

Solve: $0.15\,(40m-120)=0.5\,(60m+12)$.

Show answer
−1

Key Concepts

- **Solve an equation with variables and constants on both sides**
    1. Choose one side to be the variable side and then the other will be the constant side.

2. Collect the variable terms to the variable side, using the Addition or Subtraction Property of Equality.
3. Collect the constants to the other side, using the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting into the original equation.

- **General strategy for solving linear equations**
    1. Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
    2. Collect all the variable terms to one side of the equation. Use the Addition or Subtraction Property of Equality.
    3. Collect all the constant terms to the other side of the equation. Use the Addition or Subtraction Property of Equality.
    4. Make the coefficient of the variable term to equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
    5. Check the solution. Substitute the solution into the original equation to make sure the result is a true statement.

Practice Makes Perfect

Solve an Equation with Constants on Both Sides

In the following exercises, solve the equation for the variable.

1. $7x - 8 = 34$	2. $6x - 2 = 40$
3. $14y + 7 = 91$	4. $11w + 6 = 93$
5. $4m + 9 = -23$	6. $3a + 8 = -46$
7. $-47 = 6b + 1$	8. $-50 = 7n - 1$
9. $29 = -8x - 3$	10. $25 = -9y + 7$
11. $-14\mathrm{q} - 15 = 13$	12. $-12p - 3 = 15$

Solve an Equation with Variables on Both Sides

In the following exercises, solve the equation for the variable.

13. $9k = 8k - 11$	14. $8z = 7z - 7$
15. $6x + 27 = 9x$	16. $4x + 36 = 10x$
17. $b = -4b - 15$	18. $c = -3c - 20$
19. $7z = 39 - 6z$	20. $5q = 44 - 6q$
21. $8x + \frac{3}{4} = 7x$	22. $3y + \frac{1}{2} = 2y$
23. $-15r - 8 = -11r$	24. $-12a - 8 = -16a$

Solve an Equation with Variables and Constants on Both Sides

In the following exercises, solve the equations for the variable.

25. $4x - 17 = 3x + 2$	26. $6x - 15 = 5x + 3$
27. $21 + 6f = 7f + 14$	28. $26 + 8d = 9d + 11$
29. $8q - 5 = 5q - 20$	30. $3p - 1 = 5p - 33$
31. $9c + 7 = -2c - 37$	32. $4a + 5 = -a - 40$
33. $12x - 17 = -3x + 13$	34. $8y - 30 = -2y + 30$
35. $3y - 4 = 12 - y$	36. $2\mathrm{z} - 4 = 23 - \mathrm{z}$
37. $\frac{4}{3}m - 7 = \frac{1}{3}m - 13$	38. $\frac{5}{4}c - 3 = \frac{1}{4}c - 16$
39. $11 - \frac{1}{4}a = \frac{3}{4}a + 4$	40. $8 - \frac{2}{5}q = \frac{3}{5}q + 6$
41. $\frac{5}{4}a + 15 = \frac{3}{4}a - 5$	42. $\frac{4}{3}n + 9 = \frac{1}{3}n - 9$
43. $\frac{3}{5}p + 2 = \frac{4}{5}p - 1$	44. $\frac{1}{4}y + 7 = \frac{3}{4}y - 3$
45. $13z + 6.45 = 8z + 23.75$	46. $14n + 8.25 = 9n + 19.60$
47. $2.7w - 80 = 1.2w + 10$	48. $2.4w - 100 = 0.8w + 28$
49. $6.6x - 18.9 = 3.4x + 54.7$	50. $5.6r + 13.1 = 3.5r + 57.2$

Solve an Equation Using the General Strategy

In the following exercises, solve the linear equation using the general strategy.

51. $4\left(y+7\right)=64$	52. $5\left(x+3\right)=75$
53. $9=3\left(x-3\right)$	54. $8=4\left(x-3\right)$
55. $14\left(y-6\right)=-42$	56. $20\left(y-8\right)=-60$
57. $-7\left(3n+4\right)=14$	58. $-4\left(2n+1\right)=16$
59. $8\left(3+3\text{p}\right)=0$	60. $3\left(10+5r\right)=0$
61. $\frac{3}{5}\left(10x-5\right)=27$	62. $\frac{2}{3}\left(9c-3\right)=22$
63. $4\left(2.5v-0.6\right)=7.6$	64. $5\left(1.2u-4.8\right)=-12$
65. $0.5\left(16m+34\right)=-15$	66. $0.2\left(30n+50\right)=28$
67. $-\left(t-8\right)=17$	68. $-\left(w-6\right)=24$
69. $8\left(6b-7\right)+23=63$	70. $9\left(3a+5\right)+9=54$
71. $13+2\left(m-4\right)=17$	72. $10+3\left(z+4\right)=19$
73. $-9+6\left(5-k\right)=12$	74. $7+5\left(4-q\right)=12$
75. $18-\left(9r+7\right)=-16$	76. $15-\left(3r+8\right)=28$
77. $18-2\left(y-3\right)=32$	78. $11-4\left(y-8\right)=43$
79. $3\left(4n-1\right)-2=8n+3$	80. $9\left(p-1\right)=6\left(2p-1\right)$
81. $5\left(x-4\right)-4x=14$	82. $9\left(2m-3\right)-8=4m+7$
83. $5+6\left(3s-5\right)=-3+2\left(8s-1\right)$	84. $8\left(x-4\right)-7x=14$
85. $4\left(x-1\right)-8=6\left(3x-2\right)-7$	86. $-12+8\left(x-5\right)=-4+3\left(5x-2\right)$

Everyday Math

Making a fence 87. Jovani has a fence around the rectangular garden in his backyard. The perimeter of the fence is 150 feet. The length is 15 feet more than the width. Find the width, w, by solving the equation $150=2\left(w+15\right)+2w$.	**Concert tickets** 88. At a school concert, the total value of tickets sold was $\$1{,}506$. Student tickets sold for $\$6$ and adult tickets sold for $\$9$. The number of adult tickets sold was 5 less than 3 times the number of student tickets. Find the number of student tickets sold, s, by solving the equation $6s+9\left(3s-5\right)=1506$.
Coins 89. Rhonda has $\$1.90$ in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, n, by solving the equation $0.05n+0.10\left(2n-1\right)=1.90$.	**Fencing** 90. Micah has 74 feet of fencing to make a rectangular dog pen in his yard. He wants the length to be 25 feet more than the width. Find the length, L, by solving the equation $2L+2\left(L-25\right)=74$.

Writing Exercises

91. When solving an equation with variables on both sides, why is it usually better to choose the side with the larger coefficient as the variable side?	92. Solve the equation $10x + 14 = -2x + 38$, explaining all the steps of your solution.
93. What is the first step you take when solving the equation $3 - 7\left(y - 4\right) = 38$? Explain why this is your first step.	94. Solve the equation $\frac{1}{4}\left(8x + 20\right) = 3x - 4$ explaining all the steps of your solution as in the examples in this section.
95. Using your own words, list the steps in the General Strategy for Solving Linear Equations.	96. Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

Answers

1. 6	3.6	5. -8
7. -8	9. -4	11. -2
13. -11	15. 9	17. -3
19. 3	21. -3/4	25. 19
27. 7	29. -5	31. -4
33. 2	35. 4	37. -6
39. 7	41. -40	43. 15
45. 3.46	47. 60	49. 23
51. 9	53. 6	55. 3
57. −2	59. −1	61. 5
63. 0.52	65. 0.25	67. −9
69. 2	71. 6	73. 3/2
75. 3	77. −4	79. 2
81. 34	83. 10	85. 2
87. 30 feet	89. 8 nickels	91. Answers will vary.
93. Answers will vary.	95. Answers will vary.	

Attributions

This chapter has been adapted from “Solve Equations with Variables and Constants on Both Sides” in

5.4 Solve Equations with Fraction or Decimal Coefficients

Learning Objectives

By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Solve Equations with Fraction Coefficients

Let’s use the General Strategy for Solving Linear Equations introduced earlier to solve the equation $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x + \frac{1}{2} - \frac{1}{2} = \frac{1}{4} - \frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x = \frac{1}{4} - \frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x = \frac{1}{4} - \frac{2}{4}$
Subtract.	$\frac{1}{8}x = -\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	$\frac{8}{1} \cdot \frac{1}{8}x = \frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x = -2$

This method worked fine, but many students don’t feel very confident when they see all those fractions. So we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of *all* the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but with no fractions. This process is called *clearing the equation of fractions*. Let’s solve the same equation again, but this time use the method that clears the fractions.

EXAMPLE 1

Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Solution

Find the least common denominator of *all* the fractions in the equation.	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4} \quad \text{LCD} = 8$
Multiply both sides of the equation by that LCD, 8. This clears the fractions.	$8\left(\frac{1}{8}x + \frac{1}{2}\right) = 8\left(\frac{1}{4}\right)$
Use the Distributive Property.	$8 \cdot \frac{1}{8}x + 8 \cdot \frac{1}{2} = 8 \cdot \frac{1}{4}$
Simplify — and notice, no more fractions!	$x + 4 = 2$
Solve using the General Strategy for Solving Linear Equations.	$x + 4 - 4 = 2 - 4$
Simplify.	$x = -2$
Check: Let $x = -2$	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$ $\frac{1}{8}(-2) + \frac{1}{2} \overset{?}{=} \frac{1}{4}$ $-\frac{2}{8} + \frac{1}{2} \overset{?}{=} \frac{1}{4}$ $-\frac{2}{8} + \frac{4}{8} \overset{?}{=} \frac{1}{4}$ $\frac{2}{4} \overset{?}{=} \frac{1}{4}$ $\frac{1}{4} = \frac{1}{4}$ ✓

TRY IT 1.1

Solve: $\frac{1}{4}x + \frac{1}{2} = \frac{5}{8}$.

Show answer

$x = \frac{1}{2}$

TRY IT 1.2

Solve: $\frac{1}{6}y - \frac{1}{3} = \frac{1}{6}$.

Show answer

$y = 3$

Notice in (Figure) that once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

HOW TO: Solve Equations with Fraction Coefficients by Clearing the Fractions

1. Find the least common denominator of *all* the fractions in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions.
3. Solve using the General Strategy for Solving Linear Equations.

EXAMPLE 2

Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$.

Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the least common denominator of *all* the fractions in the equation.	$7=\frac{1}{2}x+\frac{3}{4}x-\frac{2}{3}x$ LCD = 12
Multiply both sides of the equation by 12.	$12(7)=12\cdot\frac{1}{2}x+\frac{3}{4}x-\frac{2}{3}x$
Distribute.	$12(7)=12\cdot\frac{1}{2}x+12\cdot\frac{3}{4}x-12\cdot\frac{2}{3}x$
Simplify — and notice, no more fractions!	$84=6x+9x-8x$
Combine like terms.	$84=7x$
Divide by 7.	$\frac{84}{7}=\frac{7x}{7}$
Simplify.	$12=x$
Check: Let $x=12$.	$7=\frac{1}{2}x+\frac{3}{4}x-\frac{2}{3}x$ $7\overset{?}{=}\frac{1}{2}(12)+\frac{3}{4}(12)-\frac{2}{3}(12)$ $7\overset{?}{=}6+9-8$ $7=7$ ✓

TRY IT 2.1

Solve: $6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v$.

Show answer
$v = 40$

TRY IT 2.2

Solve: $-1=\frac{1}{2}u+\frac{1}{4}u-\frac{2}{3}u$.

Show answer
$u = -12$

In the next example, we'll have variables and fractions on both sides of the equation.

EXAMPLE 3

Solve: $x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$.

Solution

Find the LCD of all the fractions in the equation.	$x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$, LCD = 6
Multiply both sides by the LCD.	$6\left(x + \frac{1}{3}\right) = 6\left(\frac{1}{6}x - \frac{1}{2}\right)$
Distribute.	$6 \cdot x + 6 \cdot \frac{1}{3} = 6 \cdot \frac{1}{6}x - 6 \cdot \frac{1}{2}$
Simplify — no more fractions!	$6x + 2 = x - 3$
Subtract x from both sides.	$6x - x + 2 = x - x - 3$
Simplify.	$5x + 2 = -3$
Subtract 2 from both sides.	$5x + 2 - 2 = -3 - 2$
Simplify.	$5x = -5$
Divide by 5.	$\frac{5x}{5} = \frac{-5}{5}$
Simplify.	$x = -1$
Check: Substitute $x = -1$.	$x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$ $(-1) + \frac{1}{3} \overset{?}{=} \frac{1}{6}(-1) - \frac{1}{2}$ $(-1) + \frac{1}{3} \overset{?}{=} -\frac{1}{6} - \frac{1}{2}$ $-\frac{3}{3} + \frac{1}{3} \overset{?}{=} -\frac{1}{6} - \frac{3}{6}$ $-\frac{2}{3} \overset{?}{=} -\frac{4}{6}$ $-\frac{2}{3} = -\frac{2}{3}$ ✓

TRY IT 3.1

Solve: $a + \frac{3}{4} = \frac{3}{8}a - \frac{1}{2}$.

Show answer

$a = -2$

TRY IT 3.2

Solve: $c+\frac{3}{4}=\frac{1}{2}c-\frac{1}{4}$.

Show answer
$c=-2$

In (Figure), we’ll start by using the Distributive Property. This step will clear the fractions right away!

EXAMPLE 4

Solve: $1=\frac{1}{2}(4x+2)$.

Solution

	$1=\frac{1}{2}(4x+2)$
Distribute.	$1=\frac{1}{2}\cdot 4x+\frac{1}{2}\cdot 2$
Simplify. Now there are no fractions to clear!	$1=2x+1$
Subtract 1 from both sides.	$1-1=2x+1-1$
Simplify.	$0=2x$
Divide by 2.	$\frac{0}{2}=\frac{2x}{2}$
Simplify.	$0=x$
Check: Let $x=0$.	$1=\frac{1}{2}(4x+2)$ $1\overset{?}{=}\frac{1}{2}(4(0)+2)$ $1\overset{?}{=}\frac{1}{2}(2)$ $1\overset{?}{=}\frac{2}{2}$ $1=1$ ✓

TRY IT 4.1

Solve: $-11 = \frac{1}{2}(6p + 2)$.

Show answer

$p = -4$

TRY IT 4.2

Solve: $8 = \frac{1}{3}(9q + 6)$.

Show answer

$q = 2$

Many times, there will still be fractions, even after distributing.

EXAMPLE 5

Solve: $\frac{1}{2}(y - 5) = \frac{1}{4}(y - 1)$.

Solution

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the y terms to the left.	$2y-10-y=y-1-y$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check: Substitute 9 for y.	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$ $\frac{1}{2}(9-5)\overset{?}{=}\frac{1}{4}(9-1)$ $\frac{1}{2}(4)\overset{?}{=}\frac{1}{4}(8)$ $2=2$ ✓

TRY IT 5.1

Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Show answer

$n = 2$

TRY IT 5.2

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Show answer

$m = -1$

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money and percent. But decimals are really another way to represent fractions. For example, $0.3 = \frac{3}{10}$ and $0.17 = \frac{17}{100}$. So, when we have an equation with decimals, we can use the same process we used to clear fractions—multiply both sides of the equation by the least common denominator.

EXAMPLE 6

Solve: $0.8x - 5 = 7$.

Solution

The only decimal in the equation is 0.8. Since $0.8 = \frac{8}{10}$, the LCD is 10. We can multiply both sides by 10 to clear the decimal.

	$0.8x - 5 = 7$
Multiply both sides by the LCD.	$10(0.8x - 5) = 10(7)$
Distribute.	$10(0.8x) - 10(5) = 10(7)$
Multiply, and notice, no more decimals!	$8x - 50 = 70$
Add 50 to get all constants to the right.	$8x - 50 + 50 = 70 + 50$
Simplify.	$8x = 120$
Divide both sides by 8.	$\frac{8x}{8} = \frac{120}{8}$
Simplify.	$x = 15$
Check: Let $x = 15$.	$0.8(15) - 5 \overset{?}{=} 7$ $12 - 5 \overset{?}{=} 7$ $7 = 7$ ✓

TRY IT 6.1

Solve: $0.6x - 1 = 11$.

Show answer
$x = 20$

TRY IT 6.2

Solve: $1.2x - 3 = 9$.

Show answer

$x = 10$

EXAMPLE 7

Solve: $0.06x + 0.02 = 0.25x - 1.5$.

Solution

Look at the decimals and think of the equivalent fractions.

$$0.06 = \frac{6}{100}, \quad 0.02 = \frac{2}{100}, \quad 0.25 = \frac{25}{100}, \quad 1.5 = 1\frac{5}{10}$$

Notice, the LCD is 100.

By multiplying by the LCD we will clear the decimals.

	$0.06x + 0.02 = 0.25x - 1.5$
Multiply both sides by 100.	$100(0.06x + 0.02) = 100(0.25x - 1.5)$
Distribute.	$100(0.06x) + 100(0.02) = 100(0.25x) - 100(1.5)$
Multiply, and now no more decimals.	$6x + 2 = 25x - 150$
Collect the variables to the right.	$6x - 6x + 2 = 25x - 6x - 150$
Simplify.	$2 = 19x - 150$
Collect the constants to the left.	$2 + 150 = 19x - 150 + 150$
Simplify.	$152 = 19x$
Divide by 19.	$\frac{152}{19} = \frac{19x}{19}$
Simplify.	$8 = x$
Check: Let $x = 8$.	
$0.06(8) + 0.02 = 0.25(8) - 1.5$ $0.48 + 0.02 = 2.00 - 1.5$ $0.50 = 0.50$ ✓	

TRY IT 7.1

Solve: $0.14h + 0.12 = 0.35h - 2.4$.

Show answer

$h = 12$

TRY IT 7.2

Solve: $0.65k - 0.1 = 0.4k - 0.35$.

Show answer

$k = -1$

The next example uses an equation that is typical of the ones we will see in the money applications in the next chapter. Notice that we will distribute the decimal first before we clear all decimals in the equation.

EXAMPLE 8

Solve: $0.25x + 0.05\left(x + 3\right) = 2.85$.

Solution

	$0.25x+0.05(x+3)=2.85$
Distribute first.	$0.25x+0.05x+0.15=2.85$
Combine like terms.	$0.30x+0.15=2.85$
To clear decimals, multiply by 100.	$100(0.30x+0.15)=100(2.85)$
Distribute.	$30x+15=285$
Subtract 15 from both sides.	$30x+15-15=285-15$
Simplify.	$30x=270$
Divide by 30.	$\frac{30x}{30}=\frac{270}{30}$
Simplify.	$x=9$
Check: Let $x=9$.	$0.25x+0.05(x+3)=2.85$ $0.25(9)+0.05(9+3)\overset{?}{=}2.85$ $2.25+0.05(12)\overset{?}{=}2.85$ $2.25+0.60\overset{?}{=}2.85$ $2.85=2.85$ ✓

TRY IT 8.1

Solve: $0.25n+0.05\left(n+5\right)=2.95$.

Show answer

$n=9$

TRY IT 8.2

Solve: $0.10d+0.05\left(d-5\right)=2.15$.

Show answer

$d=16$

Key Concepts

- **Solve equations with fraction coefficients by clearing the fractions.**
    1. Find the least common denominator of *all* the fractions in the equation.
    2. Multiply both sides of the equation by that LCD. This clears the fractions.
    3. Solve using the General Strategy for Solving Linear Equations.

Practices Makes Perfect

Solve equations with fraction coefficients

In the following exercises, solve the equation by clearing the fractions.

1. $\frac{1}{4}x-\frac{1}{2}=-\frac{3}{4}$	2. $\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}$
3. $\frac{5}{6}y-\frac{2}{3}=-\frac{3}{2}$	4. $\frac{5}{6}y-\frac{1}{3}=-\frac{7}{6}$
5. $\frac{1}{2}a+\frac{3}{8}=\frac{3}{4}$	6. $\frac{5}{8}b+\frac{1}{2}=-\frac{3}{4}$
7. $2=\frac{1}{3}x-\frac{1}{2}x+\frac{2}{3}x$	8. $2=\frac{3}{5}x-\frac{1}{3}x+\frac{2}{5}x$
9. $\frac{1}{4}m-\frac{4}{5}m+\frac{1}{2}m=-1$	10. $\frac{5}{6}n-\frac{1}{4}n-\frac{1}{2}n=-2$
11. $x+\frac{1}{2}=\frac{2}{3}x-\frac{1}{2}$	12. $x+\frac{3}{4}=\frac{1}{2}x-\frac{5}{4}$
13. $\frac{1}{3}w+\frac{5}{4}=w-\frac{1}{4}$	14. $\frac{3}{2}z+\frac{1}{3}=z-\frac{2}{3}$
15. $\frac{1}{2}x-\frac{1}{4}=\frac{1}{12}x+\frac{1}{6}$	16. $\frac{1}{2}a-\frac{1}{4}=\frac{1}{6}a+\frac{1}{12}$
17. $\frac{1}{3}b+\frac{1}{5}=\frac{2}{5}b-\frac{3}{5}$	18. $\frac{1}{3}x+\frac{2}{5}=\frac{1}{5}x-\frac{2}{5}$
19. $1=\frac{1}{6}(12x-6)$	20. $1=\frac{1}{5}(15x-10)$
21. $\frac{1}{4}(p-7)=\frac{1}{3}(p+5)$	22. $\frac{1}{5}(q+3)=\frac{1}{2}(q-3)$
23. $\frac{1}{2}(x+4)=\frac{3}{4}$	24. $\frac{1}{3}(x+5)=\frac{5}{6}$

Solve Equations with Decimal Coefficients

In the following exercises, solve the equation by clearing the decimals.

25. $0.6y + 3 = 9$

26. $0.4y - 4 = 2$

27. $3.6j - 2 = 5.2$

28. $2.1k + 3 = 7.2$

29. $0.4x + 0.6 = 0.5x - 1.2$

30. $0.7x + 0.4 = 0.6x + 2.4$

31. $0.23x + 1.47 = 0.37x - 1.05$

32. $0.48x + 1.56 = 0.58x - 0.64$

33. $0.9x - 1.25 = 0.75x + 1.75$

34. $1.2x - 0.91 = 0.8x + 2.29$

35. $0.05n + 0.10\left(n + 8\right) = 2.15$

36. $0.05n + 0.10\left(n + 7\right) = 3.55$

37. $0.10d + 0.25\left(d + 5\right) = 4.05$

38. $0.10d + 0.25\left(d + 7\right) = 5.25$

39. $0.05\left(q - 5\right) + 0.25q = 3.05$

40. $0.05\left(q - 8\right) + 0.25q = 4.10$

Everyday Math

Coins 41. Taylor has $\$2.00$ in dimes and pennies. The number of pennies is 2 more than the number of dimes. Solve the equation $0.10d + 0.01\left(d + 2\right) = 2$ for d, the number of dimes.

Stamps 42. Travis bought $\$9.45$ worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 5 less than the number of 49-cent stamps. Solve the equation $0.49s + 0.21\left(s - 5\right) = 9.45$ for s, to find the number of 49-cent stamps Travis bought.

Writing Exercises

43. Explain how to find the least common denominator of $\frac{3}{8},\ \frac{1}{6},\ \text{and}\ \frac{2}{3}$.

44. If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

45. If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

46. In the equation $0.35x + 2.1 = 3.85$, what is the LCD? How do you know?

Answers

1. x = -1	3. y = -1	5. $a = \frac{3}{4}$
7. x = 4	9. m = 20	11. x = -3
13. $w = \frac{9}{4}$	15. x = 1	17. b = 12
19. x = 1	21. p = -41	23. $x = -\frac{5}{2}$
25. y = 10	27. j = 2	29. x = 18
31. x = 18	33. x = 20	35. n = 9
37. d = 8	39. q = 11	41 d = 18
43. Answers will vary.	45.Answers will vary.	

Attributions

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5.5 Use a General Strategy to Solve Linear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Solve Equations Using the General Strategy

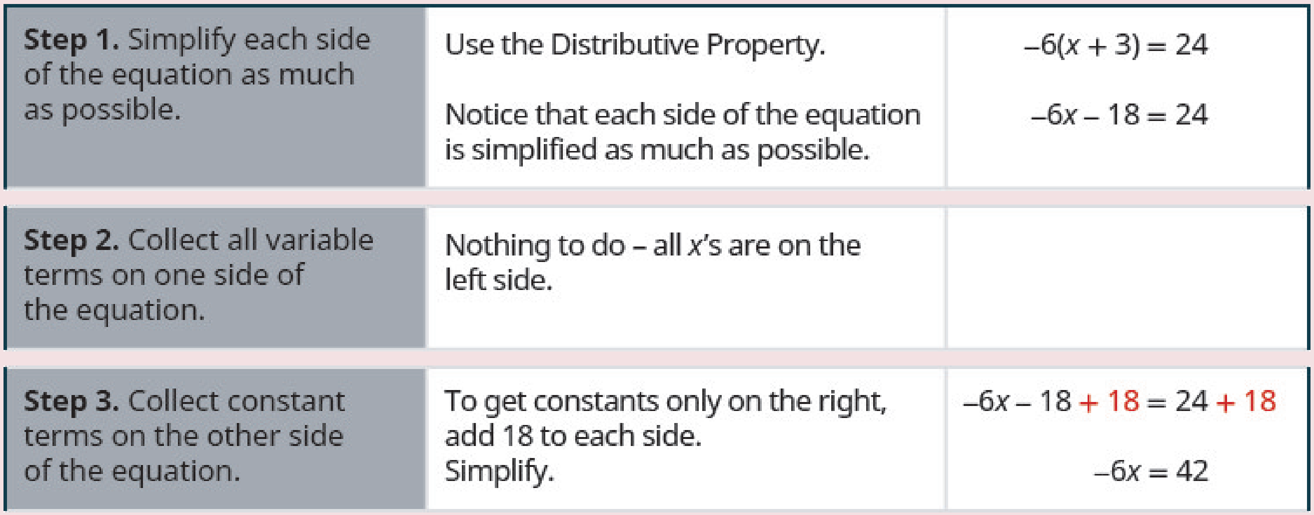
Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

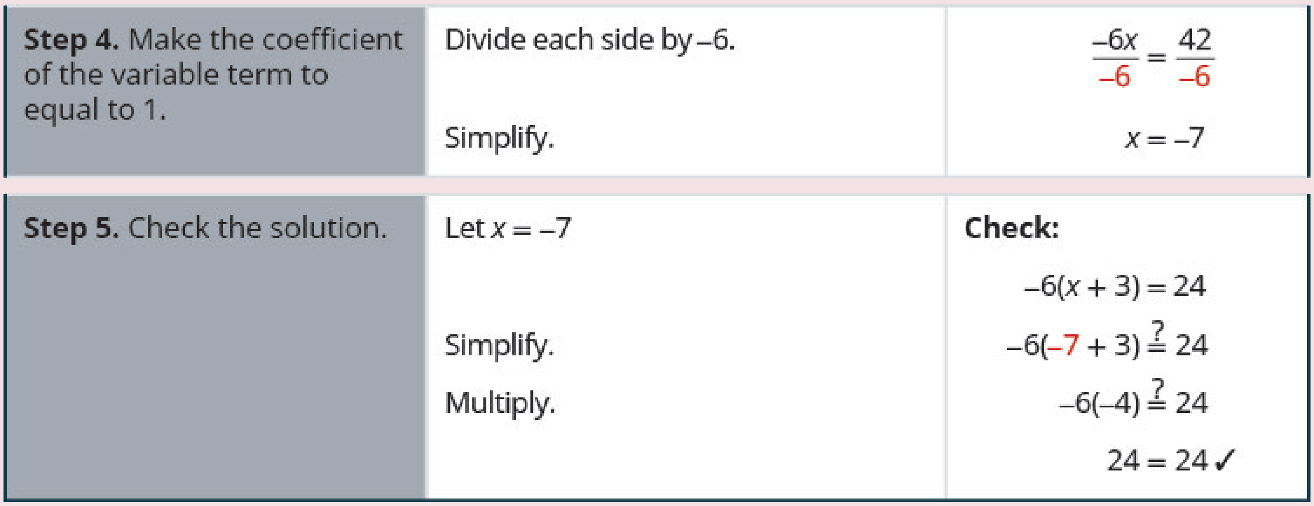
EXAMPLE 1. How to Solve Linear Equations Using the General Strategy

Solve: $-6\left(x+3\right)=24.$

Solution



Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all *x*'s are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$



Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by –6. Simplify.	$\frac{-6x}{-6}=\frac{42}{-6}$ $x=-7$
Step 5. Check the solution.	Let $x=-7$ Simplify. Multiply.	**Check:** $-6(x+3)=24$ $-6(-7+3)\overset{?}{=}24$ $-6(-4)\overset{?}{=}24$ $24=24$ ✓

TRY IT 1.1

Solve: $5\left(x+3\right)=35.$

Show answer
$x=4$

TRY IT 1.2

Solve: $6\left(y-4\right)=-18.$

Show answer
$y=1$

General strategy for solving linear equations.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.

4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2

Solve: $-(y+9)=8$.

Solution

	$-(y+9)=8$
Simplify each side of the equation as much as possible by distributing.	$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y-9+9=8+9$
Simplify.	$-y=17$
Rewrite $-y$ as $-1y$.	$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.	$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.	$y=-17$
Check: Let $y=-17$.	$-(y+9)=8$
	$-(-17+9)\overset{?}{=}8$
	$-(-8)\overset{?}{=}8$
	$8=8$ ✓

TRY IT 2.1

Solve: $-(y+8)=-2$.

Show answer
$y=-6$

TRY IT 2.2

Solve: $-\left(z+4\right)=-12$.

Show answer
$z=8$

EXAMPLE 3

Solve: $5\left(a-3\right)+5=-10$.

Solution

	$5(a-3)+5=-10$
Simplify each side of the equation as much as possible.	
Distribute.	$5a-15+5=-10$
Combine like terms.	$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.	
Add 10 to both sides to get all constant terms on the other side of the equation.	$5a-10+10=-10+10$
Simplify.	$5a=0$
Make the coefficient of the variable term to equal to 1 by dividing both sides by 5.	$\frac{5a}{5}=\frac{0}{5}$
Simplify.	$a=0$
Check:	$5(a-3)+5=-10$
Let $a=0$.	$5(0-3)+5\stackrel{?}{=}-10$
	$5(-3)+5\stackrel{?}{=}-10$
	$-15+5\stackrel{?}{=}-10$
	$-10=-10$ ✓

TRY IT 3.1

Solve: $2(m-4)+3=-1$.

Show answer
$m=2$

TRY IT 3.2

Solve: $7(n-3)-8=-15$.

Show answer
$n=2$

EXAMPLE 4

Solve: $\frac{2}{3}(6m-3)=8-m$.

Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$
Add m to get the variables only to the left.	$4m+m-2=8-m+m$
Simplify.	$5m-2=8$
Add 2 to get constants only on the right.	$5m-2+2=8+2$
Simplify.	$5m=10$
Divide by 5.	$\frac{5m}{5}=\frac{10}{5}$
Simplify.	$m=2$
Check:	$\frac{2}{3}(6m-3)=8-m$
Let $m=2$.	$\frac{2}{3}(6\cdot 2-3)\overset{?}{=}8-2$
	$\frac{2}{3}(12-3)\overset{?}{=}6$
	$\frac{2}{3}(9)\overset{?}{=}6$
	$6=6$ ✓

TRY IT 4.1

Solve: $\frac{1}{3}(6u+3)=7-u$.

Show answer

$u=2$

TRY IT 4.2

Solve: $\frac{2}{3}\left(9x-12\right)=8+2x.$

Show answer
$x=4$

EXAMPLE 5

Solve: $8-2\left(3y+5\right)=0.$

Solution

	$8-2(3y+5)=0$
Simplify—use the Distributive Property.	$8-6y-10=0$
Combine like terms.	$-6y-2=0$
Add 2 to both sides to collect constants on the right.	$-6y-2+2=0+2$
Simplify.	$-6y=2$
Divide both sides by -6.	$\frac{-6y}{-6}=\frac{2}{-6}$
Simplify.	$y=-\frac{1}{3}$
Check: Let $y=-\frac{1}{3}$.	$8-2(3y+5)=0$ $8-2\left[3\left(-\frac{1}{3}\right)+5\right]=0$ $8-2(-1+5)\overset{?}{=}0$ $8-2(4)\overset{?}{=}0$ $8-8\overset{?}{=}0$ $0=0$ ✓

TRY IT 5.1

Solve: $12-3\left(4j+3\right)=-17$.

Show answer
$j=\frac{5}{3}$

TRY IT 5.2

Solve: $-6-8\left(k-2\right)=-10$.

Show answer
$k=\frac{5}{2}$

EXAMPLE 6

Solve: $4\left(x-1\right)-2=5\left(2x+3\right)+6$.

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ to get the variables only on the right side since $10 > 4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\overset{?}{=}5\left[2\left(-\frac{9}{2}\right)+3\right]+6$
	$4\left(-\frac{11}{2}\right)-2\overset{?}{=}5(-9+3)+6$
	$-22-2\overset{?}{=}5(-6)+6$
	$-24\overset{?}{=}-30+6$
	$-24=-24$ ✓

TRY IT 6.1

Solve: $6(p-3)-7=5(4p+3)-12$.

Show answer

$p=-2$

TRY IT 6.2

Solve: $8(q+1)-5=3(2q-4)-1.$

Show answer

$q=-8$

EXAMPLE 7

Solve: $10[3-8(2s-5)]=15(40-5s).$

Solution

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to get the s's to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$
Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute $s=-2$.	$10[3-8(2(-2)-5)]\overset{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\overset{?}{=}15(40+10)$
	$10[3-8(-9)]\overset{?}{=}15(50)$
	$10[3+72]\overset{?}{=}750$
	$10[75]\overset{?}{=}750$
	$750=750$ ✓

TRY IT 7.1

Solve: $6\left[4-2\left(7y-1\right)\right]=8\left(13-8y\right)$.

Show answer

$y=-\frac{17}{5}$

TRY IT 7.2

Solve: $12\left[1-5\left(4z-1\right)\right]=3\left(24+11z\right)$.

Show answer
$z=0$

EXAMPLE 8

Solve: $0.36\left(100n+5\right)=0.6\left(30n+15\right)$.

Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract $18n$ to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$
Subtract 1.8 to get the constants to the right.	$18n+1.8-1.8=9-1.8$
Simplify.	$18n=7.2$
Divide.	$\frac{18n}{18}=\frac{7.2}{18}$
Simplify.	$n=0.4$
Check:	$0.36(100n+5)=0.6(30n+15)$
Let $n=0.4$.	$0.36(100(0.4)+5)\overset{?}{=}0.6(30(0.4)+15)$
	$0.36(40+5)\overset{?}{=}0.6(12+15)$
	$0.36(45)\overset{?}{=}0.6(27)$
	$16.2=16.2\checkmark$

TRY IT 8.1

Solve: $0.55\,(100n+8)=0.6\,(85n+14)$.

Show answer
$n=1$

TRY IT 8.2

Solve: $0.15\,(40m-120)=0.5\,(60m+12)$.

Show answer
$m=-1$

Classify Equations

Consider the equation we solved at the start of the last section, $7x+8=-13$. The solution we found was $x=-3$. This means the equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3. We showed this when we checked the solution $x=-3$ and evaluated $7x+8=-13$ for $x=-3$.

$$\begin{aligned} 7(-3)+8 &\overset{?}{=} -13 \\ -21+8 &\overset{?}{=} -13 \\ -13 &= -13 \checkmark \end{aligned}$$

If we evaluate $7x+8$ for a different value of *x*, the left side will not be -13.

The equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. Whether or not the equation $7x+8=-13$ is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

Conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

Now let’s consider the equation $2y+6=2\left(y+3\right)$. Do you recognize that the left side and the right side are equivalent? Let’s see what happens when we solve for *y*.

	$2y+6=2(y+3)$
Distribute.	$2y+6=2y+6$
Subtract $2y$ to get the y’s to one side.	$2y-2y+6=2y-2y+6$
Simplify—the y’s are gone!	$6=6$

But $6=6$ is true.

This means that the equation $2y+6=2\left(y+3\right)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an identity.

Identity

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is every real number.

What happens when we solve the equation $5z=5z-1$?

	$5z=5z-1$
Subtract $5z$ to get the constant alone on the right.	$5z-5z=5z-5z-1$
Simplify—the z’s are gone!	$0\neq-1$

But $0\neq1$.

Solving the equation $5z=5z-1$ led to the false statement $0=-1$. The equation $5z=5z-1$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

Contradiction

An equation that is false for all values of the variable is called a contradiction.

A contradiction has no solution.

EXAMPLE 9

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$$6\left(2n-1\right)+3=2n-8+5\left(2n+1\right)$$

Solution

	$6(2n-1)+3=2n-8+5(2n+1)$
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract $12n$ to get the n’s to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity. The solution is every real number.

TRY IT 9.1

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4+9\left(3x-7\right)=-42x-13+23\left(3x-2\right)$$

Show answer

identity; all real numbers

TRY IT 9.2

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$8\left(1-3x\right)+15\left(2x+7\right)=2\left(x+50\right)+4\left(x+3\right)+1$$

Show answer

identity; all real numbers

EXAMPLE 10

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$$10 + 4\left(p - 5\right) = 0$$

Solution

	$10 + 4(p - 5) = 0$
Distribute.	$10 + 4p - 20 = 0$
Combine like terms.	$4p - 10 = 0$
Add 10 to both sides.	$4p - 10 + 10 = 0 + 10$
Simplify.	$4p = 10$
Divide.	$\frac{4p}{4} = \frac{10}{4}$
Simplify.	$p = \frac{5}{2}$
The equation is true when $p = \frac{5}{2}$.	This is a conditional equation. The solution is $p = \frac{5}{2}$.

TRY IT 10.1

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$11\left(q + 3\right) - 5 = 19$$

Show answer

conditional equation; $q = \frac{9}{11}$

TRY IT 10.2

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$6 + 14\left(k - 8\right) = 95$$

Show answer

conditional equation; $k = \frac{193}{14}$

EXAMPLE 11

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$$5m + 3\left(9 + 3m\right) = 2\left(7m - 11\right)$$

Solution

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract $14m$ from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction. It has no solution.

TRY IT 11.1

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$12c + 5\left(5 + 3c\right) = 3\left(9c - 4\right)$$

Show answer

contradiction; no solution

TRY IT 11.2

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4\left(7d + 18\right) = 13\left(3d - 2\right) - 11d$$

Show answer

contradiction; no solution

Type of equation – Solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
    1. Simplify each side of the equation as much as possible.
    Use the Distributive Property to remove any parentheses.
    Combine like terms.
    2. Collect all the variable terms on one side of the equation.
    Use the Addition or Subtraction Property of Equality.
    3. Collect all the constant terms on the other side of the equation.
    Use the Addition or Subtraction Property of Equality.
    4. Make the coefficient of the variable term to equal to 1.
    Use the Multiplication or Division Property of Equality.
    State the solution to the equation.
    5. Check the solution.
    Substitute the solution into the original equation.

Glossary

conditional equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction
An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity
An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

Practice Makes Perfect

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

1. $21\left(y-5\right)=-42$
2. $15\left(y-9\right)=-60$
3. $-16\left(3n+4\right)=32$
4. $-9\left(2n+1\right)=36$
5. $5\left(8+6p\right)=0$
6. $8\left(22+11r\right)=0$
7. $-\left(t-19\right)=28$
8. $-\left(w-12\right)=30$
9. $21+2\left(m-4\right)=25$
10. $32+3\left(z+4\right)=41$
11. $-6+6\left(5-k\right)=15$
12. $51+5\left(4-q\right)=56$
13. $8\left(6t-5\right)-35=-27$
14. $2\left(9s-6\right)-62=16$
15. $-2\left(11-7x\right)+54=4$
16. $3\left(10-2x\right)+54=0$
17. $\frac{3}{5}\left(10x-5\right)=27$
18. $\frac{2}{3}\left(9c-3\right)=22$
19. $\frac{1}{4}\left(20d+12\right)=d+7$
20. $\frac{1}{5}\left(15c+10\right)=c+7$
21. $15-\left(3r+8\right)=28$
22. $18-\left(9r+7\right)=-16$
23. $-3-\left(m-1\right)=13$
24. $5-\left(n-1\right)=19$
25. $18-2\left(y-3\right)=32$
26. $11-4\left(y-8\right)=43$
27. $35-5\left(2w+8\right)=-10$
28. $24-8\left(3v+6\right)=0$
29. $-2\left(a-6\right)=4\left(a-3\right)$
30. $4\left(a-12\right)=3\left(a+5\right)$
31. $5\left(8-r\right)=-2\left(2r-16\right)$
32. $2\left(5-u\right)=-3\left(2u+6\right)$
33. $9\left(2m-3\right)-8=4m+7$
34. $3\left(4n-1\right)-2=8n+3$
35. $-15+4\left(2-5y\right)=-7\left(y-4\right)+4$
36. $12+2\left(5-3y\right)=-9\left(y-1\right)-2$
37. $5\left(x-4\right)-4x=14$
38. $8\left(x-4\right)-7x=14$
39. $-12+8\left(x-5\right)=-4+3\left(5x-2\right)$
40. $5+6\left(3s-5\right)=-3+2\left(8s-1\right)$
41. $7\left(2n-5\right)=8\left(4n-1\right)-9$
42. $4\left(u-1\right)-8=6\left(3u-2\right)-7$
43. $3\left(a-2\right)-\left(a+6\right)=4\left(a-1\right)$
44. $4\left(p-4\right)-\left(p+7\right)=5\left(p-3\right)$
45. $-\left(7m+4\right)-\left(2m-5\right)=14-\left(5m-3\right)$
46. $-\left(9y+5\right)-\left(3y-7\right)=16-\left(4y-2\right)$
47. $5\left[9-2\left(6d-1\right)\right]=11\left(4-10d\right)-139$
48. $4\left[5-8\left(4c-3\right)\right]=12\left(1-13c\right)-8$
49. $3\left[-14+2\left(15k-6\right)\right]=8\left(3-5k\right)-24$
50. $3\left[-9+8\left(4h-3\right)\right]=2\left(5-12h\right)-19$
51. $10\left[5\left(n+1\right)+4\left(n-1\right)\right]$ $=11\left[7\left(5+n\right)-\left(25-3n\right)\right]$
52. $5\left[2\left(m+4\right)+8\left(m-7\right)\right]$ $=2\left[3\left(5+m\right)-\left(21-3m\right)\right]$
53. $4\left(2.5v-0.6\right)=7.6$
54. $5\left(1.2u-4.8\right)=-12$

55. $0.2\left(p-6\right)=0.4\left(p+14\right)$	56. $0.25\left(q-6\right)=0.1\left(q+18\right)$
57. $0.5\left(16m+34\right)=-15$	58. $0.2\left(30n+50\right)=28$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

59. $15y+32=2\left(10y-7\right)-5y+46$	60. $23z+19=3\left(5z-9\right)+8z+46$
61. $9\left(a-4\right)+3\left(2a+5\right)=7\left(3a-4\right)-6a+7$	62. $5\left(b-9\right)+4\left(3b+9\right)=6\left(4b-5\right)-7b+21$
63. $24\left(3d-4\right)+100=52$	64. $18\left(5j-1\right)+29=47$
65. $30\left(2n-1\right)=5\left(10n+8\right)$	66. $22\left(3m-4\right)=8\left(2m+9\right)$
67. $18u-51=9\left(4u+5\right)-6\left(3u-10\right)$	68. $7v+42=11\left(3v+8\right)-2\left(13v-1\right)$
69. $5\left(p+4\right)+8\left(2p-1\right)=9\left(3p-5\right)-6\left(p-2\right)$	70. $3\left(6q-9\right)+7\left(q+4\right)=5\left(6q+8\right)-5\left(q+1\right)$
71. $9\left(4k-7\right)=11\left(3k+1\right)+4$	72. $12\left(6h-1\right)=8\left(8h+5\right)-4$
73. $60\left(2x-1\right)=15\left(8x+5\right)$	74. $45\left(3y-2\right)=9\left(15y-6\right)$
75. $36\left(4m+5\right)=12\left(12m+15\right)$	76. $16\left(6n+15\right)=48\left(2n+5\right)$
77. $11\left(8c+5\right)-8c=2\left(40c+25\right)+5$	78. $9\left(14d+9\right)+4d=13\left(10d+6\right)+3$

Everyday Math

79. **Coins.** Rhonda has \$1.90 in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, *n*, by solving the equation $0.05n+0.10\left(2n-1\right)=1.90.$	80. **Fencing.** Micah has 44 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, *L*, by solving the equation $2L+2\left(L-2.5\right)=44.$

Writing Exercises

81. Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.	82. Using your own words, list the steps in the general strategy for solving linear equations.
83. Solve the equation $\frac{1}{4}\left(8x+20\right)=3x-4$ explaining all the steps of your solution as in the examples in this section.	84. What is the first step you take when solving the equation $3-7\left(y-4\right)=38$? Why is this your first step?

Answers

1. $y = 3$	3. $n = -2$	5. $p = -\frac{4}{3}$
7. $t = -9$	9. $m = 6$	11. $k = \frac{3}{2}$
13. $t = 1$	15. $x = -2$	17. $x = 5$
19. $d = 1$	21. $r = -7$	23. $m = -15$
25. $y = -4$	27. $w = \frac{1}{2}$	29. $a = 4$
31. $r = 8$	33. $m = 3$	35. $y = -3$
37. $x = 34$	39. $x = -6$	41. $n = -1$
43. $a = -4$	45. $m = -4$	47. $d = -3$
49. $k = \frac{3}{5}$	51. $n = -5$	53. $v = 1$
55. $p = -34$	57. $m = -4$	59. identity; all real numbers
61. identity; all real numbers	63. conditional equation; $d = \frac{2}{3}$	65. conditional equation; $n = 7$
67. contradiction; no solution	69. contradiction; no solution	71. conditional equation; $k = 26$
73. contradiction; no solution	75. identity; all real numbers	77. identity; all real numbers
79. 8 nickels	81. Answers will vary.	83. Answers will vary.

Attributions

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5.6 Solve a Formula for a Specific Variable

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for “speed.” The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car’s cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

Distance, Rate, and Time

For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:

$$d = rt \qquad \text{where} \qquad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

HOW TO: Solve an application (with a formula).

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.

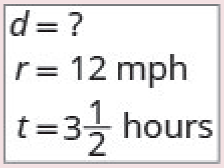
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

EXAMPLE 1

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let *d* = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	 $d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	
Does 42 miles make sense?	
Jamal rides:	
	 12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 48 miles in 4 hours. 42 miles in $3\frac{1}{2}$ hours is reasonable
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

TRY IT 1.1

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Show answer
330 miles

TRY IT 1.2

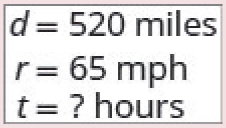
Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Show answer
7 miles

EXAMPLE 2

Rey is planning to drive from his house in Saskatoon to visit his grandmother in Winnipeg, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let *t* = time.
	 $d = 520$ miles $r = 65$ mph $t = ?$ hours *d = 600 km r = 75 km/h t = ? hours*
Step 4. Translate. Write the appropriate formula.	$d = rt$
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	$d = rt$ $520 \overset{?}{=} 65 \cdot 8$ $520 = 520$
Step 7. Answer the question with a complete sentence. Rey's trip will take 8 hours.	

TRY IT 2.1

Lee wants to drive from Kamloops to his brother’s apartment in Banff, a distance of 495 km. If he drives at a steady rate of 90 km/h, how many hours will the trip take?

Show answer
5 1/2 hours

TRY IT 2.2

Yesenia is 168 km from Toronto. If she needs to be in Toronto in 2 hours, at what rate does she need to drive?

Show answer
84 km/h

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In (Example 1) and (Example 2), we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of r and t, the rate and time. But in (Example 2), we had to find the value of t. We substituted in values of d and r and then used algebra to solve for t. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of d and r. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

EXAMPLE 3

Solve the formula $d = rt$ for t:

a. when $d = 520$ and $r = 65$
b. in general

Solution

We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

a) when $d = 520$ and $r = 65$		b) in general	
Write the formula.	$d = rt$	Write the formula.	$d = rt$
Substitute.	$520 = 65t$		
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$	Divide, to isolate t.	$\frac{d}{r} = \frac{rt}{r}$
Simplify.	$8 = t$	Simplify.	$\frac{d}{r} = t$

We say the formula $t = \frac{d}{r}$ is solved for t.

TRY IT 3.1

Solve the formula $d = rt$ for r:

a) when $d = 180 \text{ and } t = 4$ b) in general

Show answer

a) $r = 45$ b) $r = \frac{d}{t}$

TRY IT 3.2

Solve the formula $d = rt$ for r:

a) when $d = 780 \text{ and } t = 12$ b) in general

Show answer

a) $r = 65$ b) $r = \frac{d}{t}$

EXAMPLE 4

Solve the formula $A = \frac{1}{2}bh$ for h:

a) when $A = 90$ and $b = 15$ b) in general

Solution

a) when $A = 90$ and $b = 15$		b) in general	
Write the formula.	$A = \frac{1}{2}bh$	Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$		
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2} 15h$	Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2}bh$
Simplify.	$180 = 15h$	Simplify.	$2A = bh$
Solve for h.	$12 = h$	Solve for h.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$.

TRY IT 4.1

Use the formula $A = \frac{1}{2}bh$ to solve for h:

a) when $A = 170$ and $b = 17$ b) in general

Show answer

a) $h = 20$ b) $h = \frac{2A}{b}$

TRY IT 4.2

Use the formula $A = \frac{1}{2}bh$ to solve for b:

a) when $A = 62$ and $h = 31$ b) in general

Show answer

a) $b = 4$ b) $b = \frac{2A}{h}$

The formula $I = Prt$ is used to calculate simple interest, *I*, for a principal, *P*, invested at rate, *r*, for *t* years.

EXAMPLE 5

Solve the formula $I = Prt$ to find the principal, P:

a) when $I = \$5,600, r = 4\%, t = 7\ years$ b) in general

Solution

a)$I = \$5,600, r = 4\%, t = 7years$		b) in general	
Write the formula.	$I = Prt$	Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$		
Simplify.	$5600 = P(0.28)$	Simplify.	$I = P(rt)$
Divide, to isolate *P*.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$	Divide, to isolate *P*.	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20,000 = P$	Simplify.	$\frac{I}{rt} = P$
The principal is	$\$20,000$		$P = \frac{I}{rt}$

TRY IT 5.1

Use the formula $I = Prt$ to find the principal, P:

a) when $I = \$2,160, r = 6\%, t = 3\ years$ b) in general

Show answer

a) $12,000 b) $P = \frac{I}{rt}$

TRY IT 5.2

Use the formula $I = Prt$ to find the principal,P:

a) when $I = \$5,400, r = 12\%, t = 5\ years$ b) in general

Show answer

a) $9,000 b) $P = \frac{I}{rt}$

Later in this class, and in future algebra classes, you’ll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we’re given an equation with both *x* and *y* on the same side and we’ll solve it for *y*.

EXAMPLE 6

Solve the formula $3x + 2y = 18$ for *y*:

a) when $x = 4$ b) in general

Solution

a) when $x = 4$		b) in general	
	$3x + 2y = 18$		$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$		
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$	Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$	Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$
Simplify.	$y = 3$	Simplify.	$y = -\frac{3x}{2} + 9$

TRY IT 6.1

Solve the formula $3x + 4y = 10$ for *y*:

a) when $x = \frac{14}{3}$ b) in general

Show answer

a)$y = 1$b)$y = \frac{10-3x}{4}$

TY IT 6.2

Solve the formula $5x + 2y = 18$ for *y*:

a) when $x = 4$ b) in general

Show answer
a)$y=-1$b)$y=\frac{18-5x}{2}$

Now we will solve a formula in general without using numbers as a guide.

EXAMPLE 7

Solve the formula $P=a+b+c$ for a.

Solution

We will isolate a on one side of the equation.	$P=a+b+c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P-b-c=a+b+c-b-c$
Simplify.	$P-b-c=a$ $a=P-b-c$

TRY IT 7.1

Solve the formula $P=a+b+c$ for b.

Show answer
$b=P-a-c$

TRY IT 7.2

Solve the formula $P=a+b+c$ for c.

Show answer
$c=P-a-b$

EXAMPLE 8

Solve the formula $6x+5y=13$ for y.

Solution

	$6x + 5y = 13$
Subtract $6x$ from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$
Simplify.	$y = \frac{13 - 6x}{5}$

The fraction is simplified. We cannot divide $13 - 6x$ by 5

TRY IT 8.1

Solve the formula $4x + 7y = 9$ for *y*.

Show answer

$y = \frac{9-4x}{7}$

TRY IT 8.2

Solve the formula $5x + 8y = 1$ for *y*.

Show answer

$y = \frac{1-5x}{8}$

Key Concepts

- **To Solve an Application (with a formula)**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute

in the given information.

5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula: $d = rt$ where *d* = distance, *r* = rate, *t* = time.
- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

1. Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?	2. Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?
3. Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?	4. Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?
5. Marta is taking the bus from Abbotsford to Cranbrook. The distance is 774 km and the bus travels at a steady rate of 86 miles per hour. How long will the bus ride be?	6. Connor wants to drive from Vancouver to the Nakusp, a distance of 630 km. If he drives at a steady rate of 90 km/h, how many hours will the trip take?
7. Kareem wants to ride his bike from Golden, BC to Banff, AB. The distance is 140 km. If he rides at a steady rate of 20 km/h, how many hours will the trip take?	8. Aurelia is driving from Calgary to Edmonton at a rate of 85 km/h. The distance is 300 km. To the nearest tenth of an hour, how long will the trip take?
9. Alejandra is driving to Prince George, 450 km away. If she wants to be there in 6 hours, at what rate does she need to drive?	10. Javier is driving to Vernon, 240 km away. If he needs to be in Vernon in 3 hours, at what rate does he need to drive?
11. Philip got a ride with a friend from Calgary to Kelowna, a distance of 890 km. If the trip took 10 hours, how fast was the friend driving?	12. Aisha took the train from Spokane to Seattle. The distance is 280 miles and the trip took 3.5 hours. What was the speed of the train?

Solve a Formula for a Specific Variable

In the following exercises, use the formula $d = rt$.

13. Solve for t
a) when $d = 240$ and $r = 60$
b) in general

14. Solve for t
a) when $d = 350$ and $r = 70$
b) in general

15. Solve for t
a) when $d = 175$ and $r = 50$
b) in general

16. Solve for t
a) when $d = 510$ and $r = 60$
b) in general

17. Solve for r
a) when $d = 420$ and $t = 6$
b) in general

18. Solve for r
a) when $d = 204$ and $t = 3$
b) in general

19. Solve for r
a) when $d = 180$ and $t = 4.5$
b) in general

In the following exercises, use the formula $A = \frac{1}{2}bh$.

20. Solve for r
a) when $d = 160$ and $t = 2.5$
b) in general

21. Solve for h
a) when $A = 176$ and $b = 22$
b) in general

22. Solve for b
a) when $A = 126$ and $h = 18$
b) in general

23. Solve for the principal, P for
a) $I = \$5,480$,$r = 4\%$,$t = 7$ years
b) in general

24. Solve for b
a) when $A = 65$ and $h = 13$
b) in general

In the following exercises, use the formula I = Prt.

25. Solve for the time, t for
a) $I = \$2,376$,$P = \$9,000$,$r = 4.4\%$
b) in general

26. Solve for the principal, P for
a) $I = \$3,950$,$r = 6\%$,$t = 5$ years
b) in general

27. Solve the formula $2x + 3y = 12$ for y
a) when $x = 3$
b) in general

28. Solve for the time, t for
a) $I = \$624$,$P = \$6,000$,$r = 5.2\%$
b) in general

In the following exercises, solve.

29. Solve the formula $3x - y = 7$ for y
a) when $x = -2$
b) in general

30. Solve the formula $5x + 2y = 10$ for y
a) when $x = 4$
b) in general

31. Solve $a + b = 90$ for b.

32. Solve the formula $4x + y = 5$ for y
a) when $x = -3$
b) in general

33. Solve $180 = a + b + c$ for a.

34. Solve $a + b = 90$ for a.

35. Solve the formula $8x + y = 15$ for y.

36. Solve $180 = a + b + c$ for c.

37. Solve the formula $-4x + y = -6$ for y.

38. Solve the formula $9x + y = 13$ for y.

39. Solve the formula $4x + 3y = 7$ for y.

40. Solve the formula $-5x + y = -1$ for y.

41. Solve the formula $x - y = -4$ for y.

42. Solve the formula $3x + 2y = 11$ for y.

43. Solve the formula $P = 2L + 2W$ for L.

44. Solve the formula $x - y = -3$ for y.

45. Solve the formula $C = \pi d$ for d.

46. Solve the formula $P = 2L + 2W$ for W.

47. Solve the formula $V = LWH$ for L.

48. Solve the formula $C = \pi d$ for π.

49. Solve the formula $V = LWH$ for H.

Everyday Math

50. **Converting temperature.** Yon was visiting the United States and he saw that the temperature in Seattle one day was 50^o Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature.

51. **Converting temperature.** While on a tour in Greece, Tatyana saw that the temperature was 40^o Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

Writing Exercises

52. Solve the equation $5x - 2y = 10$ for x
a) when $y = 10$
b) in general
c) Which solution is easier for you, a) or b)? Why?

53. Solve the equation $2x + 3y = 6$ for y
a) when $x = -3$
b) in general
c) Which solution is easier for you, a) or b)? Why?

Answers

1. 290 miles	3. 30 miles	5. 9 hours.
7. 75 km/h	9. 3.5 hours	11. 7 hours
13. 7	15. 89 km/h	17. a) $t = 4$ b) $t = \frac{d}{r}$
19. a) $t = 3.5$ b) $t = \frac{d}{r}$	21. a) $r = 70$ b) $r = \frac{d}{t}$	23. a) $r = 40$ b) $r = \frac{d}{t}$
25. a) $h = 16$ b) $h = \frac{2A}{b}$	27. a) $b = 10$ b) $b = \frac{2A}{h}$	29. a) $P = \$13,166.67$ b) $P = \frac{I}{rt}$
31. a) $t = 2$ years b) $t = \frac{I}{\mathrm{Pr}}$	33. a) $y = -5$ b) $y = \frac{10-5x}{2}$	35. a) $y = 17$ b) $y = 5 - 4x$
37. $a = 90 - b$	39. $c = 180 - a - b$	41. $y = 13 - 9x$
43. $y = -1 + 5x$	45. $y = \frac{11-3x}{4}$	47. $y = 3 + x$
49. $W = \frac{P-2L}{2}$	51. $\pi = \frac{C}{d}$	53. $H = \frac{V}{LW}$
55. 10°C	57. Answers will vary.	

Attributions

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5.7 Use a Problem-Solving Strategy

Learning Objectives

By the end of this section, you will be able to:

- Approach word problems with a positive attitude
- Use a problem-solving strategy for word problems
- Solve number problems

Approach Word Problems with a Positive Attitude

“If you think you can… or think you can’t… you’re right.”—Henry Ford

The world is full of word problems! Will my income qualify me to rent that apartment? How much punch do I need to make for the party? What size diamond can I afford to buy my girlfriend? Should I fly or drive to my family reunion?

How much money do I need to fill the car with gas? How much tip should I leave at a restaurant? How many socks should I pack for vacation? What size turkey do I need to buy for Thanksgiving dinner, and then what time do I need to put it in the oven? If my sister and I buy our mother a present, how much does each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student below?
Negative thoughts can be barriers to success.

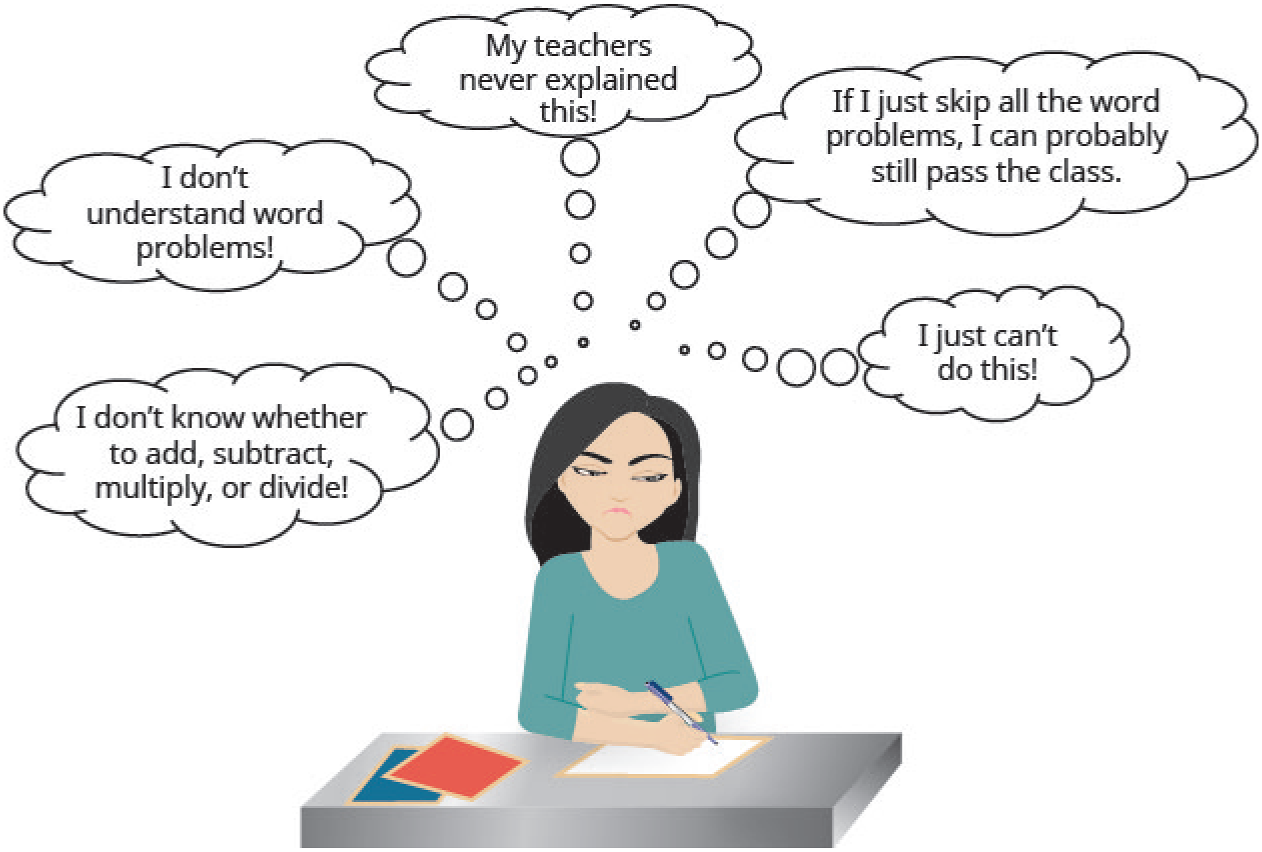

Figure .1

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts. If we take control and believe we can be successful, we will be able to master word problems! Read the positive thoughts in (Figure 2) and say them out loud.
Thinking positive thoughts is a first step towards success.

Figure .2

Think of something, outside of school, that you can do now but couldn't do 3 years ago. Is it driving a car? Snowboarding? Cooking a gourmet meal? Speaking a new language? Your past experiences with word problems happened when you were younger—now you're older and ready to succeed!

Use a Problem-Solving Strategy for Word Problems

We have reviewed translating English phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. We have also translated English sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. We restated the situation in one sentence, assigned a variable, and then wrote an equation to solve the problem. This method works as long as the situation is familiar and the math is not too complicated.

Now, we'll expand our strategy so we can use it to successfully solve any word problem. We'll list the strategy here, and then we'll use it to solve some problems. We summarize below an effective strategy for problem solving.

Use a Problem-Solving Strategy to Solve Word Problems.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

EXAMPLE 1

Pilar bought a purse on sale for $18, which is one-half of the original price. What was the original price of the purse?

Solution

Step 1. Read the problem. Read the problem two or more times if necessary. Look up any unfamiliar words in a dictionary or on the internet.

- *In this problem, is it clear what is being discussed? Is every word familiar?*

Step 2. Identify what you are looking for. Did you ever go into your bedroom to get something and then forget what you were looking for? It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

- *In this problem, the words "what was the original price of the purse" tell us what we need to find.*

Step 3. Name what we are looking for. Choose a variable to represent that quantity. We can use any letter for the variable, but choose one that makes it easy to remember what it represents.

- Let $p =$ the original price of the purse.

Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Translate the English sentence into an algebraic equation.

Reread the problem carefully to see how the given information is related. Often, there is one sentence that gives this information, or it may help to write one sentence with all the important information. Look for clue words to help translate the sentence into algebra. Translate the sentence into an equation.

Restate the problem in one sentence with all the important information.	$\underbrace{18}\quad\underbrace{\text{is}}\quad\underbrace{\text{one-half the original price.}}$
Translate into an equation.	$18 \quad = \quad \frac{1}{2} \cdot p$

Step 5. Solve the equation using good algebraic techniques. Even if you know the solution right away, using good algebraic techniques here will better prepare you to solve problems that do not have obvious answers.

Solve the equation.	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$

Step 6. Check the answer in the problem to make sure it makes sense. We solved the equation and found that $p = 36$, which means "the original price" was $36

- *Does $36 make sense in the problem? Yes, because 18 is one-half of 36, and the purse was on sale at half the original price.*

Step 7. Answer the question with a complete sentence. The problem asked "What was the original price of the purse?"

- *The answer to the question is: "The original price of the purse was $36."*

If this were a homework exercise, our work might look like this:

Pilar bought a purse on sale for $18, which is one-half the original price. What was the original price of the purse?

	Let $p =$ the original price.
	18 is one-half the original price.
	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$
Check. Is \$36 a reasonable price for a purse?	Yes.
Is 18 one half of 36?	$18 \overset{?}{=} \frac{1}{2} \cdot 36$
	$18 = 18$?
	The original price of the purse was \$36.

TRY IT 1.1

Joaquin bought a bookcase on sale for \$120, which was two-thirds of the original price. What was the original price of the bookcase?

Show answer
\$180

TRY IT 1.2

Two-fifths of the songs in Mariel’s playlist are country. If there are 16 country songs, what is the total number of songs in the playlist?

Show answer
40

Let’s try this approach with another example.

EXAMPLE 2

Ginny and her classmates formed a study group. The number of girls in the study group was three more than twice the number of boys. There were 11 girls in the study group. How many boys were in the study group?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How many boys were in the study group?
Step 3. Name. Choose a variable to represent the number of boys.	Let n = the number of boys.
Step 4. Translate. Restate the problem in one sentence with all the important information.	$\underbrace{\text{The number of girls (11)}}\quad\underbrace{\text{was}}\quad\underbrace{\text{three more than twice the number of boys}}$
Translate into an equation.	$11 \quad = \quad 2b+3$
Step 5. Solve the equation.	$11=2b+3$
Subtract 3 from each side.	$11-3=2b+3-3$
Simplify.	$8=2b$
Divide each side by 2.	$\frac{8}{2}=\frac{2b}{2}$
Simplify.	$4=b$
Step 6. Check. First, is our answer reasonable?	Yes, having 4 boys in a study group seems OK. The problem says the number of girls was 3 more than twice the number of boys. If there are four boys, does that make eleven girls? Twice 4 boys is 8. Three more than 8 is 11.
Step 7. Answer the question.	There were 4 boys in the study group.

TRY IT 2.1

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than twice the number of notebooks. He bought 7 textbooks. How many notebooks did he buy?

Show answer

2

TRY IT 2.2

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Show answer
7

Solve Number Problems

Now that we have a problem solving strategy, we will use it on several different types of word problems. The first type we will work on is “number problems.” Number problems give some clues about one or more numbers. We use these clues to write an equation. Number problems don’t usually arise on an everyday basis, but they provide a good introduction to practicing the problem solving strategy outlined above.

EXAMPLE 3

The difference of a number and six is 13. Find the number.

Solution

Step 1. Read the problem. Are all the words familiar?	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let n = the number.
Step 4. Translate. Remember to look for clue words like “difference… of… and…”	
Restate the problem as one sentence.	$\underbrace{\text{The difference of the number and 6}}\quad\underbrace{\text{is}}\quad\underbrace{13}$
Translate into an equation.	$n-6 \qquad = \quad 13$
Step 5. Solve the equation.	$n-6=13$
Simplify.	$n=19$
Step 6. Check.	
The difference of 19 and 6 is 13. It checks!	
Step 7. Answer the question.	The number is 19.

TRY IT 3.1

The difference of a number and eight is 17. Find the number.

Show answer
25

TRY IT 3.2

The difference of a number and eleven is -7. Find the number.

Show answer
4

EXAMPLE 4

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let n $=$ the number.
Step 4. Translate.	
Restate the problem as one sentence.	$\underbrace{\text{The sum of twice a number and 7}}\quad\underbrace{\text{is}}\quad\underbrace{15}$
Translate into an equation.	$2n+7 \quad = \quad 15$
Step 5. Solve the equation.	$2n+7=15$
Subtract 7 from each side and simplify.	$2n=8$
Divide each side by 2 and simplify.	$n=4$
Step 6. Check.	
Is the sum of twice 4 and 7 equal to 15?	$2\cdot4+7 \overset{?}{=} 15$ $15 = 15\checkmark$
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

TRY IT 4.1

The sum of four times a number and two is 14. Find the number.

Show answer
3

TRY IT 4.2

The sum of three times a number and seven is 25. Find the number.

Show answer
6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

EXAMPLE 5

One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name. We have two numbers to name and need a name for each.	
Choose a variable to represent the first number.	Let $n = 1^{\text{st}}$ number.
What do we know about the second number?	One number is five more than another.
	$n + 5 = 2^{\text{nd}}$ number
Step 4. Translate. Restate the problem as one sentence with all the important information.	The sum of the 1st number and the 2nd number is 21.
Translate into an equation.	$\underbrace{1^{\text{st}}\text{ number}} + \underbrace{2^{\text{nd}}\text{ number}}\ \underbrace{=}\ \underbrace{21}$
Substitute the variable expressions.	$n \quad + \quad n+5 \quad = \quad 21$
Step 5. Solve the equation.	$n + n + 5 = 21$
Combine like terms.	$2n + 5 = 21$
Subtract 5 from both sides and simplify.	$2n = 16$
Divide by 2 and simplify.	$n = 8 \quad 1^{\text{st}}$ number
Find the second number, too.	$n + 5 \quad 2^{\text{nd}}$ number
	$8 + 5$
	13
Step 6. Check.	
Do these numbers check in the problem?	
Is one number 5 more than the other?	$13 \overset{?}{=} 8 + 5$
Is thirteen 5 more than 8? Yes.	$13 = 13$?
Is the sum of the two numbers 21?	$8 + 13 \overset{?}{=} 21$
	$21 = 21$?
Step 7. Answer the question.	The numbers are 8 and 13.

TRY IT 5.1

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Show answer
9, 15

TRY IT 5.2

The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Show answer
27, 31

EXAMPLE 6

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name.	
Choose a variable.	Let $n = 1^{\text{st}}$ number.
One number is 4 less than the other.	$n - 4 = 2^{\text{nd}}$ number
Step 4. Translate.	
Write as one sentence.	The sum of the 2 numbers is negative 14.
Translate into an equation.	$\underbrace{1^{\text{st}}\text{ number}} + \underbrace{2^{\text{nd}}\text{ number}}\ \underbrace{\text{is}}\ \underbrace{\text{negative fourteen}}$
Step 5. Solve the equation.	$n \quad + \quad n-4 \quad = \quad -14$
Combine like terms.	$n + n - 4 = -14$
Add 4 to each side and simplify.	$2n - 4 = -14$
Simplify.	$2n = -10$
	$n = -5 \quad 1^{\text{st}}$ number
	$n - 4 \quad 2^{\text{nd}}$ number
	$-5 - 4$
	-9
Step 6. Check.	
Is −9 four less than −5?	$-5 - 4 \stackrel{?}{=} -9$
	$-9 = -9$?
Is their sum −14?	$-5 + (-9) \stackrel{?}{=} -14$
	$-14 = -14$?
Step 7. Answer the question.	The numbers are −5 and −9.

TRY IT 6.1

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Show answer

$-15, -8$

TRY IT 6.2

The sum of two numbers is -18. One number is 40 more than the other. Find the numbers.

Show answer

$-29, 11$

EXAMPLE 7

One number is ten more than twice another. Their sum is one. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for two numbers.
Step 3. Name.	
Choose a variable.	Let $x = 1^{\text{st}}$ number.
One number is 10 more than twice another.	$2x + 10 = 2^{\text{nd}}$ number
Step 4. Translate.	
Restate as one sentence.	Their sum is one.
	The sum of the two numbers is 1.
Translate into an equation.	$x + 2x + 10 = 1$
Step 5. Solve the equation.	
Combine like terms.	$x + 2x + 10 = 1$
Subtract 10 from each side.	$3x + 10 = 1$
Divide each side by 3.	$3x = -9$
	$x = -3$ 1^{st} number
	$2x + 10$ 2^{nd} number
	$2(-3) + 10$
	4
Step 6. Check.	
Is ten more than twice −3 equal to 4?	$2\left(-3\right) + 10 \overset{?}{=} 4$
	$-6 + 10 \overset{?}{=} 4$
	$4 = 4$?
Is their sum 1?	$-3 + 4 \overset{?}{=} 1$
	$1 = 1$?
Step 7. Answer the question.	The numbers are −3 and −4.

TRY IT 7.1

One number is eight more than twice another. Their sum is negative four. Find the numbers.

Show answer

$-4, 0$

TRY IT 7.2

One number is three more than three times another. Their sum is -5. Find the numbers.

Show answer

$-3, -2$

Some number problems involve consecutive integers.*Consecutive integers* are integers that immediately follow each other.

Examples of consecutive integers are:

$$1, 2, 3, 4$$

$$-10, -9, -8, -7$$
$$150, 151, 152, 153$$

Notice that each number is one more than the number preceding it. So if we define the first integer as *n*, the next consecutive integer is $n + 1$. The one after that is one more than $n + 1$, so it is $n + 1 + 1$, which is $n + 2$.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ integer} \\ n+1 & 2^{\text{nd}} \text{ consecutive integer} \\ n+2 & 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{array}$$

EXAMPLE 8

The sum of two consecutive integers is 47. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two consecutive integers
Step 3. Name each number.	Let $n = 1^{\text{st}}$ integer.
	$n + 1 =$ next consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the integers is 47.
Translate into an equation.	$n + n + 1 = 47$
Step 5. Solve the equation.	$n + n + 1 = 47$
Combine like terms.	$2n + 1 = 47$
Subtract 1 from each side.	$2n = 46$
Divide each side by 2.	$n = 23$ 1^{st} integer
	$n + 1$ next consecutive integer
	$23 + 1$
	24
Step 6. Check.	$23 + 24 \stackrel{?}{=} 47$ $47 = 47\checkmark$
Step 7. Answer the question.	The two consecutive integers are 23 and 24.

TRY IT 8.1

The sum of two consecutive integers is 95. Find the numbers.

Show answer
47, 48

TRY IT 8.2

The sum of two consecutive integers is -31. Find the numbers.

Show answer

$-16, -15$

EXAMPLE 9

Find three consecutive integers whose sum is -42.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive integers
Step 3. Name each of the three numbers.	Let $n = 1^{\text{st}}$ integer.
	$n + 1 = 2^{\text{nd}}$ consecutive integer
	$n + 2 = 3^{\text{rd}}$ consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three integers is −42.
Translate into an equation.	$n + n + 1 + n + 2 = -42$
Step 5. Solve the equation.	$n + n + 1 + n + 2 = -42$
Combine like terms.	$3n + 3 = -42$
Subtract 3 from each side.	$3n = -45$
Divide each side by 3.	$n = -15 \quad 1^{\text{st}}$ integer
	$n + 1 \quad 2^{\text{nd}}$ integer
	$-15 + 1$
	-14
	$n + 2 \quad 3^{\text{rd}}$ integer
	$-15 + 2$
	-13
Step 6. Check.	$-13 + (-14) + (-15) \overset{?}{=} -42$ $-42 = -42 ✓$
Step 7. Answer the question.	The three consecutive integers are −13, −14, and −15.

TRY IT 9.1

Find three consecutive integers whose sum is -96.

Show answer

$-33, -32, -31$

TRY IT 9.2

Find three consecutive integers whose sum is -36.

Show answer

$-13, -12, -11$

Now that we have worked with consecutive integers, we will expand our work to include consecutive even integers and consecutive odd integers. *Consecutive even integers* are even integers that immediately follow one another. Examples of consecutive even integers are:

$$18, 20, 22$$

$$64, 66, 68$$
$$-12, -10, -8$$

Notice each integer is 2 more than the number preceding it. If we call the first one *n*, then the next one is $n + 2$. The next one would be $n + 2 + 2$ or $n + 4$.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ even integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer} \;.\;.\;.\; \text{etc.} \end{array}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 77, 79, and 81

$$77, 79, 81$$

$$n, n+2, n+4$$

$$\begin{array}{ll} n & 1^{\text{st}} \text{ odd integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer} \;.\;.\;.\; \text{etc.} \end{array}$$

Does it seem strange to add 2 (an even number) to get from one odd integer to the next? Do you get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you don’t have to do anything different. The pattern is still the same—to get from one odd or one even integer to the next, add 2

EXAMPLE 10

Find three consecutive even integers whose sum is 84

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive even integers
Step 3. Name the integers.	Let $n = 1^{\text{st}}$ even integer. $n + 2 = 2^{\text{nd}}$ consecutive even integer $n + 4 = 3^{\text{rd}}$ consecutive even integer
Step 4. Translate.	
Restate as one sentence.	The sume of the three even integers is 84.
Translate into an equation.	$n + n + 2 + n + 4 \ = \ 84$
Step 5. Solve the equation.	
Combine like terms.	$n + n + 2 + n + 4 \ = \ 84$
Subtract 6 from each side.	$3n + 6 \ = \ 84$
Divide each side by 3.	$3n \ = \ 78$
	$n = 26 \quad 1^{\text{st}}$ integer $n + 2 \quad 2^{\text{nd}}$ integer $26 + 2$ 28 $n + 4 \quad 3^{\text{rd}}$ integer $26 + 4$ 30
Step 6. Check.	$26 + 28 + 30 \stackrel{?}{=} 84$ $84 = 84\checkmark$
Step 7. Answer the question.	The three consecutive integers are 26, 28, and 30.

TRY IT 10.1

Find three consecutive even integers whose sum is 102

Show answer
32, 34, 36

TRY IT 10.2

Find three consecutive even integers whose sum is -24.

Show answer
$-10, -8, -6$

EXAMPLE 11

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How much does the husband earn?
Step 3. Name.	
Choose a variable to represent the amount the husband earns.	Let $h =$ the amount the husband earns.
The wife earns $16,000 less than twice that.	$2h - 16,000$ the amount the wife earns.
Step 4. Translate.	Together the husband and wife earn $110,000.
Restate the problem in one sentence with all the important information.	$\underbrace{\text{The amount the husband earns}}\ \text{plus}\ \underbrace{\text{the amount the wife earns}}\ \text{is}\ \$110{,}000$
Translate into an equation.	$h + 2h - 16{,}000 = 110{,}000$
Step 5. Solve the equation.	h + 2h − 16,000 = 110,000
Combine like terms.	3h − 16,000 = 110,000
Add 16,000 to both sides and simplify.	3h = 126,000
Divide each side by 3.	h = 42,000
	$42,000 amount husband earns
	2h − 16,000 amount wife earns
	2(42,000) − 16,000
	84,000 − 16,000
	68,000
Step 6. Check.	If the wife earns $68,000 and the husband earns $42,000 is the total $110,000? Yes!
Step 7. Answer the question.	The husband earns $42,000 a year.

TRY IT 11.1

According to the National Automobile Dealers Association, the average cost of a car in 2014 was 28,500. This was 1,500 less than 6 times the cost in 1975. What was the average cost of a car in 1975?

Show answer
5,000

TRY IT 11.2

The Canadian Real Estate Association (CREA) data shows that the median price of new home in the Canada in December 2018 was $470,000. This was $14,000 more than 19 times the price in December 1967. What was the median price of a new home in December 1967?

Show answer
$24,000

Key Concepts

- **Problem-Solving Strategy**
    1. **Read** the problem. Make sure all the words and ideas are understood.
    2. **Identify** what we are looking for.
    3. **Name** what we are looking for. Choose a variable to represent that quantity.
    4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
    5. **Solve** the equation using good algebra techniques.
    6. **Check** the answer in the problem and make sure it makes sense.
    7. **Answer** the question with a complete sentence.
- **Consecutive Integers**
  Consecutive integers are integers that immediately follow each other.

$$\begin{array}{ll} n & 1^{\text{st}}\text{ integer} \\ n+1 & 2^{\text{nd}}\text{ integer consecutive integer} \\ n+2 & 3^{\text{rd}}\text{ consecutive integer . . . etc.} \end{array}$$

  Consecutive even integers are even integers that immediately follow one another.

$$\begin{array}{ll} n & 1^{\text{st}}\text{ integer} \\ n+2 & 2^{\text{nd}}\text{ integer consecutive integer} \\ n+4 & 3^{\text{rd}}\text{ consecutive integer . . . etc.} \end{array}$$

  Consecutive odd integers are odd integers that immediately follow one another.

$$\begin{array}{ll} n & 1^{\text{st}}\text{ integer} \\ n+2 & 2^{\text{nd}}\text{ integer consecutive integer} \\ n+4 & 3^{\text{rd}}\text{ consecutive integer . . . etc.} \end{array}$$

Practice Makes Perfect

Use the Approach Word Problems with a Positive Attitude

In the following exercises, prepare the lists described.

1. List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.	2. List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

Use a Problem-Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

3. Two-thirds of the children in the fourth-grade class are girls. If there are 20 girls, what is the total number of children in the class?	4. Three-fifths of the members of the school choir are women. If there are 24 women, what is the total number of choir members?
5. Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?	6. One-fourth of the candies in a bag of M&M's are red. If there are 23 red candies, how many candies are in the bag?
7. There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.	8. There are 18 Cub Scouts in Pack 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.
9. Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?	10. Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?
11. Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?	12. Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?
13. Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than three times the amount invested in bonds. How much does Laurie have invested in bonds?	14. Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Number Problems

In the following exercises, solve each number word problem.

15. The sum of a number and eight is 12. Find the number.

16. The sum of a number and nine is 17. Find the number.

17. The difference of a number and 12 is three. Find the number.

18. The difference of a number and eight is four. Find the number.

19. The sum of three times a number and eight is 23. Find the number.

20. The sum of twice a number and six is 14. Find the number.

21.The difference of twice a number and seven is 17. Find the number.

22. The difference of four times a number and seven is 21. Find the number.

23. Three times the sum of a number and nine is 12. Find the number.

24. Six times the sum of a number and eight is 30. Find the number.

25. One number is six more than the other. Their sum is 42. Find the numbers.

26. One number is five more than the other. Their sum is 33. Find the numbers.

27. The sum of two numbers is 20. One number is four less than the other. Find the numbers.

28. The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

29. The sum of two numbers is -45. One number is nine more than the other. Find the numbers.

30. The sum of two numbers is -61. One number is 35 more than the other. Find the numbers.

31. The sum of two numbers is -316. One number is 94 less than the other. Find the numbers.

32. The sum of two numbers is -284. One number is 62 less than the other. Find the numbers.

33. One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

34. One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

35. One number is five more than another. If their sum is increased by nine, the result is 60. Find the numbers.

36. One number is eight more than another. If their sum is increased by 17, the result is 95. Find the numbers.

37. One number is one more than twice another. Their sum is -5. Find the numbers.

38. One number is six more than five times another. Their sum is six. Find the numbers.

39. The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

40. The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

41. The sum of two consecutive integers is 77. Find the integers.

42. The sum of two consecutive integers is 89. Find the integers.

43. The sum of two consecutive integers is -23. Find the integers.

44. The sum of two consecutive integers is -37. Find the integers.

45. The sum of three consecutive integers is 78. Find the integers.

46. The sum of three consecutive integers is 60. Find the integers.

47. Find three consecutive integers whose sum is -36.

48. Find three consecutive integers whose sum is -3.

49. Find three consecutive even integers whose sum is 258.

50. Find three consecutive even integers whose sum is 222.

51. Find three consecutive odd integers whose sum is 171.

52. Find three consecutive odd integers whose sum is 291.

53. Find three consecutive even integers whose sum is -36.

54. Find three consecutive even integers whose sum is -84.

55. Find three consecutive odd integers whose sum is -213.

56. Find three consecutive odd integers whose sum is -267.

Everyday Math

57. **Sale Price.** Patty paid $35 for a purse on sale for $10 off the original price. What was the original price of the purse?

58. **Sale Price.** Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

59. **Buying in Bulk.** Minh spent $6.25 on five sticker books to give his nephews. Find the cost of each sticker book.

60. **Buying in Bulk.** Alicia bought a package of eight peaches for $3.20. Find the cost of each peach.

61. **Price before Sales Tax.** Tom paid $1,166.40 for a new refrigerator, including $86.40 tax. What was the price of the refrigerator?

62. **Price before Sales Tax.** Kenji paid $2,279 for a new living room set, including $129 tax. What was the price of the living room set?

Writing Exercises

63. What has been your past experience solving word problems?

64. When you start to solve a word problem, how do you decide what to let the variable represent?

65. What are consecutive odd integers? Name three consecutive odd integers between 50 and 60.

66. What are consecutive even integers? Name three consecutive even integers between -50 and -40.

Answers

1. Answers will vary	3. 30	5. 125
7. 6	9. 58	11. $750
13. $13,500	15. 4	17. 15
19. 5	21. 12	23. -5
25. 18, 24	27. 8, 12	29. $-18, -27$
31. $-111, -205$	33. 32, 46	35. 23, 28
37. $-2, -3$	39. 4, 10	41. 38, 39
43. $-11, -12$	45. 25, 26, 27	47. $-11, -12, -13$
49. 84, 86, 88	51. 55, 57, 59	53. $-10, -12, -14$
55. $-69, -71, -73$	57. $45	59. $1.25
61. $1080	63. Answers will vary	65. Consecutive odd integers are odd numbers that immediately follow each other. An example of three consecutive odd integers between 50 and 60 would be 51, 53, and 55.

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5.8 Chapter Review

Review Exercises

Verify a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.

1. $w - 8 = 5,\ w = 3$	2. $x + 16 = 31,\ x = 15$
3. $4a = 72,\ a = 18$	4.$-9n = 45,\ n = 54$

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction Property of Equality.

5. $y + 2 = -6$	6. $x + 7 = 19$
7. $n + 3.6 = 5.1$	8. $a + \frac{1}{3} = \frac{5}{3}$

In the following exercises, solve each equation using the Addition Property of Equality.

9. $x - 9 = -4$	10. $u - 7 = 10$
11. $p - 4.8 = 14$	12. $c - \frac{3}{11} = \frac{9}{11}$

In the following exercises, solve each equation.

13. $y + 16 = -9$	14. $n - 12 = 32$
15. $d - 3.9 = 8.2$	16. $f + \frac{2}{3} = 4$

Solve Equations That Require Simplification

In the following exercises, solve each equation.

17. $7x+10-6x+3=5$

18. $y+8-15=-3$

19. $8\left(3p+5\right)-23\left(p-1\right)=35$

20. $6\left(n-1\right)-5n=-14$

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

21. Four less than n is 13.

22. The sum of -6 and m is 25.

Translate and Solve Applications

In the following exercises, translate into an algebraic equation and solve.

23. Tan weighs 146 pounds. Minh weighs 15 pounds more than Tan. How much does Minh weigh?

24. Rochelle’s daughter is 11 years old. Her son is 3 years younger. How old is her son?

25. Elissa earned $152.84 this week, which was $21.65 more than she earned last week. How much did she earn last week?

26. Peter paid $9.75 to go to the movies, which was $46.25 less than he paid to go to a concert. How much did he pay for the concert?

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the division and multiplication properties of equality and check the solution.

27. $13a=-65$

28. $8x=72$

29. $-y=4$

30. $0.25p=5.25$

31. $\frac{y}{-10}=30$

32. $\frac{n}{6}=18$

33. $\frac{5}{8}u=\frac{15}{16}$

34. $36=\frac{3}{4}x$

35. $\frac{c}{9}=36$

36. $-18m=-72$

37. $\frac{11}{12}=\frac{2}{3}y$

38. $0.45x=6.75$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

39. $24x + 8x - 11x = -7 - 14$	40. $5r - 3r + 9r = 35 - 2$
41. $-9\left(d - 2\right) - 15 = -24$	42. $\frac{11}{12}n - \frac{5}{6}n = 9 - 5$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

43. The quotient of *b* and and 9 is -27.	44. 143 is the product of -11 and *y*.
45. The difference of *s* and one-twelfth is one fourth.	46. The sum of *q* and one-fourth is one.

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

47. Janet gets paid \$24 per hour. She heard that this is $\frac{3}{4}$ of what Adam is paid. How much is Adam paid per hour?	48. Ray paid \$21 for 12 tickets at the county fair. What was the price of each ticket?

Solve an Equation with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.

49. $10w - 5 = 65$	50. $8p + 7 = 47$
51. $32 = -4 - 9n$	52. $3x + 19 = -47$

Solve an Equation with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.

53. $5a + 21 = 2a$	54. $7y = 6y - 13$
55. $4x - \frac{3}{8} = 3x$	56. $k = -6k - 35$

Solve an Equation with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

57. $5n-20=-7n-80$	58. $12x-9=3x+45$
59. $\frac{5}{8}c-4=\frac{3}{8}c+4$	60. $4u+16=-19-u$

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

61. $9\left(2p-5\right)=72$	62. $6\left(x+6\right)=24$
63. $8+3\left(n-9\right)=17$	64. - $\left(s+4\right)=18$
65. $\frac{1}{3}\left(6m+21\right)=m-7$	66. $23-3\left(y-7\right)=8$
67. $0.25\left(q-8\right)=0.1\left(q+7\right)$	68. $4\left(3.5y+0.25\right)=365$
69. $5+7\left(2-5x\right)=2\left(9x+1\right)-\left(13x-57\right)$	70. $8\left(r-2\right)=6\left(r+10\right)$
71. $2\left[-16+5\left(8k-6\right)\right]=8\left(3-4k\right)-32$	72. $\left(9n+5\right)-\left(3n-7\right)=20-\left(4n-2\right)$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

73. $9u+32=15\left(u-4\right)-3\left(2u+21\right)$	74. $17y-3\left(4-2y\right)=11\left(y-1\right)+12y-1$
75. $21\left(c-1\right)-19\left(c+1\right)=2\left(c-20\right)$	76. $-8\left(7m+4\right)=-6\left(8m+9\right)$

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

77. $\frac{1}{3}x+\frac{1}{5}x=8$	78. $\frac{2}{5}n-\frac{1}{10}=\frac{7}{10}$
79. $\frac{1}{2}\left(k-3\right)=\frac{1}{3}\left(k+16\right)$	80. $\frac{3}{4}a-\frac{1}{3}=\frac{1}{2}a-\frac{5}{6}$
81. $\frac{5y-1}{3}+4=\frac{-8y+4}{6}$	82. $\frac{3x-2}{5}=\frac{3x+4}{8}$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

83. $0.36u + 2.55 = 0.41u + 6.8$	84. $0.8x - 0.3 = 0.7x + 0.2$

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

85. Mallory is taking the bus from Edmonton to North Battleford. The distance is 300 miles and the bus travels at a steady rate of 60 miles per hour. How long will the bus ride be?	86. Natalie drove for $7\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?
87. Link rode his bike at a steady rate of 15 miles per hour for $2\frac{1}{2}$ hours. How much distance did he travel?	88. Aaron's friend drove him from Williams Lake to Kamloops. The distance is 187 miles and the trip took 2.75 hours. How fast was Aaron's friend driving?

Solve a Formula for a Specific Variable

In the following exercises, solve.

89. Use the formula. $d = rt$ to solve for r a) when when $d = 451$ and $t = 5.5$ b) in general	90. Use the formula. $d = rt$ to solve for t a) when $d = 510$ and $r = 60$ b) in general
91. Use the formula $A = \frac{1}{2}bh$ to solve for h a) when $A = 153$ and $b = 18$ b) in general	92. Use the formula $A = \frac{1}{2}bh$ to solve for b a) when $A = 390$ and $h = 26$ b) in general
93. Solve the formula $4x + 3y = 6$ for y a) when $x = -2$ b) in general	94. Use the formula $I = Prt$ to solve for the principal, P for a) $I = \$2,501$,$r = 4.1\%$,$t = 5$ years b) in general
95. Solve the formula $V = LWH$ for H.	96. Solve $180 = a + b + c$ for c.

Everyday Math

97. Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

Review Exercises Answers

1. no	3. yes
5. $y = -8$	7. $n = 1.5$
9. $x = 5$	11. $p = 18.8$
13. $y = -25$	15. $d = 12.1$
17. $x = -8$	19. $p = -28$
21. $n - 4 = 13; n = 17$	23. 161 pounds
25. $131.19	27. $a = -5$
29. $y = -4$	31. $y = -300$
33. $u = \frac{3}{2}$	35. $c = 324$
37. $y = \frac{11}{8}$	39. $x = -1$
41. $d = 3$	43. $\frac{b}{9} = -27; b = -243$
45. $s - \frac{1}{12} = \frac{1}{4}; s = \frac{1}{3}$	47. $32
49. $w = 7$	51. $n = -4$
53. $a = -7$	55. $x = \frac{3}{8}$
57. $n = -5$	59. $c = 32$
61. $p = \frac{13}{2}$	63. $n = 12$
65. $m = -14$	67. $q = 18$
69. $x = -1$	71. $k = \frac{3}{4}$
73. contradiction; no solution	75. identity; all real numbers
77. $x = 15$	79. $k = 41$
81. $y = -1$	83. $u = -85$
85. 5 hours	87. 37.5 miles
89. a)$r = 82 \text{ mph}$; b)$r = \frac{D}{t}$	91. a)$h = 17$b)$h = \frac{2A}{b}$
93. a)$y = \frac{14}{3}$b)$y = \frac{6-4x}{3}$	95. $H = \frac{V}{LW}$

Practice Test

Determine whether each number is a solution to the equation $3x+5=20$.

1. a) 5 b) $\frac{23}{5}$

In the following exercises, solve each equation.

2. $n-18=31$	3. $9c=144$
4. $4y-8=16$	5. $-8x-15+9x-1=-21$
6. $-15a=120$	7. $\frac{2}{3}x=6$
8. $x-3.8=8.2$	9. $10y=-5y-60$
10. $8n-2=6n-12$	11. $9m-2-4m-m=42-8$
12. $-5\left(2x-1\right)=45$	13. - $\left(d-9\right)=23$
14. $\frac{1}{4}\left(12m-28\right)=6-2\left(3m-1\right)$	15. $2\left(6x-5\right)-8=-22$
16.$8\left(3a-5\right)-7\left(4a-3\right)=20-3a$	17. $\frac{1}{4}p-\frac{1}{3}=\frac{1}{2}$
18. $0.1d+0.25\left(d+8\right)=4.1$	19. $14n-3\left(4n+5\right)=-9+2\left(n-8\right)$
20. $9\left(3u-2\right)-4\left[6-8\left(u-1\right)\right]=3\left(u-2\right)$	21. Solve the formula $x-2y=5$ for *y* a) when $x=-3$ b) in general
22. Samuel paid $25.82 for gas this week, which was $3.47 less than he paid last week. How much had he paid last week?	

Practice Test Answers

1. a) yes b) no	2. $n = 49$
3. $c = 16$	4. $y = 6$
5. $x = -5$	6. $a = 8$
7. $x = 9$	8. $x = 12$
9. $y = -4$	10. $n = -5$
11. $m = 9$	12. $x = -4$
13. $d = -14$	14. $m = \frac{5}{3}$
15. $x = -\frac{1}{3}$	16. $a = -39$
17. $p = \frac{10}{3}$	18. $d = 6$
19. contradiction; no solution	20. $u = \frac{17}{14}$
21. a) $y = 4$ b) $y = \frac{5-x}{2}$	22. $\$29.29$

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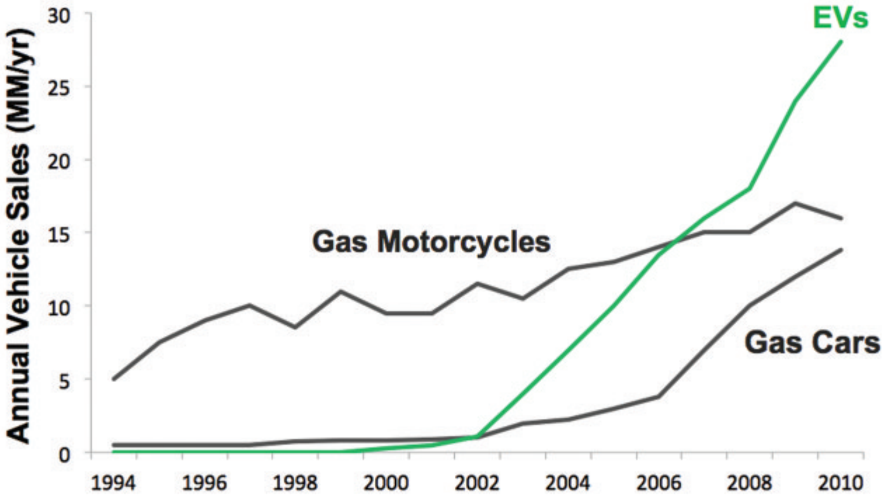
CHAPTER 6 Linear Equations and Graphing

This graph illustrates the annual vehicle sales of gas motorcycles, gas cars, and electric vehicles from 1994 to 2010. It is a line graph with *x*– and *y*-axes, one of the most common types of graphs. (credit: Steve Jurvetson, Flickr)



Graphs are found in all areas of our lives—from commercials showing you which cell phone carrier provides the best coverage, to bank statements and news articles, to the boardroom of major corporations. In this chapter, we will study the rectangular coordinate system, which is the basis for most consumer graphs. We will look at linear graphs, slopes of lines, and equations of lines.

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6.1 Use the Rectangular Coordinate System

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to a linear equation in two variables

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a **rectangular coordinate system**. The rectangular coordinate system is also called the *xy*-plane or the ‘coordinate plane.’

The horizontal number line is called the *x-axis*. The vertical number line is called the *y-axis*. The *x*-axis and the *y*-axis together form the rectangular coordinate system. These axes divide a plane into four regions, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See (Figure 1).

‘Quadrant’ has the root ‘quad,’ which means ‘four.’

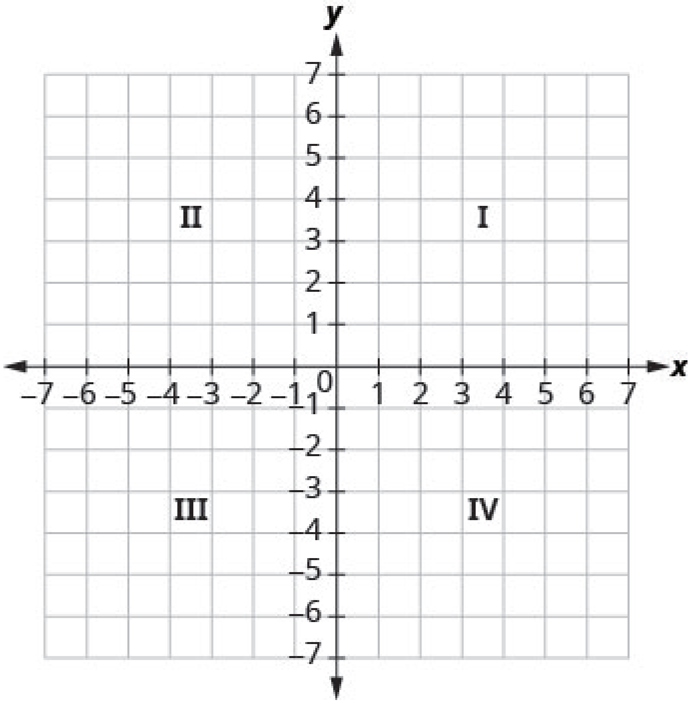
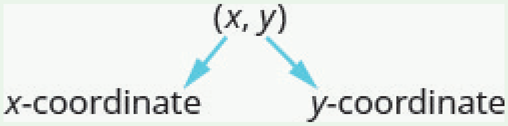


Figure .1

In the rectangular coordinate system, every point is represented by an *ordered pair*. The first number in the ordered pair is the ***x*-coordinate** of the point, and the second number is the ***y*-coordinate** of the point.

Ordered pair

An ordered pair, (x, y), gives the coordinates of a point in a rectangular coordinate system.



The first number is the x-coordinate.

The second number is the y-coordinate.

The phrase ‘ordered pair’ means the order is important. What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

The origin

The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$.

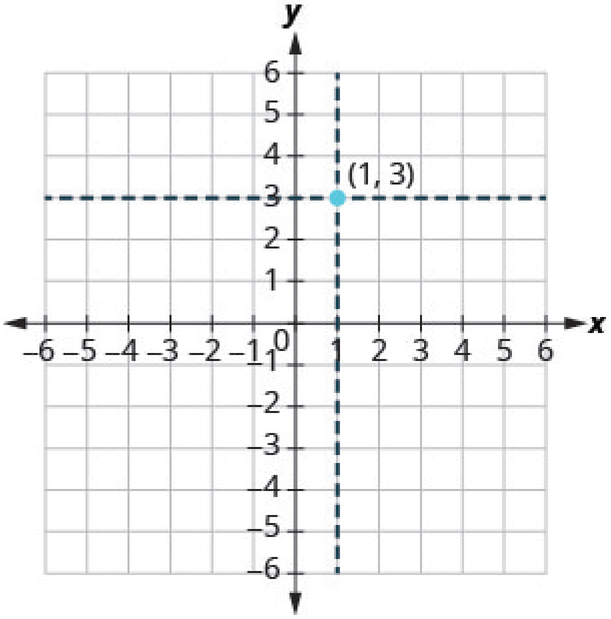


Figure .2

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

EXAMPLE 1

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:
A $(-5, 4)$ B $(-3, -4)$ C $(2, -3)$ D $(-2, 3)$ E $\left(3, \frac{5}{2}\right)$.

Solution

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate.

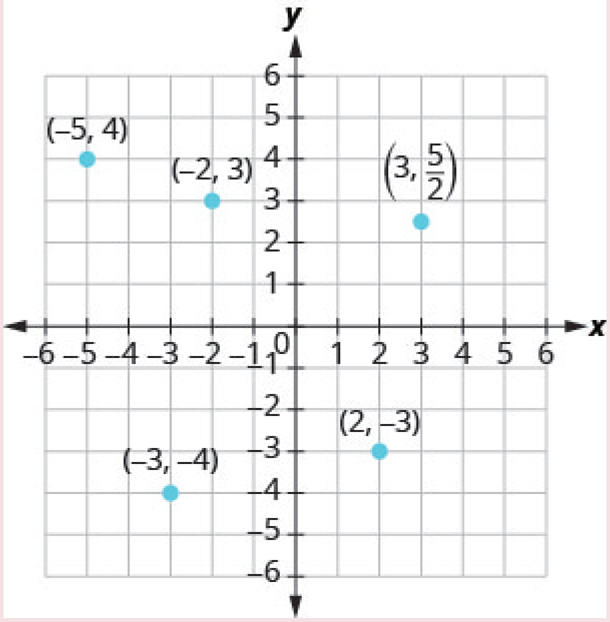
A. Since $x = -5$, the point is to the left of the *y*-axis. Also, since $y = 4$, the point is above the *x*-axis. The point $(-5, 4)$ is in Quadrant II.

B. Since $x = -3$, the point is to the left of the *y*-axis. Also, since $y = -4$, the point is below the *x*-axis. The point $(-3, -4)$ is in Quadrant III.

C. Since $x = 2$, the point is to the right of the *y*-axis. Since $y = -3$, the point is below the *x*-axis. The point $(2, -3)$ is in Quadrant lV.

D. Since $x = -2$, the point is to the left of the *y*-axis. Since $y = 3$, the point is above the *x*-axis. The point $(-2, 3)$ is in Quadrant II.

E. Since $x = 3$, the point is to the right of the *y*-axis. Since $y = \frac{5}{2}$, the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.

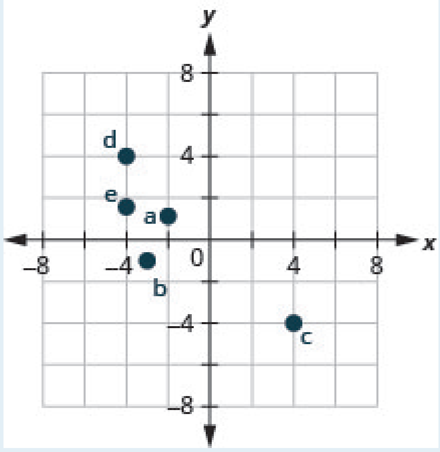


TRY IT 1.1

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

A $(-2, 1)$ B $(-3, -1)$ C $(4, -4)$ D $(-4, 4)$ E $\left(-4, \frac{3}{2}\right)$.

Show answer

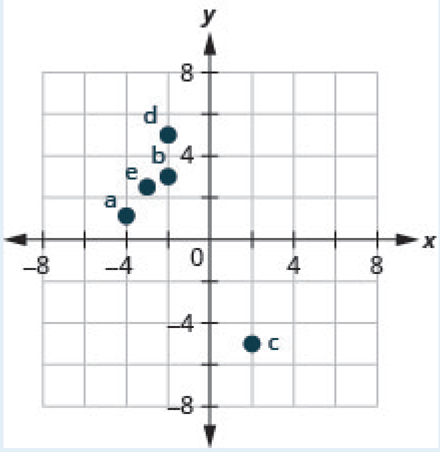


TRY IT 1.2

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

A $(-4,1)$ B $(-2,3)$ C $(2,-5)$ D $(-2,5)$ E $\left(-3,\frac{5}{2}\right)$

Show answer



How do the signs affect the location of the points? You may have noticed some patterns as you graphed the points in the previous example.

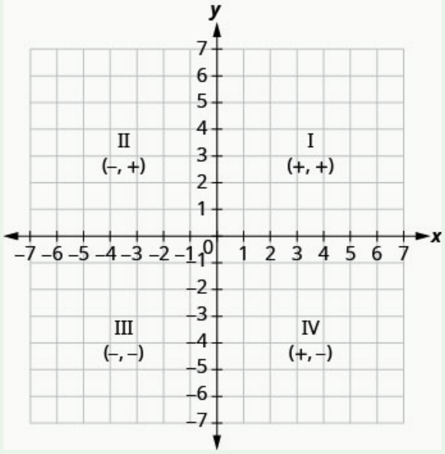
For the point in (Figure 2) in Quadrant IV, what do you notice about the signs of the coordinates? What about the signs of the coordinates of points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point $(-2,5)$ is located? In which quadrant is $(2,-5)$ located?

Quadrants

We can summarize sign patterns of the quadrants in this way.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$



What if one coordinate is zero as shown in (Figure 3)? Where is the point $(0, 4)$ located? Where is the point $(-2, 0)$ located?

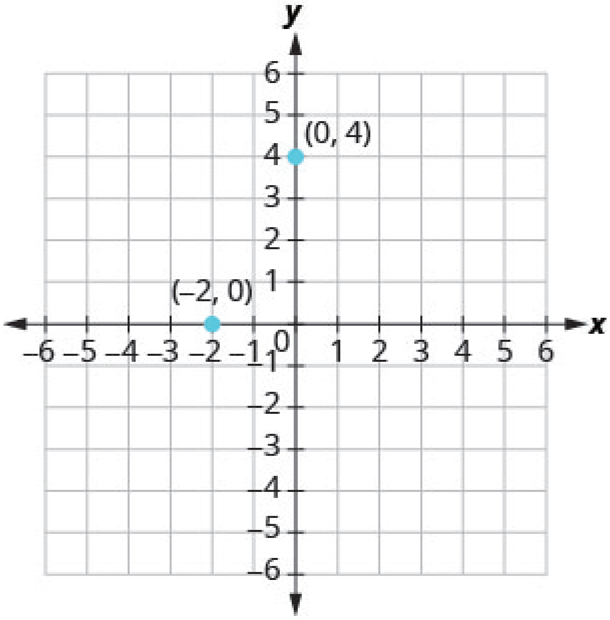


Figure .3

The point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.

Points on the axes

Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.

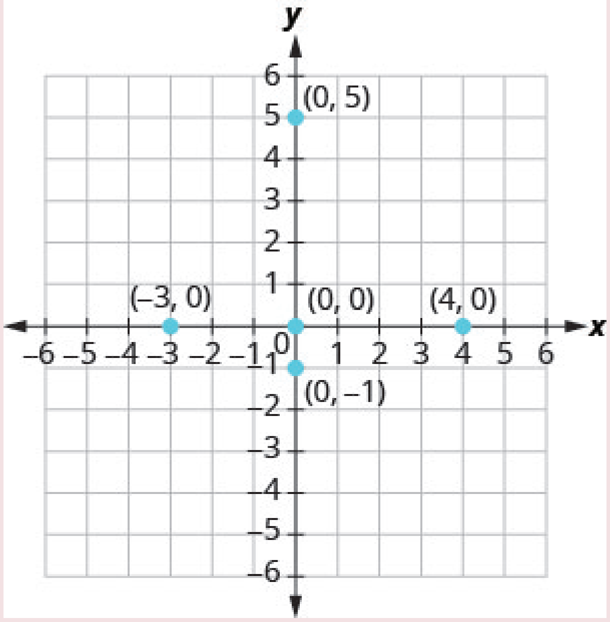
Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.

EXAMPLE 2

Plot each point:A $(0, 5)$ B $(4, 0)$ C $(-3, 0)$ D $(0, 0)$ E $(0, -1)$.

Solution

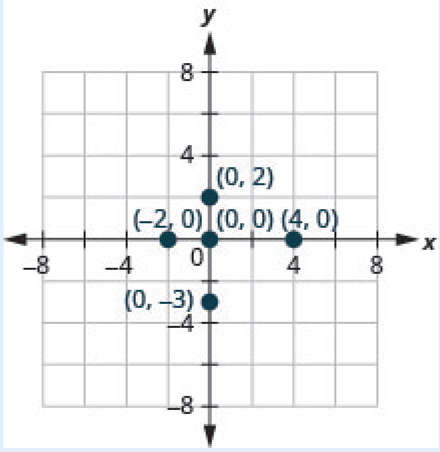
A. Since $x = 0$, the point whose coordinates are $(0, 5)$ is on the *y*-axis.
B. Since $y = 0$, the point whose coordinates are $(4, 0)$ is on the *x*-axis.
C. Since $y = 0$, the point whose coordinates are $(-3, 0)$ is on the *x*-axis.
D. Since $x = 0$ and $y = 0$, the point whose coordinates are $(0, 0)$ is the origin.
E. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the *y*-axis.



TRY IT 2.1

Plot each point: A $(4, 0)$ B $(-2, 0)$ C $(0, 0)$ D $(0, 2)$ E $(0, -3)$.

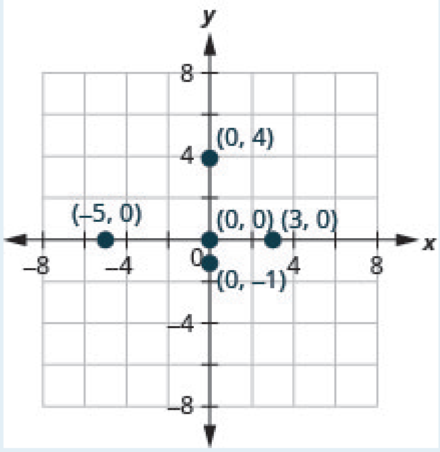
Show answer



TRY IT 2.2

Plot each point: A $(-5, 0)$ B $(3, 0)$ C $(0, 0)$ D $(0, -1)$ E $(0, 4)$.

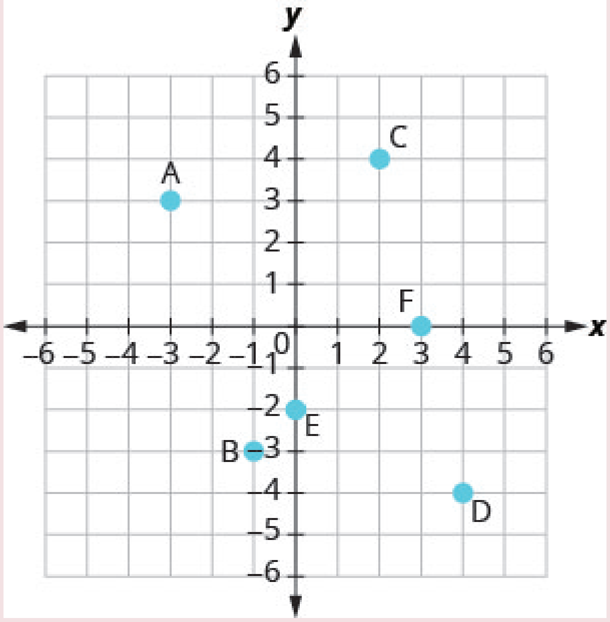
Show answer



In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the *x*-coordinate of a point on a graph, read the number on the *x*-axis directly above or below the point. To identify the *y*-coordinate of a point, read the number on the *y*-axis directly to the left or right of the point. Remember, when you write the ordered pair use the correct order, (x, y).

EXAMPLE 3

Name the ordered pair of each point shown in the rectangular coordinate system.



Solution

Point A is above -3 on the *x*-axis, so the *x*-coordinate of the point is -3.

- The point is to the left of 3 on the *y*-axis, so the *y*-coordinate of the point is 3.
- The coordinates of the point are $(-3, 3)$.

Point B is below -1 on the *x*-axis, so the *x*-coordinate of the point is -1.

- The point is to the left of -3 on the *y*-axis, so the *y*-coordinate of the point is -3.
- The coordinates of the point are $(-1, -3)$.

Point C is above 2 on the *x*-axis, so the *x*-coordinate of the point is 2

- The point is to the right of 4 on the *y*-axis, so the *y*-coordinate of the point is 4.
- The coordinates of the point are $(2, 4)$.

Point D is below 4 on the *x*-axis, so the *x*-coordinate of the point is 4

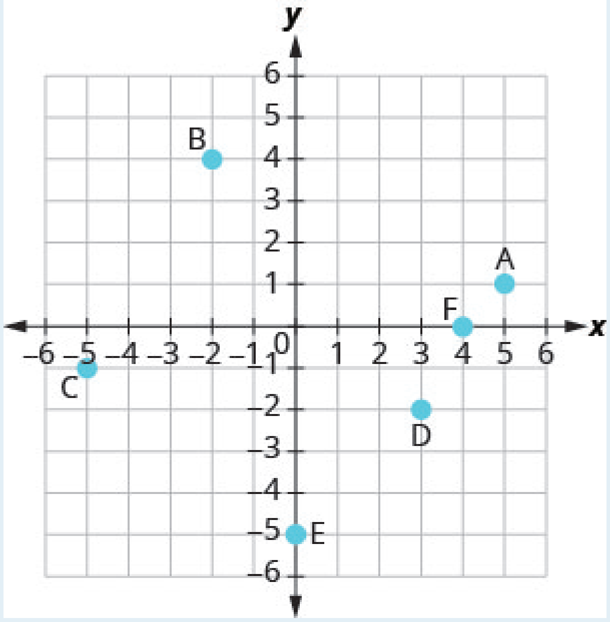
- The point is to the right of -4 on the *y*-axis, so the *y*-coordinate of the point is -4.
- The coordinates of the point are $(4, -4)$.

Point E is on the *y*-axis at $y = -2$. The coordinates of point E are $(0, -2)$.

Point F is on the *x*-axis at $x = 3$. The coordinates of point F are $(3, 0)$.

TRY IT 3.1

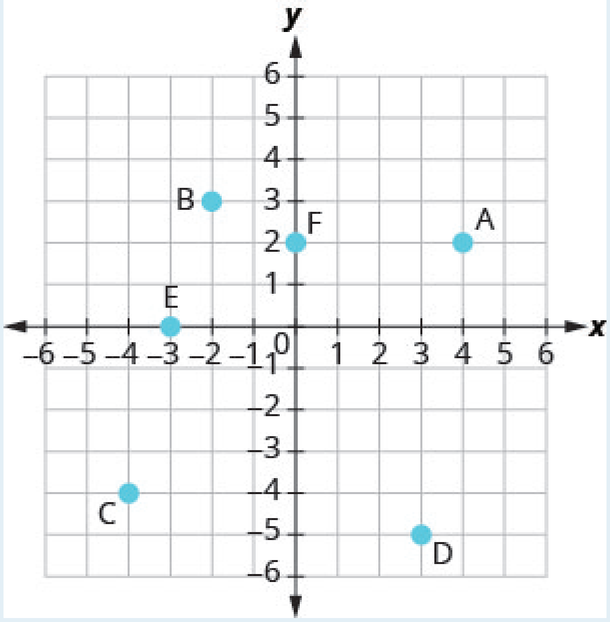
Name the ordered pair of each point shown in the rectangular coordinate system.



Show answer
A: $(5,1)$ B: $(-2,4)$ C: $(-5,-1)$ D: $(3,-2)$ E: $(0,-5)$ F: $(4,0)$

TRY IT 3.2

Name the ordered pair of each point shown in the rectangular coordinate system.



Show answer

A: $(4, 2)$ B: $(-2, 3)$ C: $(-4, -4)$ D: $(3, -5)$ E: $(-3, 0)$ F: $(0, 2)$

Verify Solutions to an Equation in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. The process of solving an equation ended with a statement like $x = 4$. (Then, you checked the solution by substituting back into the equation.) Here's an example of an equation in one variable, and its one solution.

$$\begin{aligned} 3x + 5 &= 17 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. Equations of this form are called **linear equations in two variables**.

Linear equation

An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation **in two variables**.

Notice the word *line* in **linear**. Here is an example of a linear equation in two variables, x and y.

$$Ax + By = C$$

$$x + 4y = 8$$

$$A = 1,\ B = 4,\ C = 8$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

	$y = -3x + 5$
Add to both sides.	$y + 3x = -3x + 5 + 3x$
Simplify.	$y + 3x = 5$
Use the Commutative Property to put it in $Ax + By = C$ form.	$3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in *standard form*.

Standard Form of Linear Equation

A linear equation is in standard form when it is written $Ax + By = C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a **solution** of the linear equation $Ax + By = C$, if the equation is a true statement when the *x*– and *y*-values of the ordered pair are substituted into the equation.

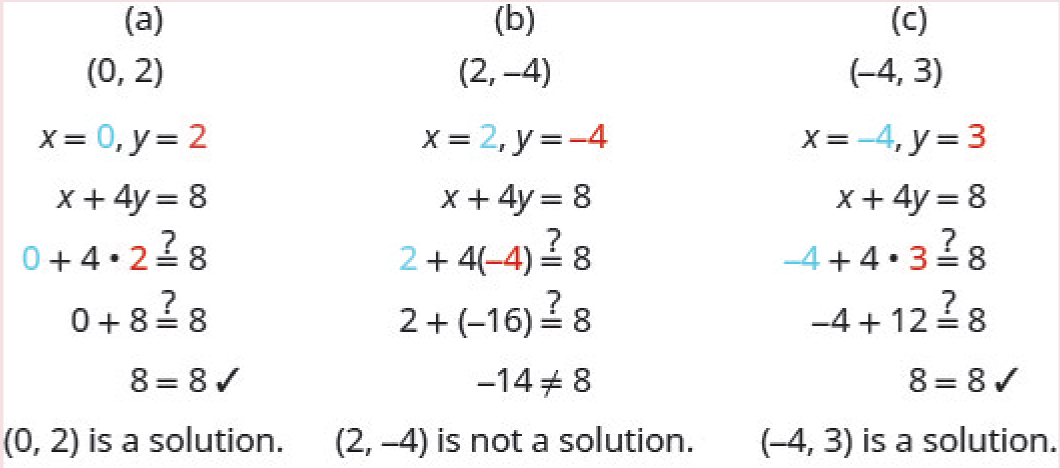
EXAMPLE 4

Determine which ordered pairs are solutions to the equation $x + 4y = 8$.

A $(0, 2)$ B $(2, -4)$ C $(-4, 3)$

Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if the result is a true statement.



(a)	(b)	(c)
(0, 2)	(2, –4)	(–4, 3)
$x = 0, y = 2$	$x = 2, y = -4$	$x = -4, y = 3$
$x + 4y = 8$	$x + 4y = 8$	$x + 4y = 8$
$0 + 4 \cdot 2 \overset{?}{=} 8$	$2 + 4(-4) \overset{?}{=} 8$	$-4 + 4 \cdot 3 \overset{?}{=} 8$
$0 + 8 \overset{?}{=} 8$	$2 + (-16) \overset{?}{=} 8$	$-4 + 12 \overset{?}{=} 8$
$8 = 8$ ✓	$-14 \neq 8$	$8 = 8$ ✓
(0, 2) is a solution.	(2, –4) is not a solution.	(–4, 3) is a solution.

TRY IT 4.1

Which of the following ordered pairs are solutions to $2x + 3y = 6$?
A $(3, 0)$ B $(2, 0)$ C $(6, -2)$

Show answer
A, C

TRY IT 4.2

Which of the following ordered pairs are solutions to the equation $4x - y = 8$? A $(0, 8)$ B $(2, 0)$ C $(1, -4)$

Show answer
B, C

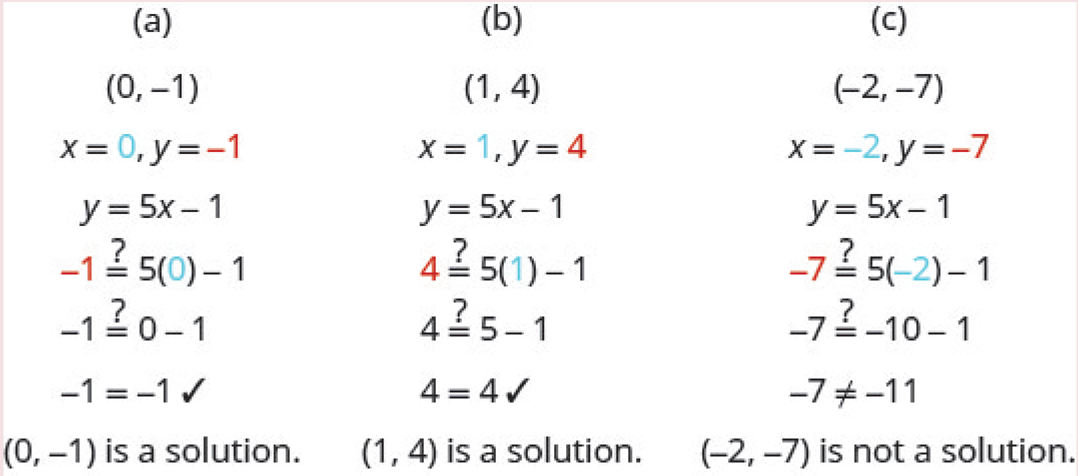
EXAMPLE 5

Which of the following ordered pairs are solutions to the equation $y = 5x - 1$?

A $(0, -1)$ B $(1, 4)$ C $(-2, -7)$

Solution

Substitute the *x*– and *y*-values from each ordered pair into the equation and determine if it results in a true statement.



(a)	(b)	(c)
$(0, -1)$	$(1, 4)$	$(-2, -7)$
$x = 0, y = -1$	$x = 1, y = 4$	$x = -2, y = -7$
$y = 5x - 1$	$y = 5x - 1$	$y = 5x - 1$
$-1 \overset{?}{=} 5(0) - 1$	$4 \overset{?}{=} 5(1) - 1$	$-7 \overset{?}{=} 5(-2) - 1$
$-1 \overset{?}{=} 0 - 1$	$4 \overset{?}{=} 5 - 1$	$-7 \overset{?}{=} -10 - 1$
$-1 = -1$ ✓	$4 = 4$ ✓	$-7 \neq -11$
$(0, -1)$ is a solution.	$(1, 4)$ is a solution.	$(-2, -7)$ is not a solution.

TRY IT 5.1

Which of the following ordered pairs are solutions to the equation $y = 4x - 3$? A $(0, 3)$ B $(1, 1)$ C $(-1, -1)$

Show answer
B

TRY IT 5.2

Which of the following ordered pairs are solutions to the equation $y = -2x + 6$? A $(0, 6)$ B $(1, 4)$ C $(-2, -2)$

Show answer
A, B

Complete a Table of Solutions to a Linear Equation in Two Variables

In the examples above, we substituted the *x*– and *y*-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do you find the ordered pairs if they are not given? It's easier than you might think—you can just pick a value for x and then solve the equation for y. Or, pick a value for y and then solve for x.

We'll start by looking at the solutions to the equation $y = 5x - 1$ that we found in (Example 5). We can summarize this information in a table of solutions, as shown in (Table 1).

Table 1

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$

To find a third solution, we'll let $x = 2$ and solve for y.

	$y = 5x - 1$
Substitute $x = 2$.	$y = 5(2) - 1$
Multiply.	$y = 10 - 1$
Simplify.	$y = 9$

The ordered pair $(2, 9)$ is a solution to $y = 5x - 1$. We will add it to (Table 2).

Table 2

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$
2	9	$(2, 9)$

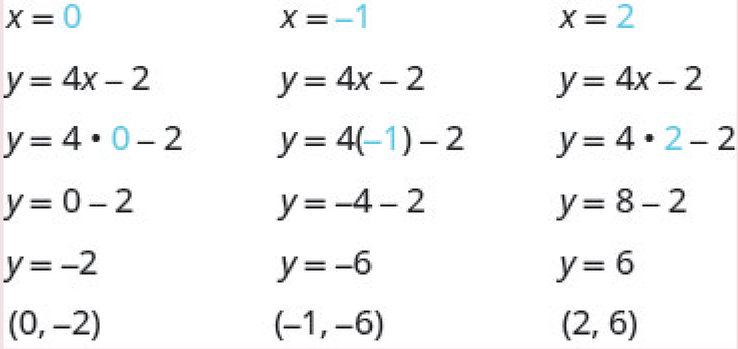
We can find more solutions to the equation by substituting in any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are infinitely many solutions of this equation.

EXAMPLE 6

Complete the table to find three solutions to the equation $y = 4x - 2$.

$y = 4x - 2$		
x	y	(x, y)
0		
-1		
2		

Solution
Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$.



$x = 0$	$x = -1$	$x = 2$
$y = 4x - 2$	$y = 4x - 2$	$y = 4x - 2$
$y = 4 \cdot 0 - 2$	$y = 4(-1) - 2$	$y = 4 \cdot 2 - 2$
$y = 0 - 2$	$y = -4 - 2$	$y = 8 - 2$
$y = -2$	$y = -6$	$y = 6$
$(0, -2)$	$(-1, -6)$	$(2, 6)$

The results are summarized in the table below.

$y = 4x - 2$		
x	y	(x, y)
0	-2	$(0, -2)$
-1	-6	$(-1, -6)$
2	6	$(2, 6)$

TRY IT 6.1

Complete the table to find three solutions to this equation: $y = 3x - 1$.

$y=3x-1$		
x	y	(x,y)
0		
-1		
2		

Show answer

$y=3x-1$		
x	y	(x,y)
0	-1	$(0,-1)$
-1	-4	$(-1,-4)$
2	5	$(2,5)$

TRY IT 6.2

Complete the table to find three solutions to this equation: $y=6x+1$.

$y=6x+1$		
x	y	(x,y)
0		
1		
-2		

Show answer

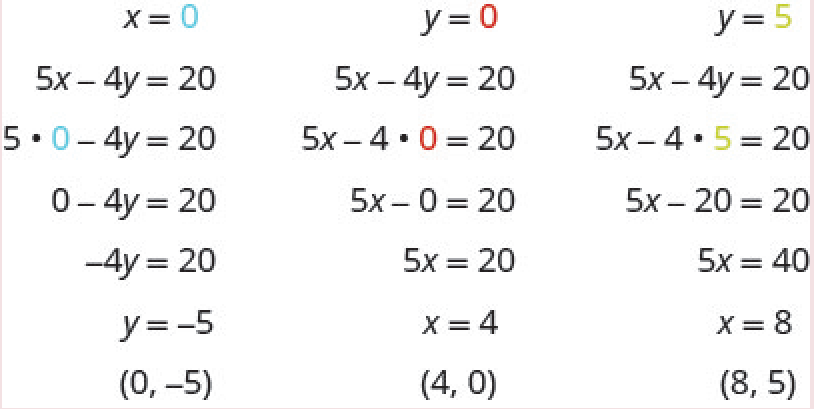
$y = 6x + 1$		
x	y	(x, y)
0	1	$(0, 1)$
1	7	$(1, 7)$
-2	-11	$(-2, -11)$

EXAMPLE 7

Complete the table to find three solutions to the equation $5x - 4y = 20$.

$5x - 4y = 20$		
x	y	(x, y)
0		
	0	
	5	

Solution
Substitute the given value into the equation $5x - 4y = 20$ and solve for the other variable. Then, fill in the values in the table.



$x = 0$	$y = 0$	$y = 5$
$5x - 4y = 20$	$5x - 4y = 20$	$5x - 4y = 20$
$5 \cdot 0 - 4y = 20$	$5x - 4 \cdot 0 = 20$	$5x - 4 \cdot 5 = 20$
$0 - 4y = 20$	$5x - 0 = 20$	$5x - 20 = 20$
$-4y = 20$	$5x = 20$	$5x = 40$
$y = -5$	$x = 4$	$x = 8$
$(0, -5)$	$(4, 0)$	$(8, 5)$

The results are summarized in the table below.

$5x - 4y = 20$		
x	y	(x, y)
0	-5	$(0, -5)$
4	0	$(4, 0)$
8	5	$(8, 5)$

TRY IT 7.1

Complete the table to find three solutions to this equation: $2x - 5y = 20$.

$2x - 5y = 20$		
x	y	(x, y)
0		
	0	
-5		

Show answer

$2x - 5y = 20$		
x	y	(x, y)
0	-4	$(0, -4)$
10	0	$(10, 0)$
-5	-6	$(-5, -6)$

TRY IT 7.2

Complete the table to find three solutions to this equation: $3x - 4y = 12$.

$3x-4y=12$		
x	y	(x,y)
0		
	0	
-4		

Show answer

$3x-4y=12$		
x	y	(x,y)
0	-3	$(0,-3)$
4	0	$(4,0)$
-4	-6	$(-4,-6)$

Find Solutions to a Linear Equation

To find a solution to a linear equation, you really can pick *any* number you want to substitute into the equation for x or y. But since you'll need to use that number to solve for the other variable it's a good idea to choose a number that's easy to work with.

When the equation is in *y*-form, with the *y* by itself on one side of the equation, it is usually easier to choose values of x and then solve for y.

EXAMPLE 8

Find three solutions to the equation $y=-3x+2$.

Solution

We can substitute any value we want for x or any value for y. Since the equation is in *y*-form, it will be easier to substitute in values of x. Let's pick $x=0$, $x=1$, and $x=-1$.

	$x = 0$	$x = 1$	$x = -1$
Substitute the value into the equation.	$y = -3x + 2$	$y = -3x + 2$	$y = -3x + 2$
Simplify.	$y = -3 \cdot 0 + 2$	$y = -3 \cdot 1 + 2$	$y = -3(-1) + 2$
Simplify.	$y = 0 + 2$	$y = -3 + 2$	$y = 3 + 2$
Write the ordered pair.	$y = 2$	$y = -1$	$y = 5$
Check.	(0, 2)	(1, −1)	(−1, 5)
	$y = -3x + 2$	$y = -3x + 2$	$y = -3x + 2$
	$2 \overset{?}{=} -3 \cdot 0 + 2$	$-1 \overset{?}{=} -3 \cdot 1 + 2$	$5 \overset{?}{=} -3\,(-1) + 2$
	$2 \overset{?}{=} 0 + 2$	$-1 \overset{?}{=} -3 + 2$	$5 \overset{?}{=} 3 + 2$
	$2 = 2\checkmark$	$-1 = -1\checkmark$	$5 = 5\checkmark$

So, $(0, 2)$, $(1, -1)$ and $(-1, 5)$ are all solutions to $y = -3x + 2$. We show them in table below.

$y = -3x + 2$		
x	y	(x, y)
0	2	$(0, 2)$
1	-1	$(1, -1)$
-1	5	$(-1, 5)$

TRY IT 8.1

Find three solutions to this equation: $y = -2x + 3$.

Show answer
Answers will vary.

TRY IT 8.2

Find three solutions to this equation: $y = -4x + 1$.

Show answer
Answers will vary

We have seen how using zero as one value of x makes finding the value of y easy. When an equation is in standard form, with both the x and y on the same side of the equation, it is usually easier to first find one solution when $x = 0$ find a second solution when $y = 0$, and then find a third solution.

EXAMPLE 9

Find three solutions to the equation $3x + 2y = 6$.

Solution
We can substitute any value we want for x or any value for y. Since the equation is in standard form, let’s pick first $x = 0$, then $y = 0$, and then find a third point.

	$x = 0$	$y = 0$	$x = 1$
	$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
Substitute the value into the equation.	$3(0) + 2y = 6$	$3x + 2(0) = 6$	$3(1) + 2y = 6$
Simplify.	$0 + 2y = 6$	$3x + 0 = 6$	$3 + 2y = 6$
Solve.	$2y = 6$	$3x = 6$	$2y = 3$
	$y = 3$	$x = 2$	$y = \frac{3}{2}$
Write the ordered pair.	(0, 3)	(2, 0)	$\left(1, \frac{3}{2}\right)$
Check.	$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
	$3 \cdot 0 + 2 \cdot 3 \overset{?}{=} 6$	$3 \cdot 2 + 2 \cdot 0 \overset{?}{=} 6$	$3 \cdot 1 + 2 \cdot \frac{3}{2} \overset{?}{=} 6$
	$0 + 6 \overset{?}{=} 6$	$6 + 0 \overset{?}{=} 6$	$3 + 3 \overset{?}{=} 6$
	$6 = 6 \checkmark$	$6 = 6$?	$6 = 6$?

So $(0, 3)$, $(2, 0)$, and $\left(1, \frac{3}{2}\right)$ are all solutions to the equation $3x + 2y = 6$. We can list these three solutions in the table below.

$3x + 2y = 6$		
x	y	(x, y)
0	3	$(0, 3)$
2	0	$(2, 0)$
1	$\frac{3}{2}$	$(1, \frac{3}{2})$

EXAMPLE 9.1

Find three solutions to the equation $2x + 3y = 6$.

Show answer
Answers will vary.

TRY IT 9.2

Find three solutions to the equation $4x + 2y = 8$.

Show answer
Answers will vary.

Glossary

linear equation
A linear equation is of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair
An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system.

origin
The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

quadrant
The *x*-axis and the *y*-axis divide a plane into four regions, called quadrants.

rectangular coordinate system
A grid system is used in algebra to show a relationship between two variables; also called the *xy*-plane or the ‘coordinate plane.’

***x*-coordinate**
The first number in an ordered pair (x, y).

***y*-coordinate**
The second number in an ordered pair (x, y).

Practice Makes Perfect

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

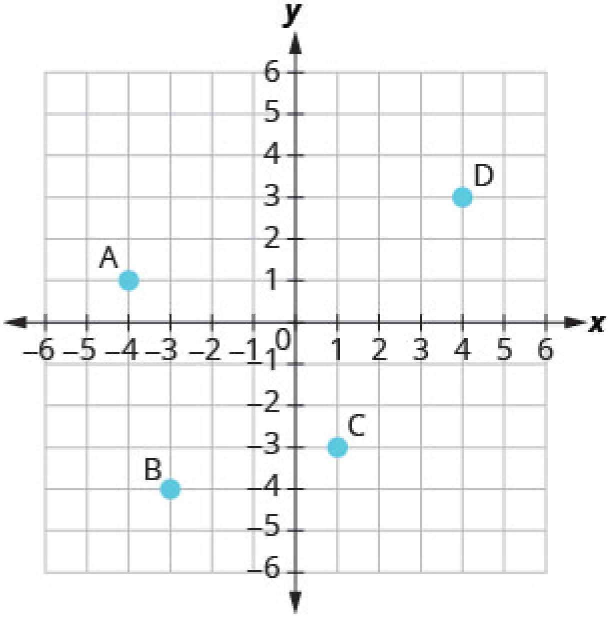
1.A $(-4, 2)$ B $(-1, -2)$ C $(3, -5)$ D $(-3, 5)$ E $\left(\frac{5}{3}, 2\right)$	2. A $(-2, -3)$ B $(3, -3)$ C $(-4, 1)$ D $(4, -1)$ E $\left(\frac{3}{2}, 1\right)$
3. A $(3, -1)$ B $(-3, 1)$ C $(-2, 2)$ D $(-4, -3)$ E $\left(1, \frac{14}{5}\right)$	4. A $(-1, 1)$ B $(-2, -1)$ C $(2, 1)$ D $(1, -4)$ E $\left(3, \frac{7}{2}\right)$

In the following exercises, plot each point in a rectangular coordinate system.

5. A $(-2, 0)$ B $(-3, 0)$ C $(0, 0)$ D $(0, 4)$ E $(0, 2)$	6. A $(0, 1)$ B $(0, -4)$ C $(-1, 0)$ D $(0, 0)$ E $(5, 0)$
7. A $(0, 0)$ B $(0, -3)$ C $(-4, 0)$ D $(1, 0)$ E $(0, -2)$	8. A $(-3, 0)$ B $(0, 5)$ C $(0, -2)$ D $(2, 0)$ E $(0, 0)$

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.

9.



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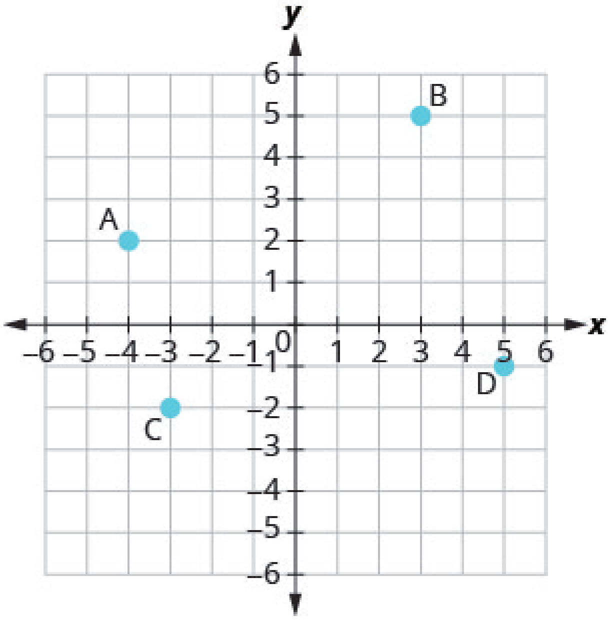
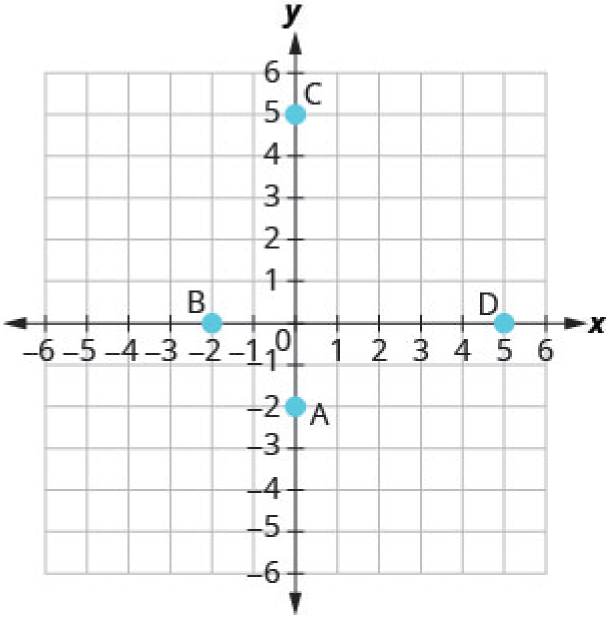


11.



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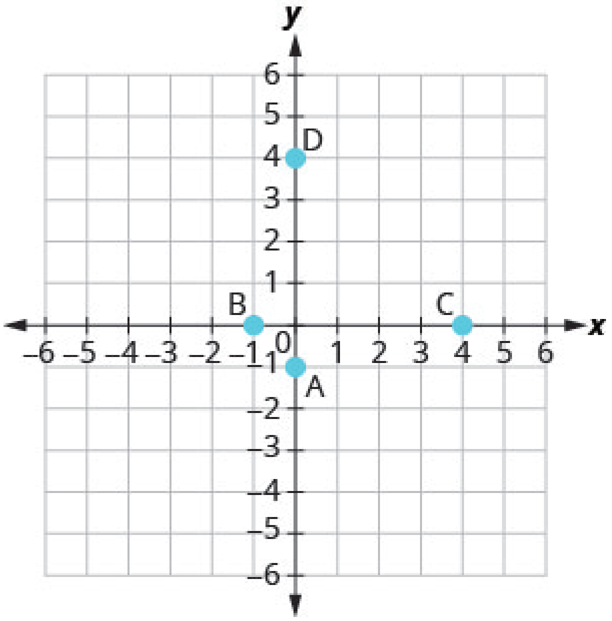


Verify Solutions to an Equation in Two Variables

In the following exercises, which ordered pairs are solutions to the given equations?

13. $2x+y=6$

A $(1,4)$
B $(3,0)$
C $(2,3)$

14. $x+3y=9$

A $(0,3)$
B $(6,1)$
C $(-3,-3)$

15. $4x-2y=8$

A $(3,2)$
B $(1,4)$
C $(0,-4)$

16. $3x-2y=12$

A $(4,0)$
B $(2,-3)$
C $(1,6)$

17. $y=4x+3$

A $(4,3)$
B $(-1,-1)$
C $\left(\frac{1}{2},5\right)$

18. $y=2x-5$

A $(0,-5)$
B $(2,1)$
C $\left(\frac{1}{2},-4\right)$

19. $y=\frac{1}{2}x-1$

A $(2,0)$
B $(-6,-4)$
C $(-4,-1)$

20. $y=\frac{1}{3}x+1$

A $(-3,0)$
B $(9,4)$
C $(-6,-1)$

Complete a Table of Solutions to a Linear Equation

In the following exercises, complete the table to find solutions to each linear equation.

21. $y = 2x - 4$

x	y	(x, y)
0		
2		
-1		

22. $y = 3x - 1$

x	y	(x, y)
0		
2		
-1		

23. $y = -x + 5$

x	y	(x, y)
0		
3		
-2		

24. $y = -x + 2$

x	y	(x, y)
0		
3		
-2		

25. $y = \frac{1}{3}x + 1$

x	y	(x, y)
0		
3		
6		

26. $y = \frac{1}{2}x + 4$

x	y	(x, y)
0		
2		
4		

27. $y = -\frac{3}{2}x - 2$

x	y	(x, y)
0		
2		
-2		

28. $y = -\frac{2}{3}x - 1$

x	y	(x, y)
0		
3		
-3		

29. $x + 3y = 6$

x	y	(x, y)
0		
3		
	0	

30. $x + 2y = 8$

x	y	(x, y)
0		
4		
	0	

31. $2x - 5y = 10$

x	y	(x, y)
0		
10		
	0	

32. $3x - 4y = 12$

x	y	(x, y)
0		
8		
	0	

Find Solutions to a Linear Equation

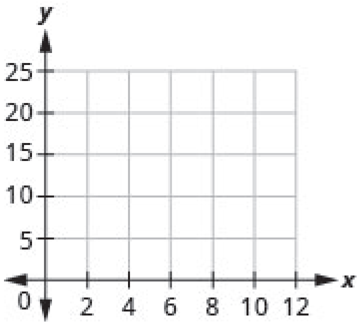
In the following exercises, find three solutions to each linear equation.

33. $y = 5x - 8$	34. $y = 3x - 9$
35. $y = -4x + 5$	36. $y = -2x + 7$
37. $x + y = 8$	38. $x + y = 6$
39. $x + y = -2$	40. $x + y = -1$
41. $3x + y = 5$	42. $2x + y = 3$
43. $4x - y = 8$	44. $5x - y = 10$
45. $2x + 4y = 8$	46. $3x + 2y = 6$
47. $5x - 2y = 10$	48. $4x - 3y = 12$

Everyday Math

49. Weight of a baby. Mackenzie recorded her baby's weight every two months. The baby's age, in months, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

a) Plot the points on a coordinate plane.

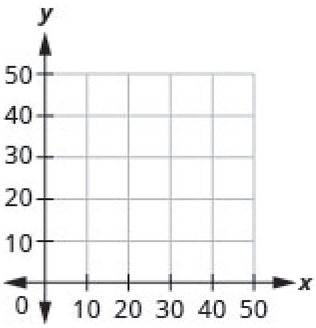


b) Why is only Quadrant I needed?

Age x	**Weight** y	(x, y)
0	7	(0, 7)
2	11	(2, 11)
4	15	(4, 15)
6	16	(6, 16)
8	19	(8, 19)
10	20	(10, 20)
12	21	(12, 21)

50. Weight of a child. Latresha recorded her son's height and weight every year. His height, in inches, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

a) Plot the points on a coordinate plane.



b) Why is only Quadrant I needed?

Height x	**Weight** y	(x, y)
28	22	(28, 22)
31	27	(31, 27)
33	33	(33, 33)
37	35	(37, 35)
40	41	(40, 41)
42	45	(42, 45)

Writing Exercises

51. Explain in words how you plot the point $(4, -2)$ in a rectangular coordinate system.

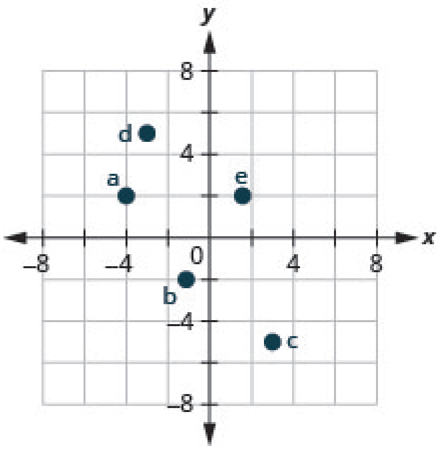
52. How do you determine if an ordered pair is a solution to a given equation?

53. Is the point $(-3, 0)$ on the *x*-axis or *y*-axis? How do you know?

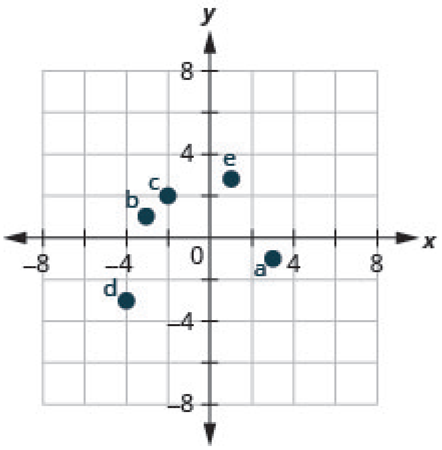
54. Is the point $(0, 8)$ on the *x*-axis or *y*-axis? How do you know?

Answers

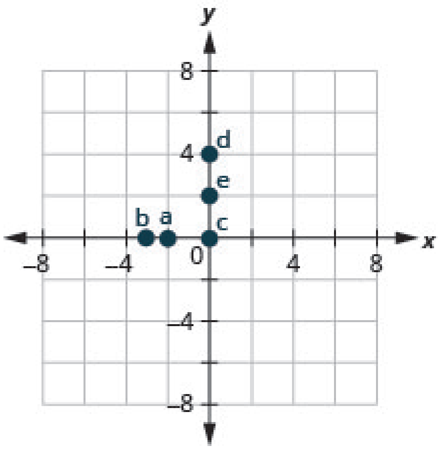
1.



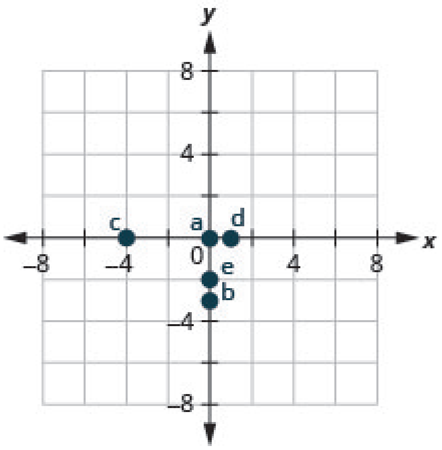
3.



5.



7.



9. A: $(-4,1)$ B: $(-3,-4)$ C: $(1,-3)$ D: $(4,3)$

11. A: $(0,-2)$ B: $(-2,0)$ C: $(0,5)$ D: $(5,0)$

13. A, B

15. A, C

17. B, C

19. A, B

21.

x	y	(x,y)
0	-4	$(0,-4)$
2	0	$(2,0)$
-1	-6	$(-1,-6)$

23.

x	y	(x,y)
0	5	$(0,5)$
3	2	$(3,2)$
-2	7	$(-2,7)$

25.

x	y	(x, y)
0	1	$(0, 1)$
3	2	$(3, 2)$
6	3	$(6, 3)$

25.

x	y	(x, y)
0	1	$(0, 1)$
3	2	$(3, 2)$
6	3	$(6, 3)$

27.

x	y	(x, y)
0	-2	$(0, -2)$
2	-5	$(2, -5)$
-2	1	$(-2, 1)$

29.

x	y	(x, y)
0	2	$(0, 2)$
3	4	$(3, 1)$
6	0	$(6, 0)$

31.

x	y	(x, y)
0	-2	$(0, -2)$
10	2	$(10, 2)$
5	0	$(5, 0)$

33. Answers will vary.

35. Answers will vary.

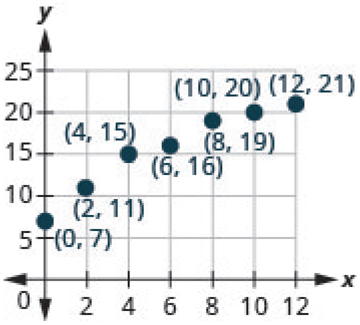
37. Answers will vary.

39. Answers will vary.

41. Answers will vary.

43. Answers will vary.

45. Answers will vary.

47. Answers will vary.	49. a)  b) Age and weight are only positive.
51. Answers will vary.	53. Answers will vary.

Attributions

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6.2 Graph Linear Equations in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Recognize the relationship between the solutions of an equation and its graph.
- Graph a linear equation by plotting points.
- Graph vertical and horizontal lines.

Recognize the Relationship Between the Solutions of an Equation and its Graph

In the previous section, we found several solutions to the equation $3x+2y=6$. They are listed in the table below. So, the ordered pairs $(0,3)$, $(2,0)$, and $\left(1,\frac{3}{2}\right)$ are some solutions to the equation $3x+2y=6$. We can plot these solutions in the rectangular coordinate system as shown in (Figure 1).

$3x+2y=6$		
x	y	(x,y)
0	3	$(0,3)$
2	0	$(2,0)$
1	$\frac{3}{2}$	$\left(1,\frac{3}{2}\right)$

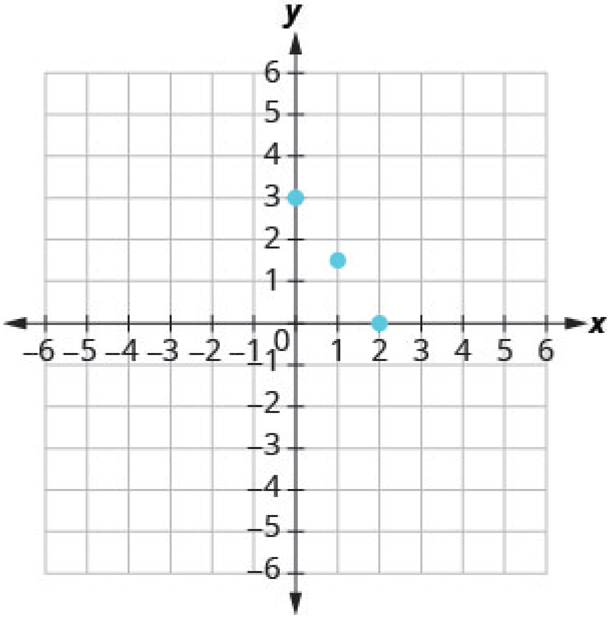


Figure .1

Notice how the points line up perfectly? We connect the points with a line to get the graph of the equation $3x + 2y = 6$. See (Figure 2). Notice the arrows on the ends of each side of the line. These arrows indicate the line continues.

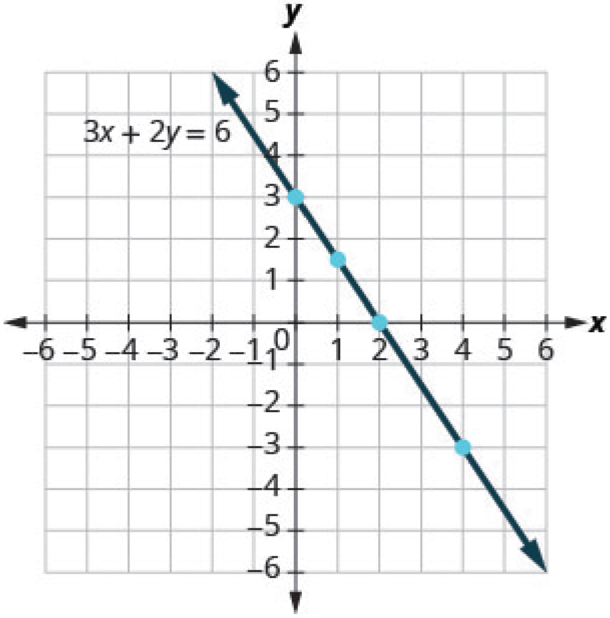


Figure .2

Every point on the line is a solution of the equation. Also, every solution of this equation is a point on this line. Points *not* on the line are not solutions.

Notice that the point whose coordinates are $(-2, 6)$ is on the line shown in (Figure 3). If you substitute $x = -2$ and $y = 6$ into the equation, you find that it is a solution to the equation.

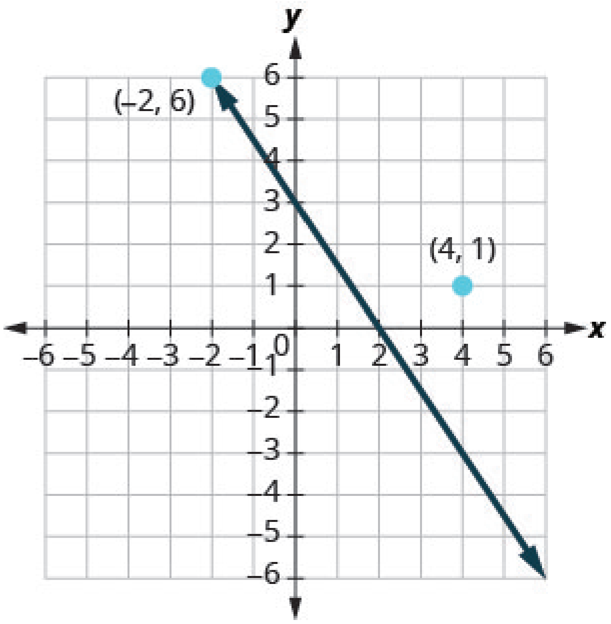


Figure .3

Test $(-2, 6)$

$$3x + 2y = 6$$

$$3(-2) + 2(6) = 6$$

$$-6 + 12 = 6$$

$$6 = 6 \checkmark$$

So the point $(-2, 6)$ is a solution to the equation $3x + 2y = 6$. (The phrase “the point whose coordinates are $(-2, 6)$” is often shortened to “the point $(-2, 6)$.”)

What about $(4, 1)$?

$$3x + 2y = 6$$

$$3 \cdot 4 + 2 \cdot 1 = 6$$

$$12 + 2 \overset{?}{=} 6$$

$$14 \neq 6$$

So $(4, 1)$ is not a solution to the equation $3x + 2y = 6$. Therefore, the point $(4, 1)$ is not on the line. See (Figure 2). This is an example of the saying, “A picture is worth a thousand words.” The line shows you *all* the solutions to the equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the *graph* of the equation $3x + 2y = 6$.

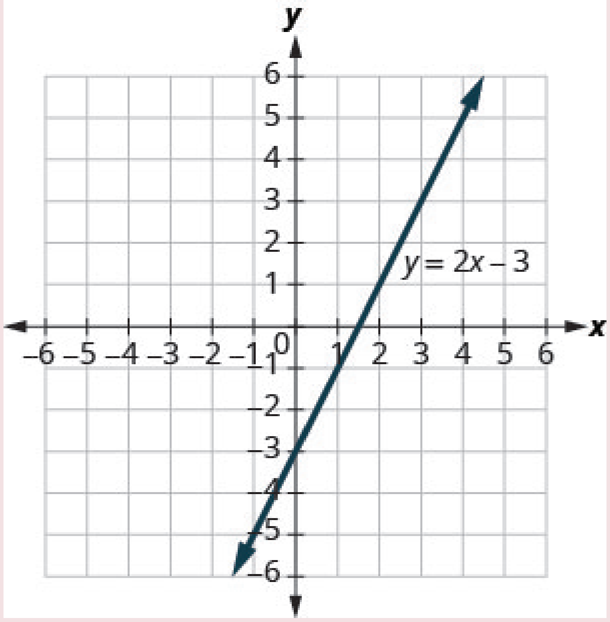
Graph of a linear equation

The graph of a linear equation $Ax + By = C$ is a line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

EXAMPLE 1

The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

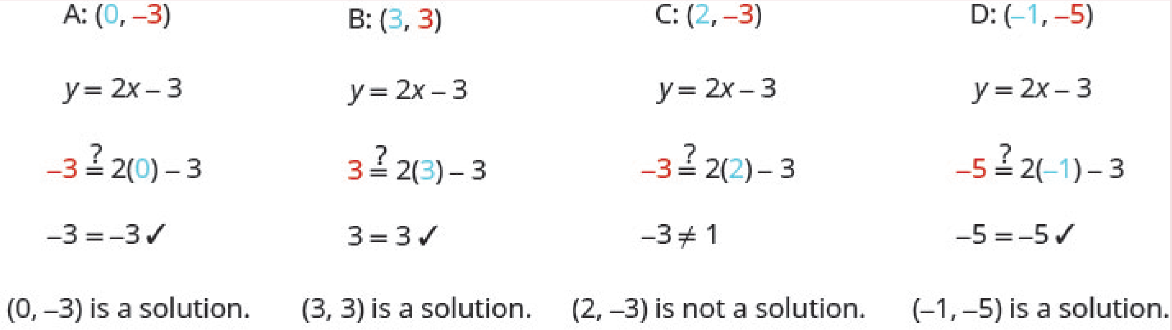
a) Is the ordered pair a solution to the equation?
b) Is the point on the line?

A $(0, -3)$ B $(3, 3)$ C $(2, -3)$ D $(-1, -5)$

Solution

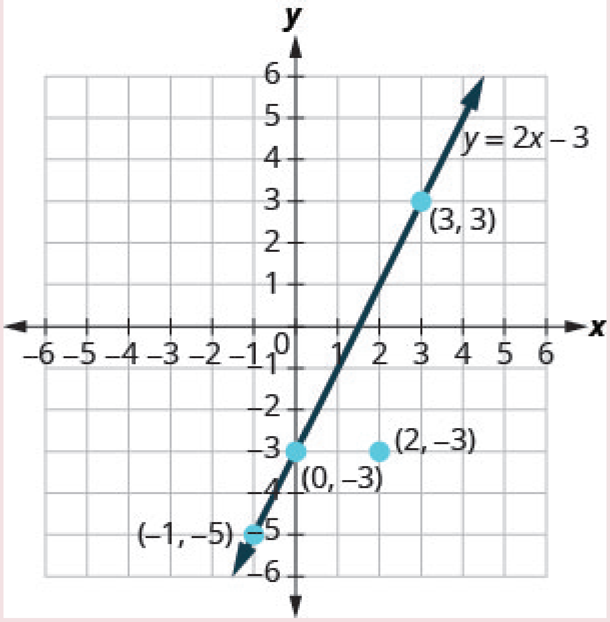
Substitute the *x*– and *y*– values into the equation to check if the ordered pair is a solution to the equation.

a.



A: $(0, -3)$	B: $(3, 3)$	C: $(2, -3)$	D: $(-1, -5)$
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
(0, –3) is a solution.	(3, 3) is a solution.	(2, –3) is not a solution.	(–1, –5) is a solution.

b. Plot the points A $(0, 3)$, B $(3, 3)$, C $(2, -3)$, and D $(-1, -5)$.



The points $(0, 3)$, $(3, 3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.

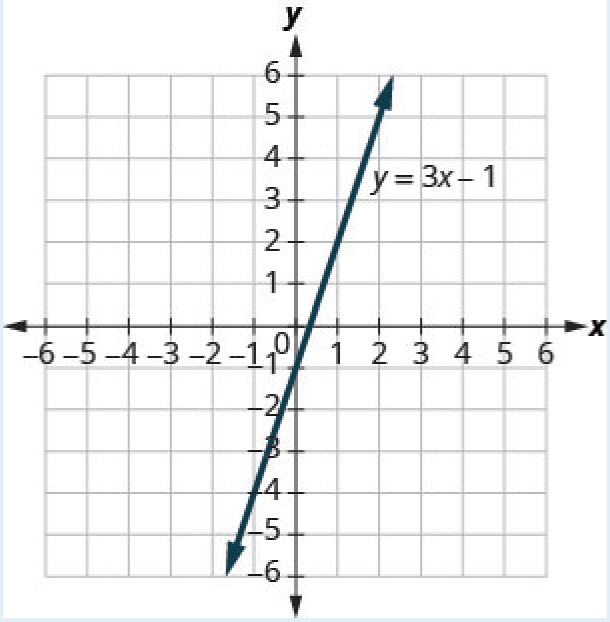
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

TRY IT 1.1

Use the graph of $y = 3x - 1$ to decide whether each ordered pair is:

- a solution to the equation.
- on the line.

a) $(0, -1)$ b) $(2, 5)$



Show answer

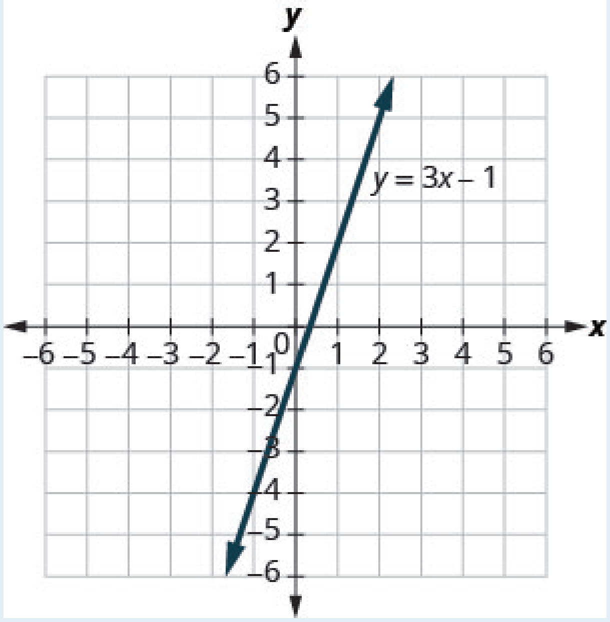
a) yes, yes b) yes, yes

TRY IT 1.2

Use graph of $y = 3x - 1$ to decide whether each ordered pair is:

- a solution to the equation
- on the line

a) $(3, -1)$ b) $(-1, -4)$



Show answer
a) no, no b) yes, yes

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The method we used to graph $3x + 2y = 6$ is called plotting points, or the Point–Plotting Method.

EXAMPLE 2

How To Graph an Equation By Plotting Points

Graph the equation $y = 2x + 1$ by plotting points.

Solution

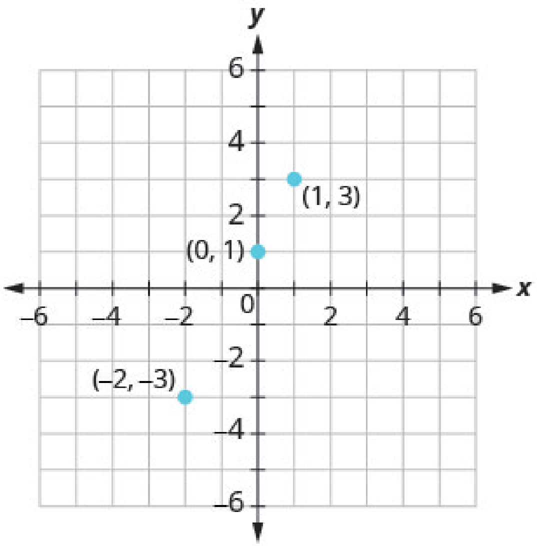
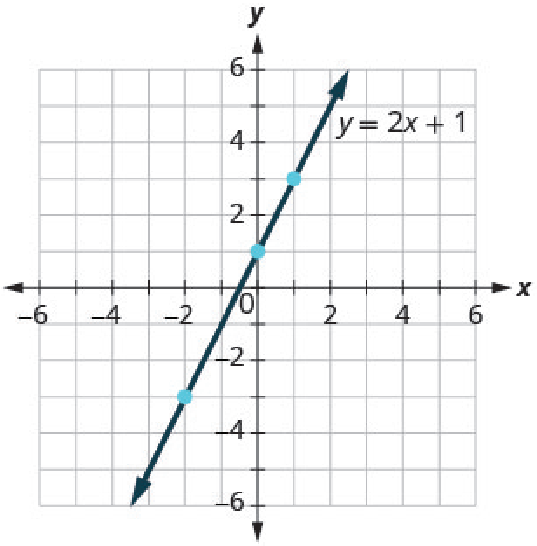
<table>
<tr>
<td>Step 1. Find three points whose coordinates are solutions to the equation.</td>
<td>You can choose any values for x or y.

In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.</td>
<td>$y = 2x + 1$

$x = 0$
$y = 2x + 1$
$y = 2 \cdot 0 + 1$
$y = 0 + 1$
$y = 1$

$x = 1$
$y = 2x + 1$
$y = 2 \cdot 1 + 1$
$y = 2 + 1$
$y = 3$

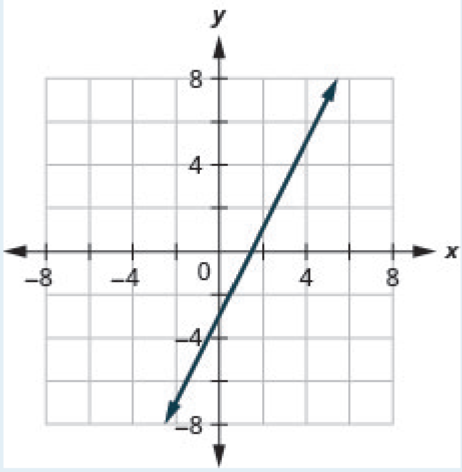
$x = -2$
$y = 2x + 1$
$y = 2(-2) + 1$
$y = -4 + 1$
$y = -3$</td>
</tr>
<tr>
<td>Organize the solutions in a table.</td>
<td>Put the three solutions in a table.</td>
<td>
<table>
<tr><th colspan="3">$y = 2x + 1$</th></tr>
<tr><th>x</th><th>y</th><th>(x, y)</th></tr>
<tr><td>0</td><td>1</td><td>(0, 1)</td></tr>
<tr><td>1</td><td>3</td><td>(1, 3)</td></tr>
<tr><td>–2</td><td>–3</td><td>(–2, –3)</td></tr>
</table>
</td>
</tr>
</table>

Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!	Plot: (0, 1), (1, 3), (–2, –3). Do the points line up? Yes, the points line up.	
Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

TRY IT 2.1

Graph the equation by plotting points: $y = 2x - 3$.

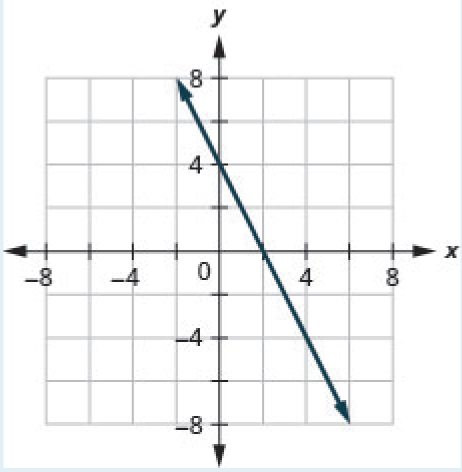
Show answer



TRY IT 2.2

Graph the equation by plotting points: $y = -2x + 4$.

Show answer



HOW TO: Graph a linear equation by plotting points.

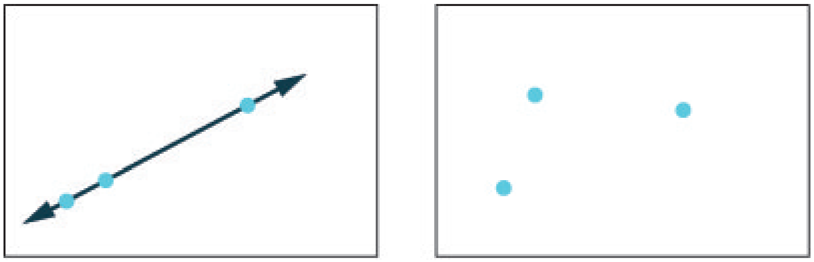
The steps to take when graphing a linear equation by plotting points are summarized below.

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of

the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between part (a) and part (b) in (Figure 4).



(a) (b)

Figure .4

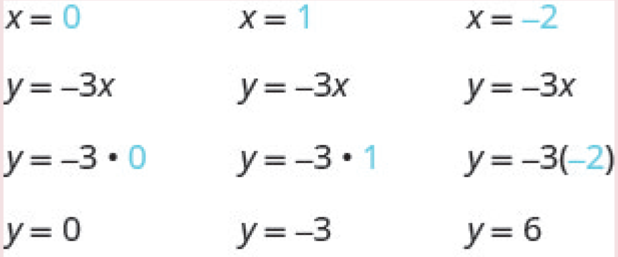
Let's do another example. This time, we'll show the last two steps all on one grid.

EXAMPLE 3

Graph the equation $y = -3x$.

Solution

Find three points that are solutions to the equation. Here, again, it's easier to choose values for x. Do you see why?

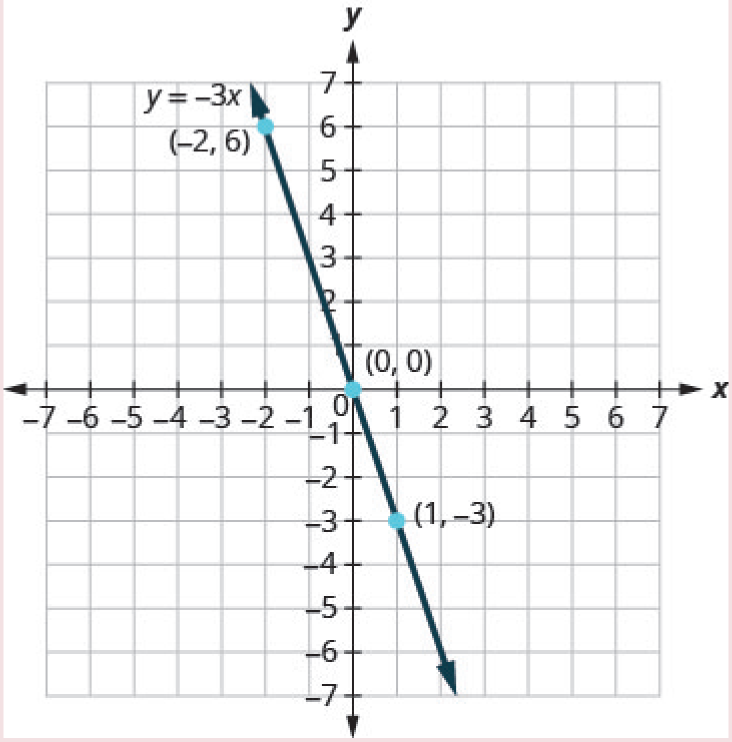


$x = 0$	$x = 1$	$x = -2$
$y = -3x$	$y = -3x$	$y = -3x$
$y = -3 \cdot 0$	$y = -3 \cdot 1$	$y = -3(-2)$
$y = 0$	$y = -3$	$y = 6$

We list the points in the table below.

$y=-3x$		
x	y	(x,y)
0	0	$(0,0)$
1	-3	$(1,-3)$
-2	6	$(-2,6)$

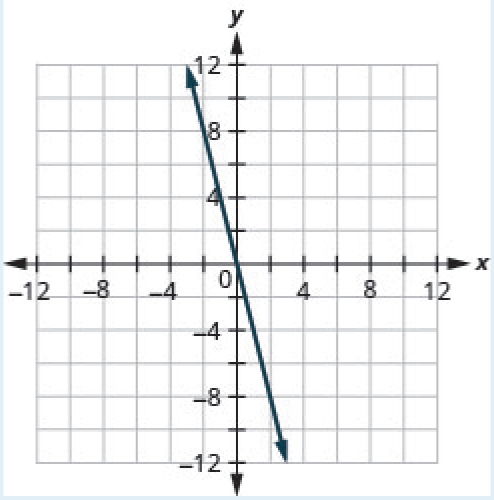
Plot the points, check that they line up, and draw the line.



TRY IT 3.1

Graph the equation by plotting points: $y=-4x$.

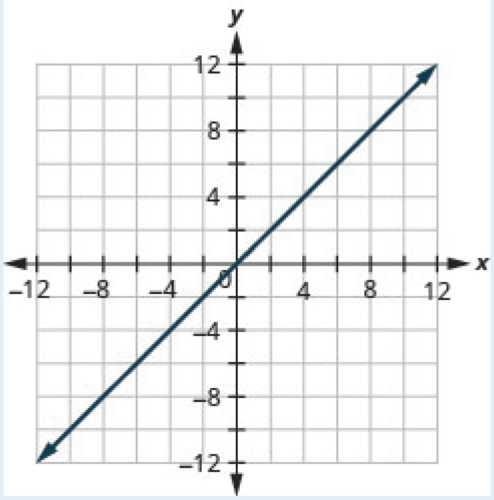
Show answer



EXAMPLE 3.2

Graph the equation by plotting points: $y = x$.

Show answer



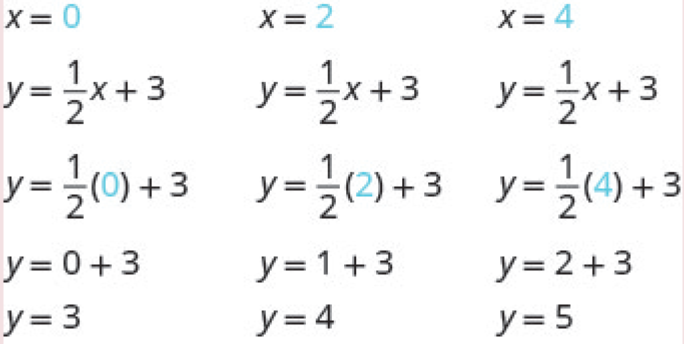
When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the math is easier if we make 'good' choices for the values of x. This way we will avoid fraction answers, which are hard to graph precisely.

EXAMPLE 4

Graph the equation $y = \frac{1}{2}x + 3$.

Solution

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of 2 a good choice for values of x?

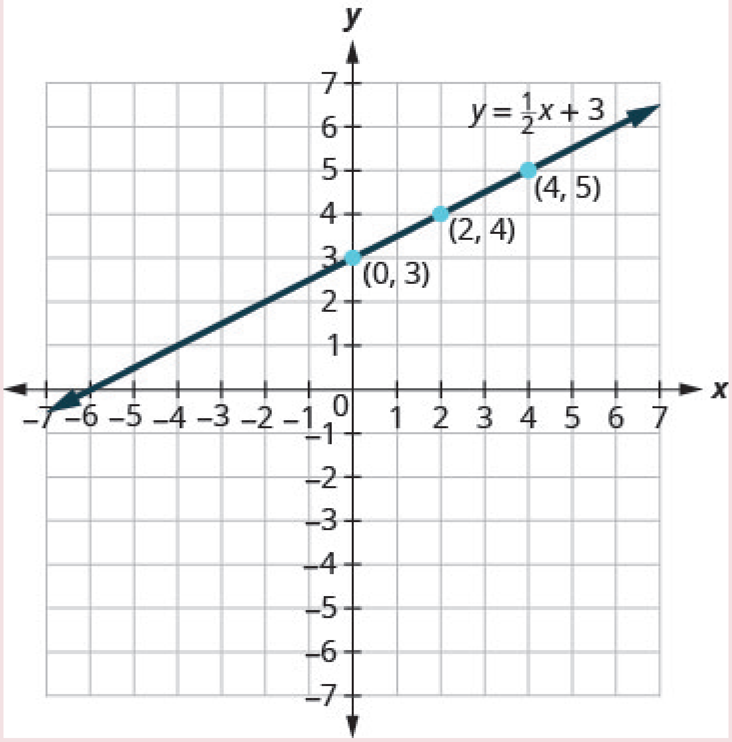


$$\begin{array}{lll}
x = 0 & x = 2 & x = 4 \\
y = \frac{1}{2}x + 3 & y = \frac{1}{2}x + 3 & y = \frac{1}{2}x + 3 \\
y = \frac{1}{2}(0) + 3 & y = \frac{1}{2}(2) + 3 & y = \frac{1}{2}(4) + 3 \\
y = 0 + 3 & y = 1 + 3 & y = 2 + 3 \\
y = 3 & y = 4 & y = 5
\end{array}$$

The points are shown in the table below.

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

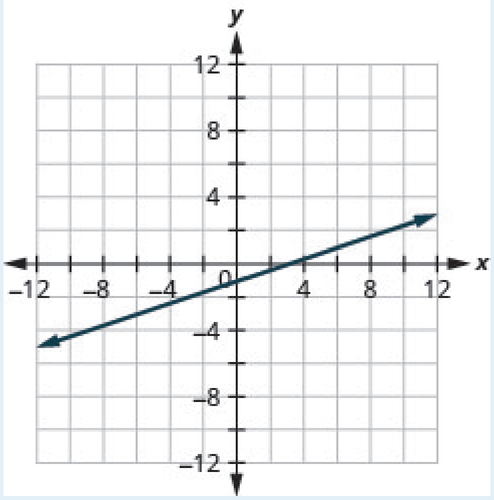
Plot the points, check that they line up, and draw the line.



TRY IT 4. 1

Graph the equation $y = \frac{1}{3}x - 1$.

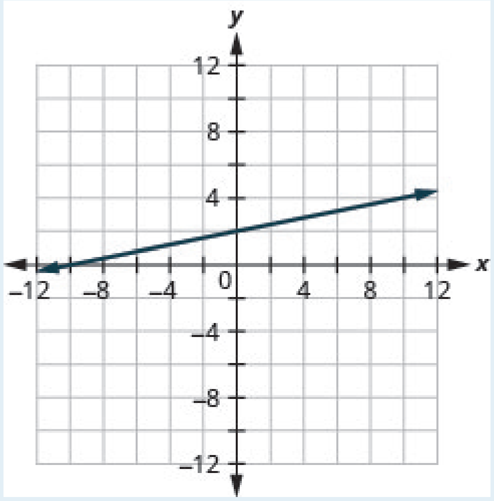
Show answer



TRY IT 4.2

Graph the equation $y = \frac{1}{4}x + 2$.

Show answer



So far, all the equations we graphed had y given in terms of x. Now we'll graph an equation with x and y on the same side. Let's see what happens in the equation $2x + y = 3$. If $y = 0$ what is the value of x?

$$y = 0$$

$$2x + y = 3$$

$$2x + 0 = 3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$\left(\frac{3}{2}, 0\right)$$

This point has a fraction for the *x*– coordinate and, while we could graph this point, it is hard to be precise graphing fractions. Remember in the example $y = \frac{1}{2}x + 3$, we carefully chose values for x so as not to graph fractions at all. If we solve the equation $2x + y = 3$ for y, it will be easier to find three solutions to the equation.

$$\begin{aligned} 2x + y &= 3 \\ y &= -2x + 3 \end{aligned}$$

The solutions for $x = 0$, $x = 1$, and $x = -1$ are shown in the table below. The graph is shown in (Figure 5).

$2x + y = 3$		
x	y	(x, y)
0	3	$(0, 3)$
1	1	$(1, 1)$
-1	5	$(-1, 5)$

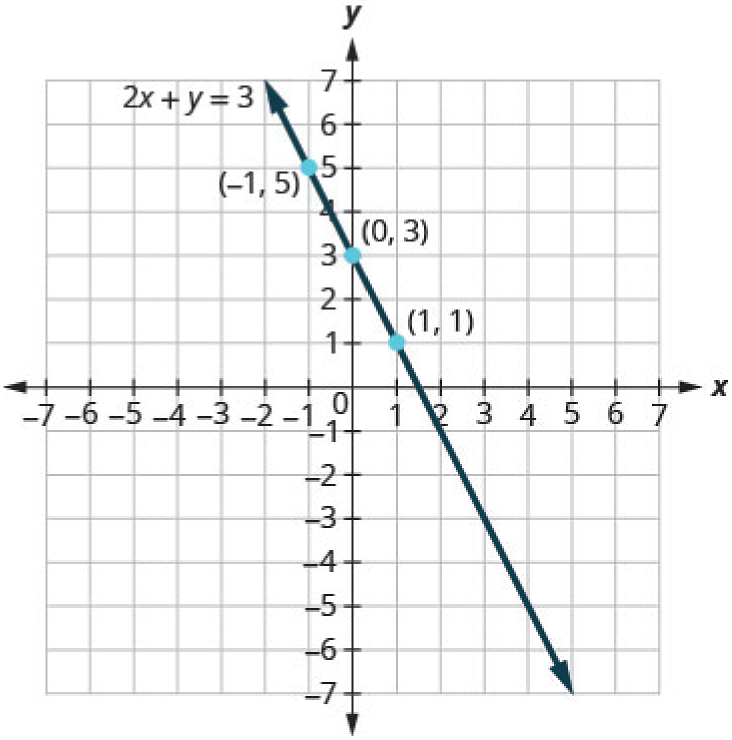


Figure .5

Can you locate the point $\left(\frac{3}{2}, 0\right)$, which we found by letting $y = 0$, on the line?

EXAMPLE 5

Graph the equation $3x + y = -1$.

Solution

Find three points that are solutions to the equation.	$3x + y \quad = \quad -1$
First, solve the equation for y.	$y \quad = \quad -3x - 1$

We'll let x be 0, 1, and -1 to find 3 points. The ordered pairs are shown in the table below. Plot the points, check that they line up, and draw the line. See (Figure 6).

$3x + y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	-4	$(1, -4)$
-1	2	$(-1, 2)$

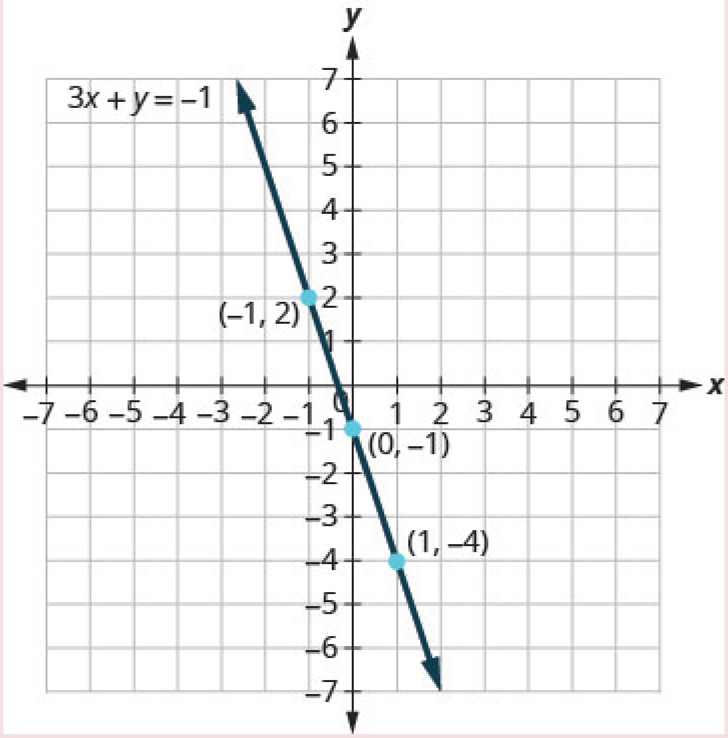
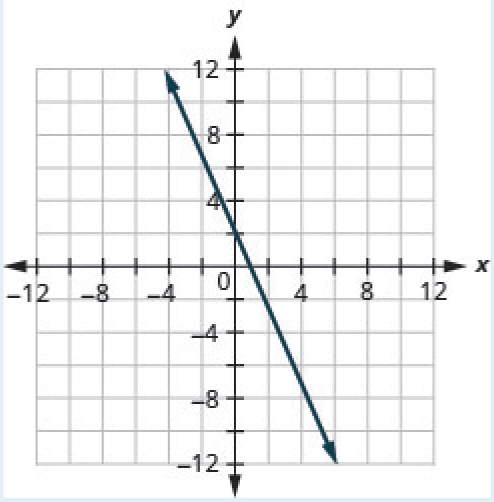


Figure .6

EXAMPLE 5.1

Graph the equation $2x + y = 2$.

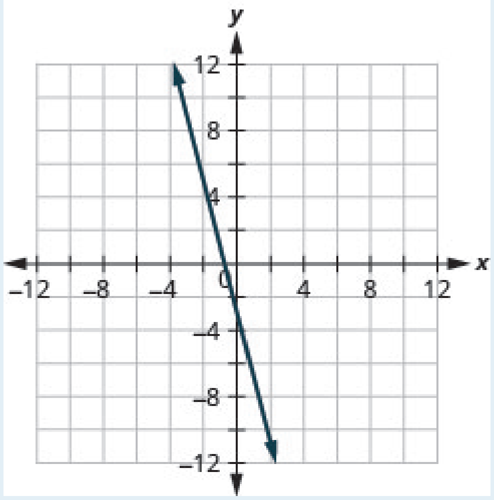
Show answer



TRY IT 5.2

Graph the equation $4x + y = -3$.

Show answer



If you can choose any three points to graph a line, how will you know if your graph matches the one shown in the answers in the book? If the points where the graphs cross the *x*– and *y*-axis are the same, the graphs match!

The equation in (Example 5) was written in standard form, with both x and y on the same side. We solved that equation for y in just one step. But for other equations in standard form it is not that easy to solve for y, so we will leave them in standard form. We can still find a first point to plot by letting $x = 0$ and solving for y. We can plot a second point by letting $y = 0$ and then solving for x. Then we will plot a third point by using some other value for x or y.

EXAMPLE 6

Graph the equation $2x-3y=6$.

Solution

Find three points that are solutions to the equation.	$2x-3y = 6$
First, let $x=0$.	$2(0)-3y = 6$
Solve for y.	$-3y = 6$ $y = -2$
Now let $y=0$.	$2x-3(0) = 6$
Solve for x.	$2x = 6$ $x = 3$
We need a third point. Remember, we can choose any value for x or y. We'll let $x=6$.	$2(6)-3y = 6$
Solve for y.	$12-3y = 6$ $-3y = -6$ $y = 2$

We list the ordered pairs in the table below. Plot the points, check that they line up, and draw the line. See (Figure 7).

$2x-3y=6$		
x	y	(x,y)
0	-2	$(0,-2)$
3	0	$(3,0)$
6	2	$(6,2)$

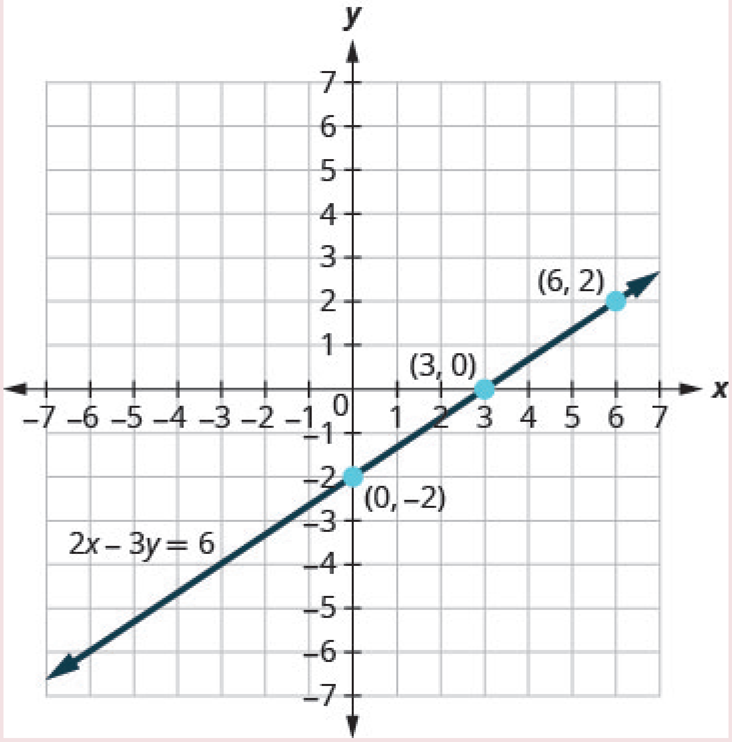
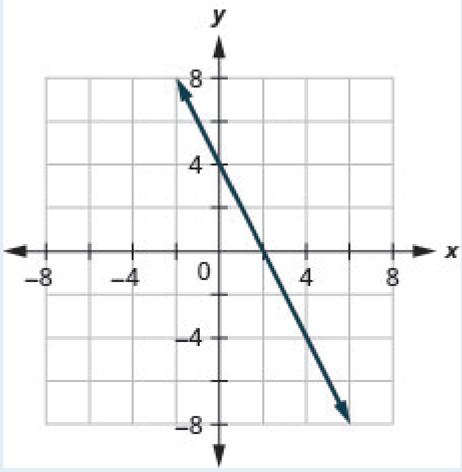


Figure .7

TRY IT 6.1

Graph the equation $4x + 2y = 8$.

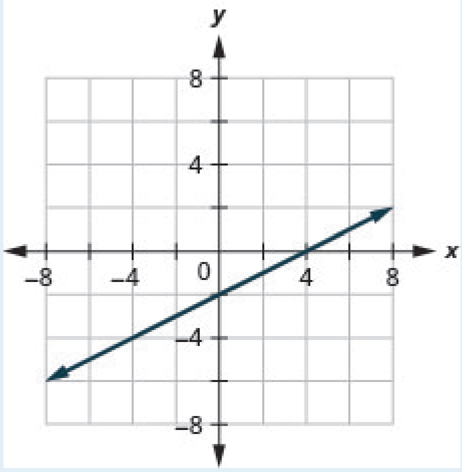
Show answer



TRY IT 6.2

Graph the equation $2x - 4y = 8$.

Show answer



Graph Vertical and Horizontal Lines

Can we graph an equation with only one variable? Just x and no y, or just y without an x? How will we make a table of values to get the points to plot?

Let's consider the equation $x = -3$. This equation has only one variable, x. The equation says that x is *always* equal to -3, so its value does not depend on y. No matter what y is, the value of x is always -3.

So to make a table of values, write -3 in for all the x values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the *y*-coordinates. See the table below.

$x = -3$		
x	y	(x, y)
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Plot the points from the table and connect them with a straight line. Notice in (Figure 8) that we have graphed a *vertical line*.

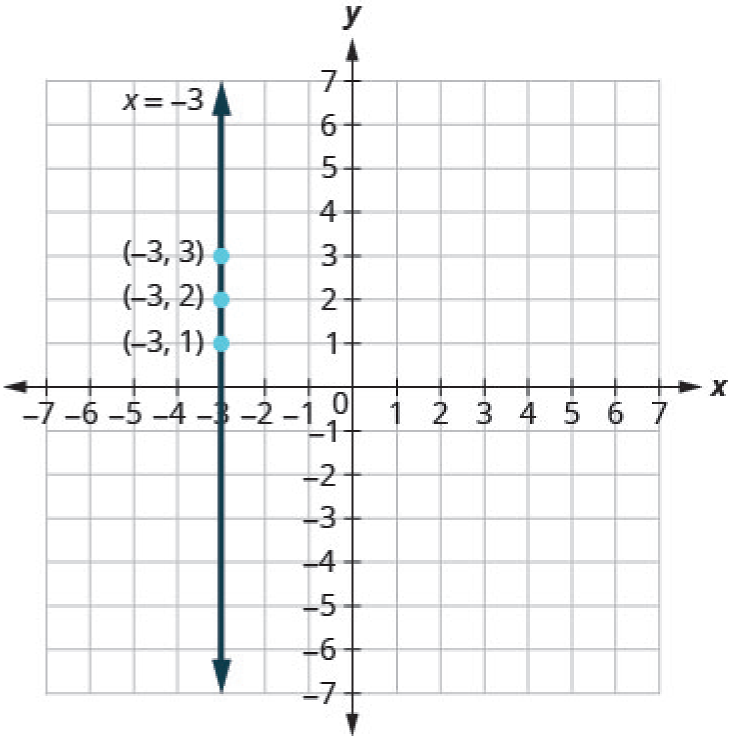


Figure .8

Vertical line

A vertical line is the graph of an equation of the form $x = a$.

The line passes through the *x*-axis at $(a, 0)$.

EXAMPLE 7

Graph the equation $x = 2$.

Solution
The equation has only one variable, x, and x is always equal to 2. We create the table below where x is always 2 and then put in any values for y. The graph is a vertical line passing through the *x*-axis at 2. See (Figure 9).

$x = 2$		
x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$

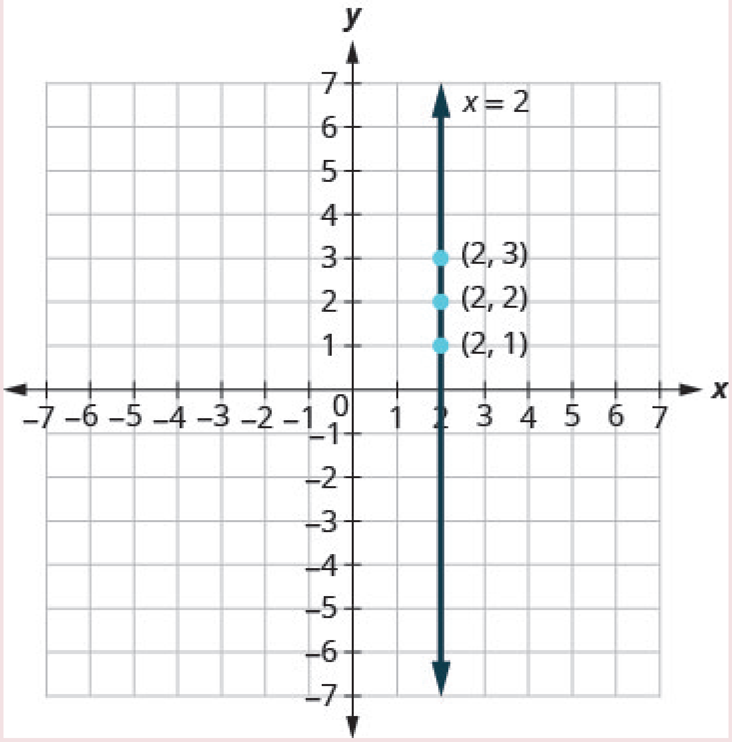
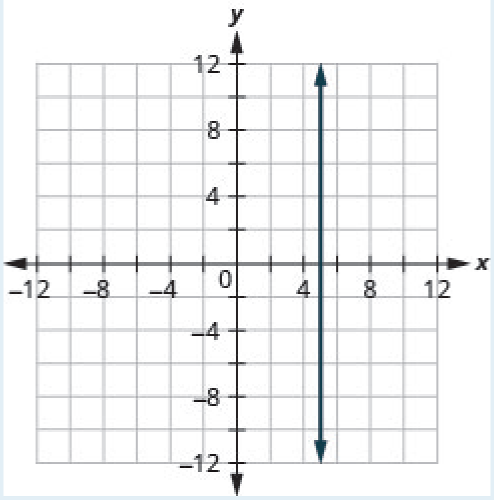


Figure .9

TRY IT 7.1

Graph the equation $x = 5$.

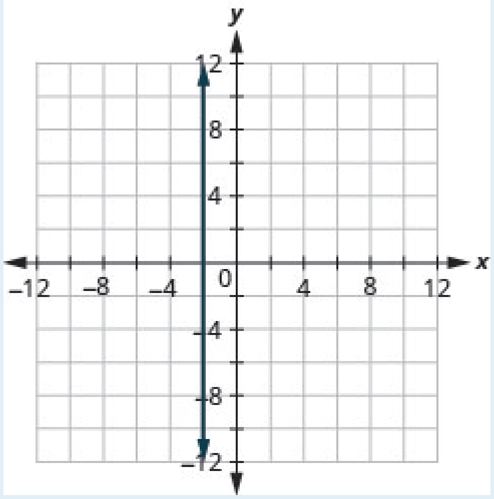
Show answer



TRY IT 7.2

Graph the equation $x = -2$.

Show answer



What if the equation has y but no x? Let's graph the equation $y = 4$. This time the *y*– value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in the table below and then choose any values for x. We'll use 0, 2, and 4 for the *x*-coordinates.

$y = 4$		
x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

The graph is a horizontal line passing through the *y*-axis at 4. See (Figure 10).

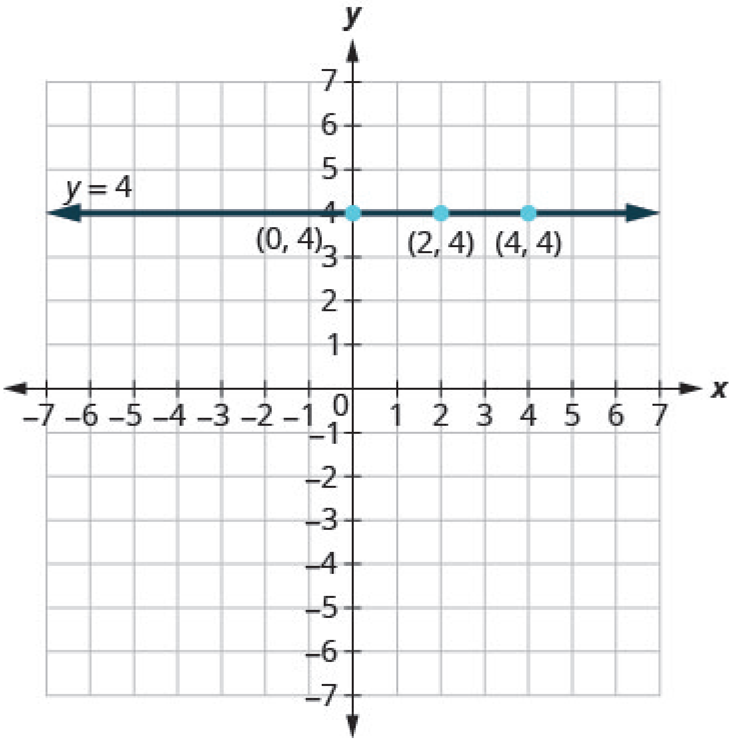


Figure .10

Horizontal line

A horizontal line is the graph of an equation of the form $y = b$.
The line passes through the *y*-axis at $(0, b)$.

EXAMPLE 8

Graph the equation $y = -1$.

Solution

The equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the table below have the same *y*-coordinate. The graph is a horizontal line passing through the *y*-axis at -1, as shown in (Figure 11).

$y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$

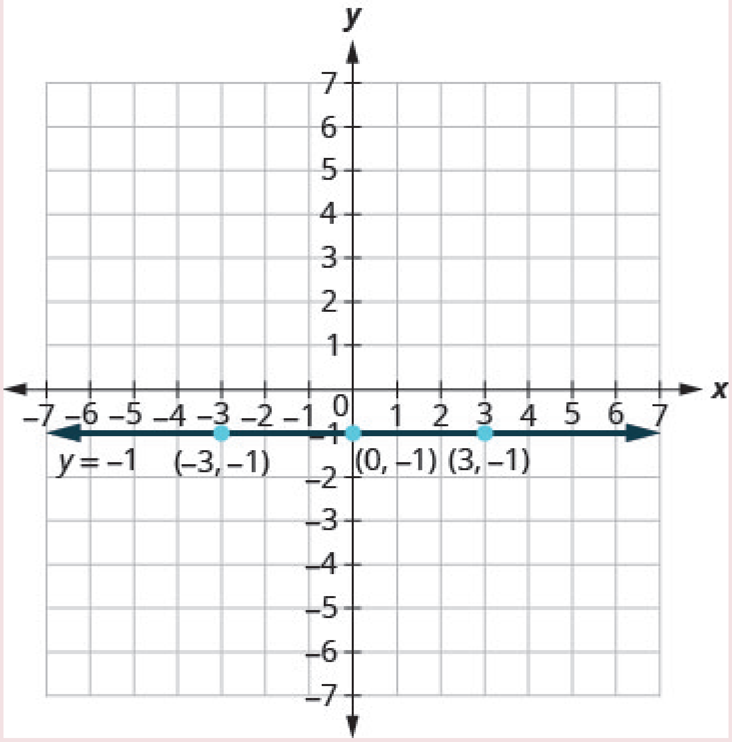
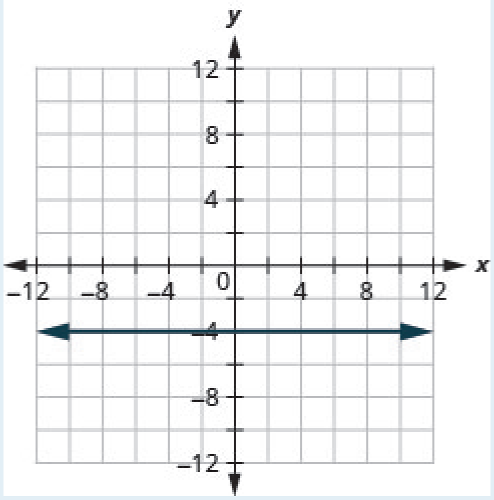


Figure .11

TRY IT 8.1

Graph the equation $y = -4$.

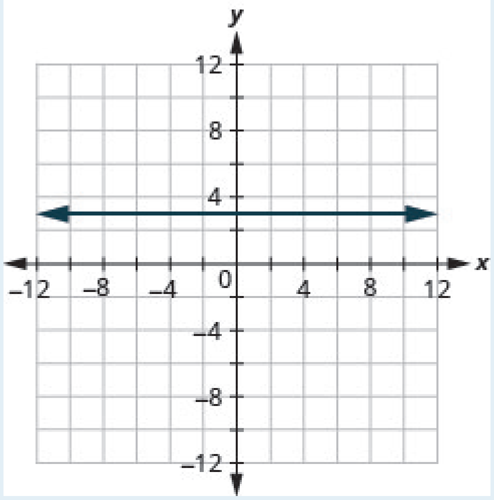
Show answer



TRY IT 8.2

Graph the equation $y = 3$.

Show answer



The equations for vertical and horizontal lines look very similar to equations like $y = 4x$. What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x. The *y*-coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant. The *y*-coordinate is always 4. It does not depend on the value of x. See the table below.

$y = 4x$				$y = 4$		
x	y	(x, y)		x	y	(x, y)
0	0	$(0, 0)$		0	4	$(0, 4)$
1	4	$(1, 4)$		1	4	$(1, 4)$
2	8	$(2, 8)$		2	4	$(2, 4)$

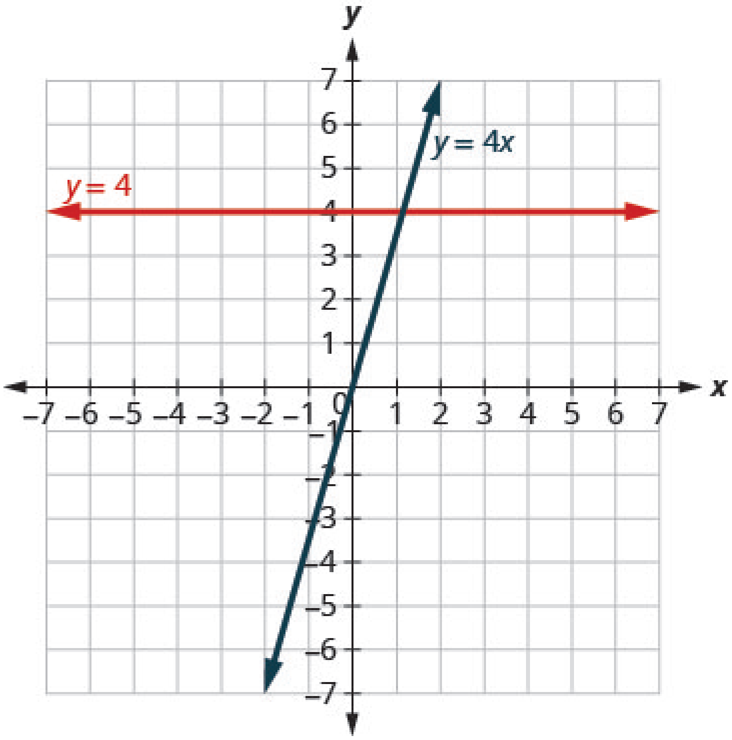


Figure .12

Notice, in (Figure 12), the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

EXAMPLE 9

Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution
Notice that the first equation has the variable x, while the second does not. See the table below. The two graphs are shown in (Figure 13).

$y=-3x$				$y=-3$		
x	y	(x,y)		x	y	(x,y)
0	0	$(0,0)$		0	-3	$(0,-3)$
1	-3	$(1,-3)$		1	-3	$(1,-3)$
2	-6	$(2,-6)$		2	-3	$(2,-3)$

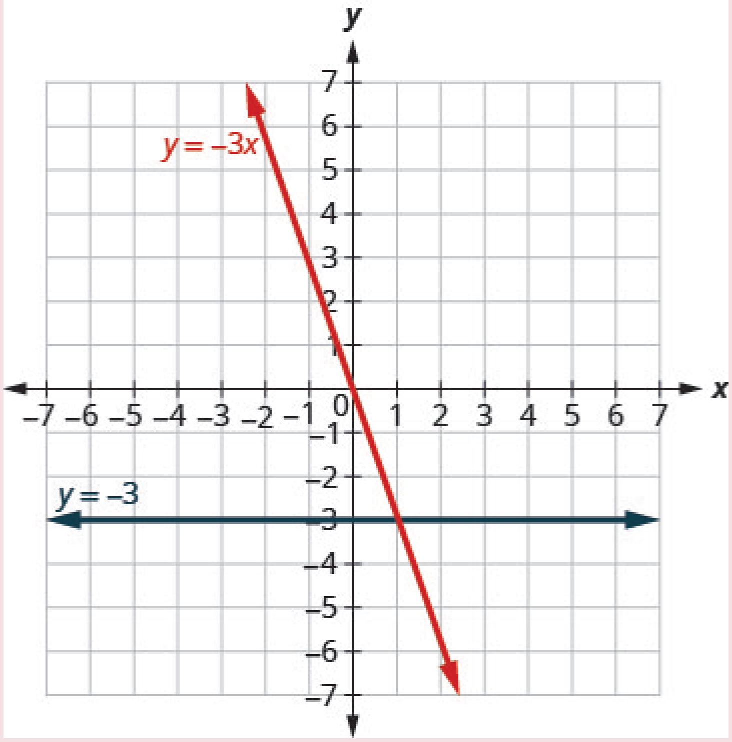
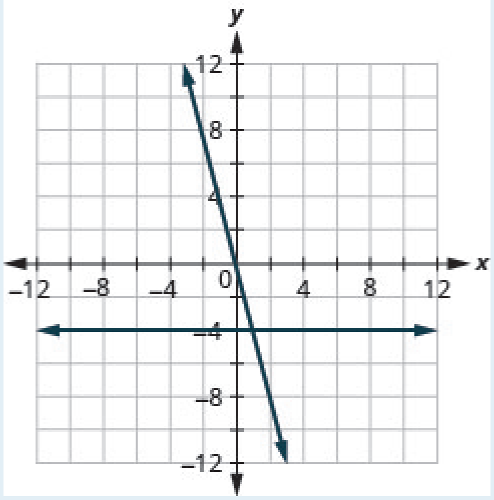


Figure .13

TRY IT 9.1

Graph $y=-4x$ and $y=-4$ in the same rectangular coordinate system.

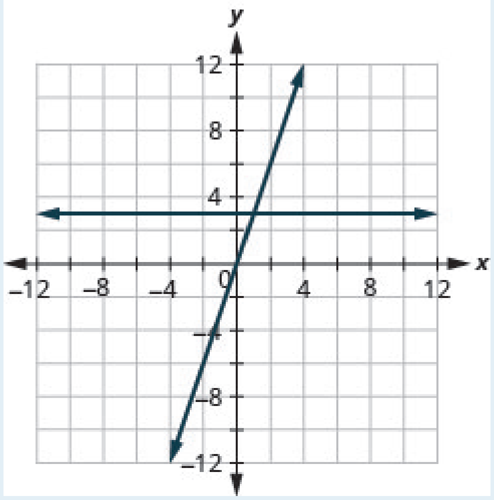
Show answer



TRY IT 9.2

Graph $y = 3$ and $y = 3x$ in the same rectangular coordinate system.

Show answer



Key Concepts

- **Graph a Linear Equation by Plotting Points**
    1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
    2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!

3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

Glossary

graph of a linear equation
The graph of a linear equation $Ax+By=C$ is a straight line. Every point on the line is a solution of the equation. Every solution of this equation is a point on this line.

horizontal line
A horizontal line is the graph of an equation of the form $y=b$. The line passes through the *y*-axis at $(0,b)$.

vertical line
A vertical line is the graph of an equation of the form $x=a$. The line passes through the *x*-axis at $(a,0)$.

Practice Makes Perfect

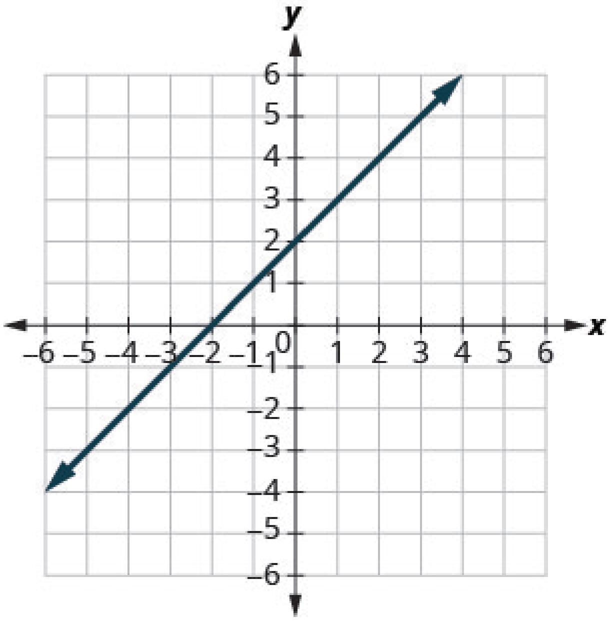
Recognize the Relationship Between the Solutions of an Equation and its Graph

In the following exercises, for each ordered pair, decide:

a) Is the ordered pair a solution to the equation? b) Is the point on the line?

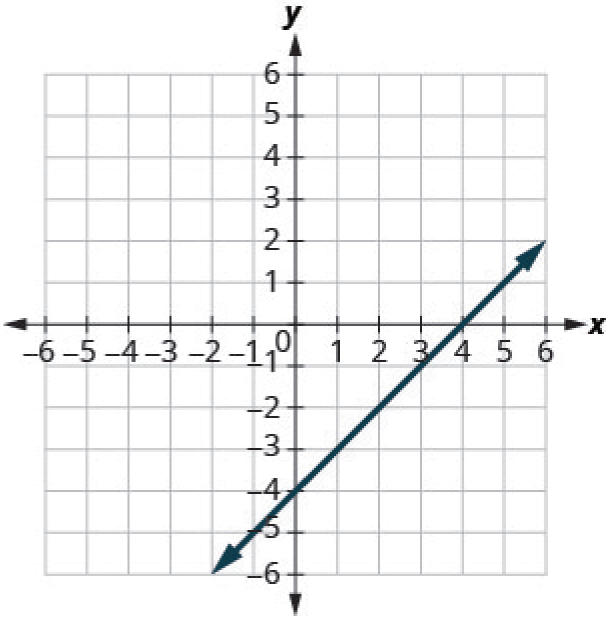
1. $y = x + 2$

a) $(0, 2)$
b) $(1, 2)$
c) $(-1, 1)$
d) $(-3, -1)$



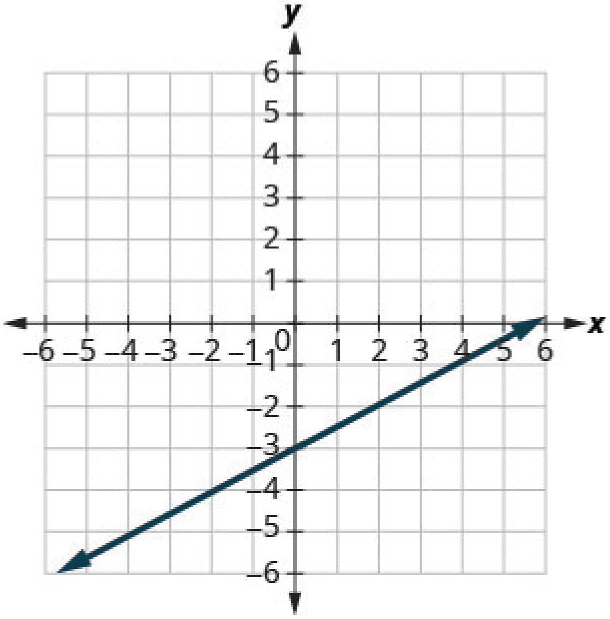
2. $y = x - 4$

a) $(0, -4)$
b) $(3, -1)$
c) $(2, 2)$
d) $(1, -5)$



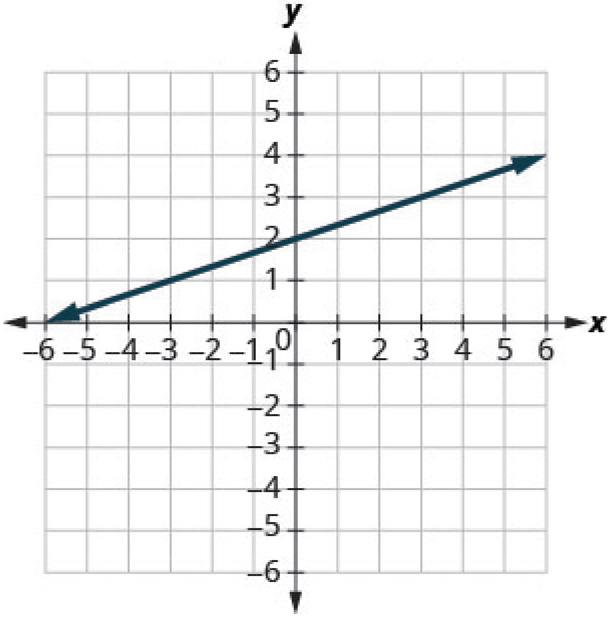
3. $y = \frac{1}{2}x - 3$

a) $(0, -3)$
b) $(2, -2)$
c) $(-2, -4)$
d) $(4, 1)$



4. $y = \frac{1}{3}x + 2$

a) $(0, 2)$
b) $(3, 3)$
c) $(-3, 2)$
d) $(-6, 0)$



Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

5. $y = 3x - 1$	6. $y = 2x + 3$
7. $y = -3x + 3$	8. $y = -3x + 1$
9. $y = x + 2$	10. $y = x - 3$
11. $y = -x - 3$	12. $y = -x - 2$
13. $y = 2x$	14. $y = 3x$
15. $y = 3x$	16. $y = -2x$
17. $y = \frac{1}{2}x + 2$	18. $y = \frac{1}{3}x - 1$
19. $y = \frac{4}{3}x - 5$	20. $y = \frac{3}{2}x - 3$
21. $y = -\frac{2}{5}x + 1$	22. $y = -\frac{4}{5}x - 1$
23. $y = -\frac{3}{2}x + 2$	24. $y = -\frac{5}{3}x + 4$
25. $x + y = 6$	26. $x + y = 4$
27. $x + y = -3$	28. $x + y = -3$
29. $x - y = 2$	30. $x - y = 1$
31. $x - y = -1$	32. $x - y = -3$
33. $3x + y = 7$	34. $5x + y = 6$
35. $2x + y = -3$	36. $4x + y = -5$
37. $\frac{1}{3}x + y = 2$	38. $\frac{1}{2}x + y = 3$
39. $\frac{2}{5}x + y = -4$	40. $\frac{3}{4}x - y = 6$
41. $2x + 3y = 12$	42. $4x + 2y = 12$
43. $3x - 4y = 12$	44. $2x - 5y = 10$
45. $x - 6y = 3$	46. $x - 4y = 2$
47. $3x + y = 2$	48. $3x + 5y = 5$

Graph Vertical and Horizontal Lines

In the following exercises, graph each equation.

49. $x = 4$	50. $x = 3$
51. $x = -2$	52. $x = -5$
53. $y = 3$	54. $y = 1$
55. $y = -5$	56. $y = -2$
57. $x = \frac{7}{3}$	58. $x = \frac{5}{4}$
59. $y = -\frac{15}{4}$	60. $y = -\frac{5}{3}$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

61. $y = 2x$ and $y = 2$	62. $y = 5x$ and $y = 5$
63. $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$	64. $y = -\frac{1}{3}x$ and $y = -\frac{1}{3}$

Mixed Practice

In the following exercises, graph each equation.

65. $y = 4x$	66. $y = 2x$
67. $y = -\frac{1}{2}x + 3$	68. $y = \frac{1}{4}x - 2$
69. $y = -x$	70. $y = x$
71. $x - y = 3$	72. $x + y = -5$
73. $4x + y = 2$	74. $2x + y = 6$
75. $y = -1$	76. $y = 5$
77. $2x + 6y = 12$	78. $5x + 2y = 10$
79. $x = 3$	80. $x = -4$

Everyday Math

81. **Motor home cost.** The Stonechilds rented a motor home for one week to go on vacation. It cost them \$594 plus \$0.32 per mile to rent the motor home, so the linear equation $y = 594 + 0.32x$ gives the cost, y, for driving x miles. Calculate the rental cost for driving 400, 800, and 1200 miles, and then graph the line.

82. **Weekly earnings.** At the art gallery where he works, Archisma gets paid \$200 per week plus 15% of the sales he makes, so the equation $y = 200 + 0.15x$ gives the amount, y, he earns for selling x dollars of artwork. Calculate the amount Archisma earns for selling \$900, \$1600, and \$2000, and then graph the line.

Writing Exercises

83. Explain how you would choose three *x*– values to make a table to graph the line $y = \frac{1}{5}x - 2$.

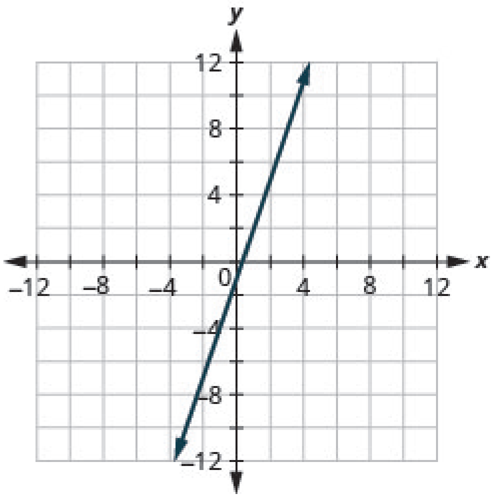
84. What is the difference between the equations of a vertical and a horizontal line?

Answers

1. a) yes; no b) no; no c) yes; yes d) yes; yes

3. a) yes; yes b) yes; yes c) yes; yes d) no; no

5.



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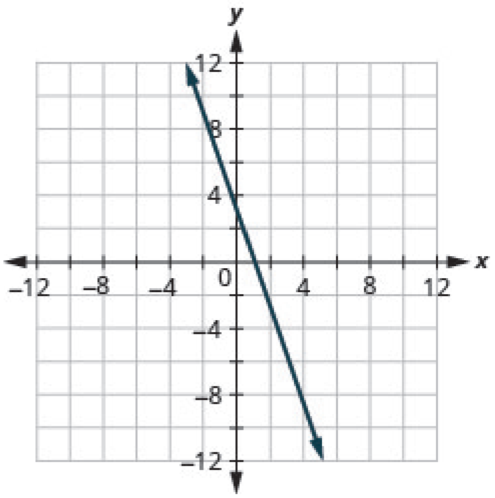

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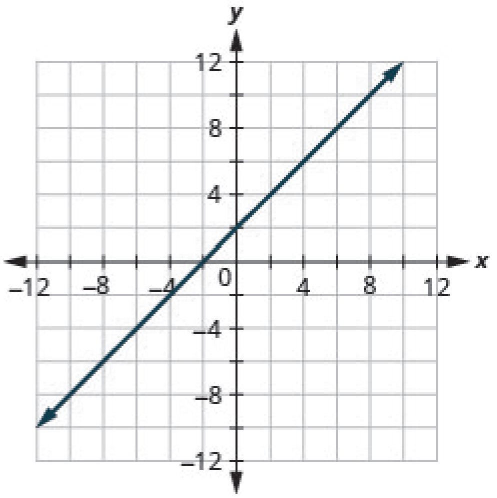

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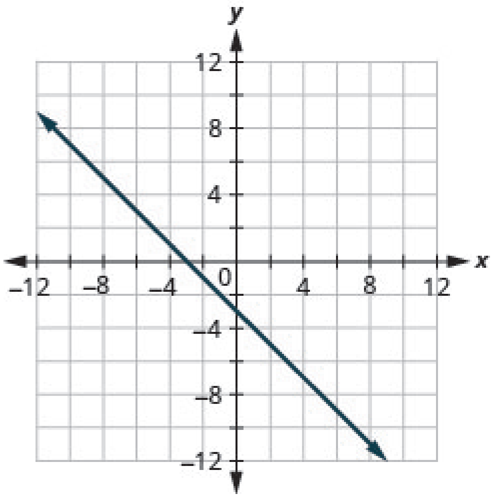

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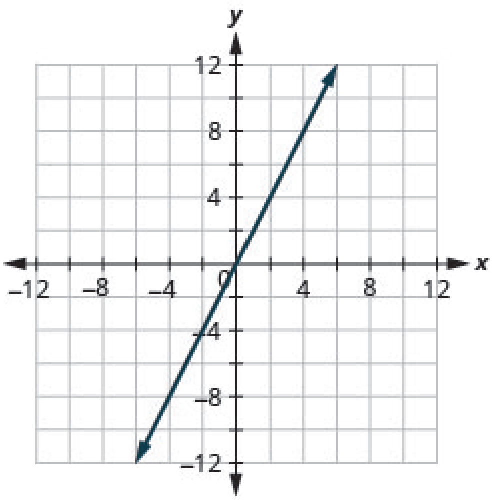

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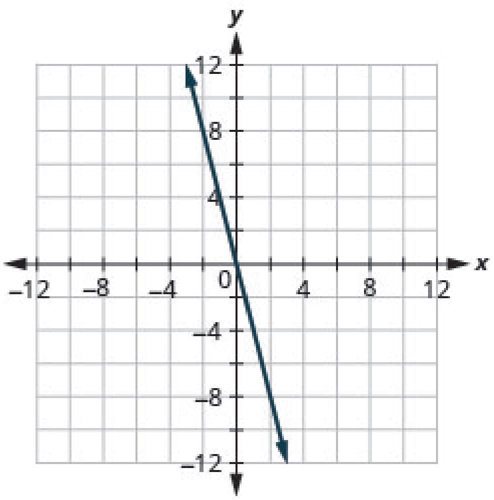

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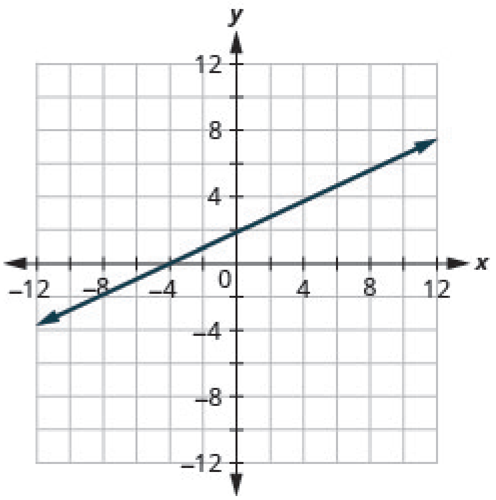

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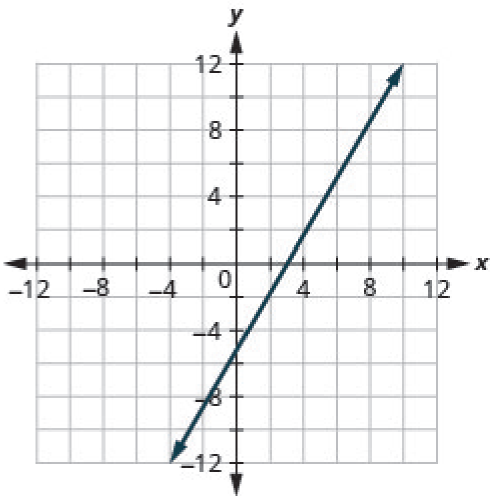

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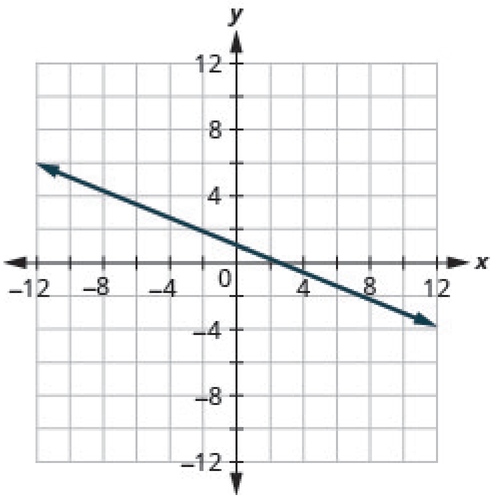

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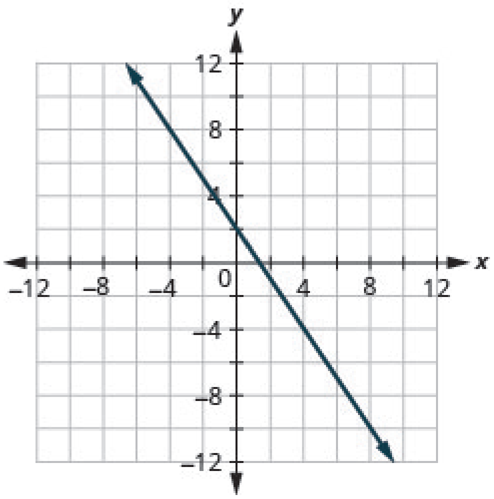

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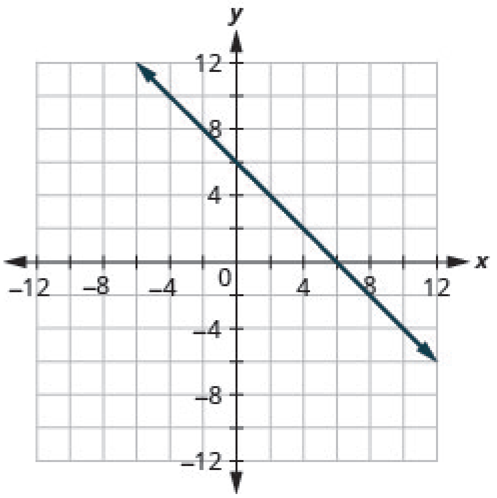

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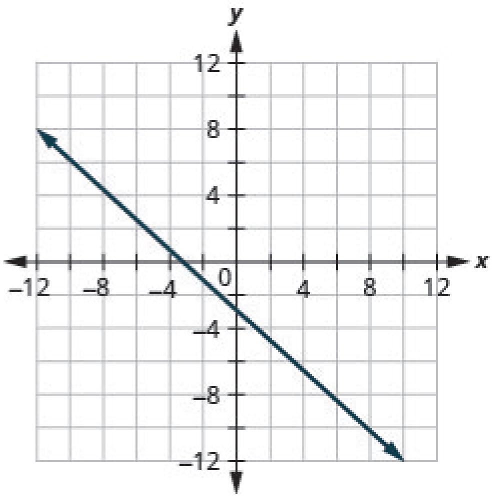

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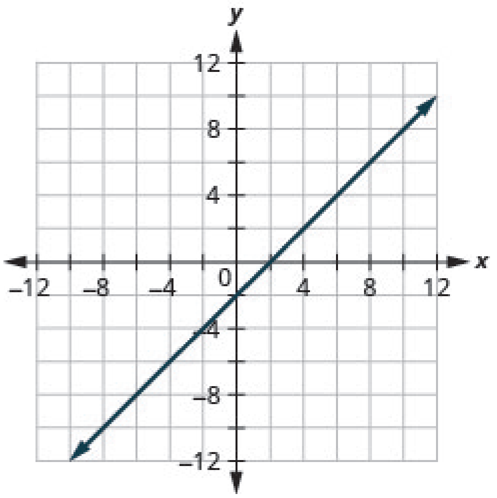

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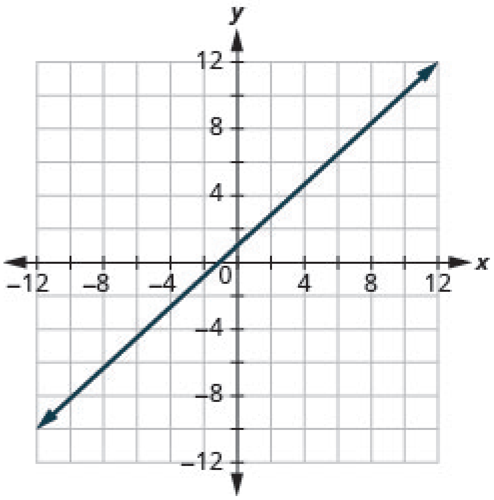

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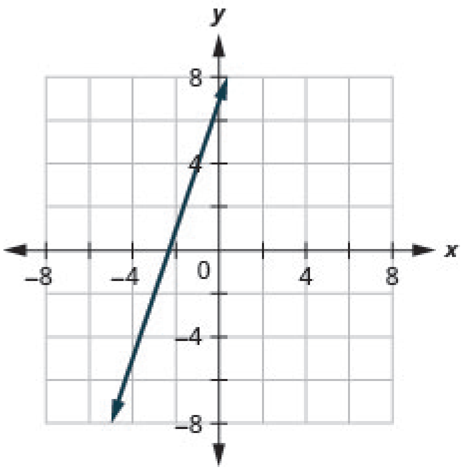

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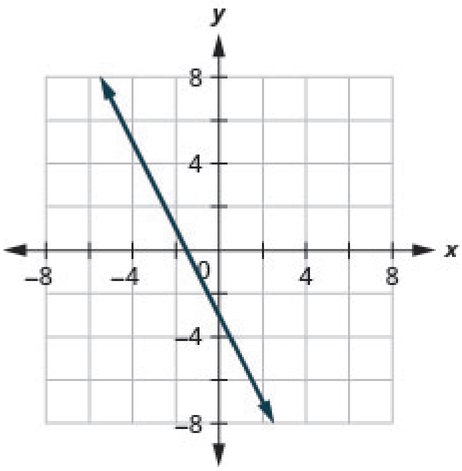
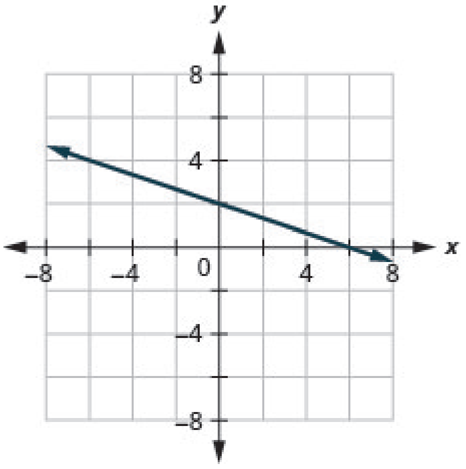
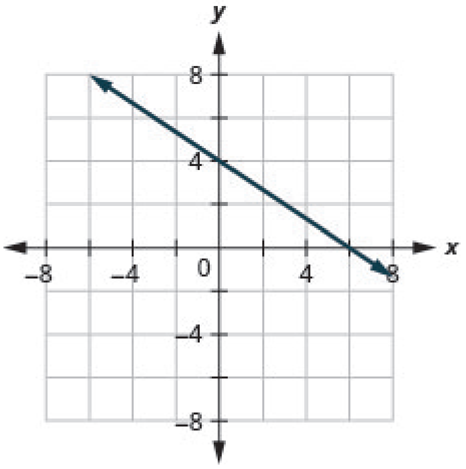
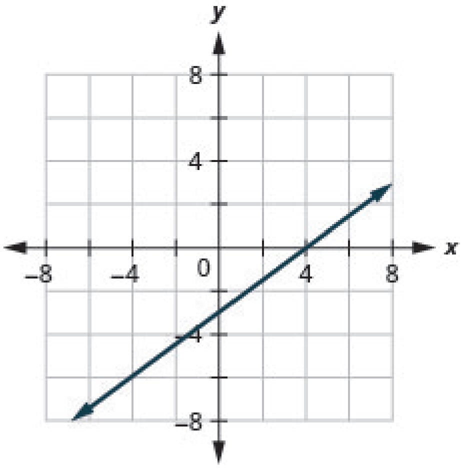
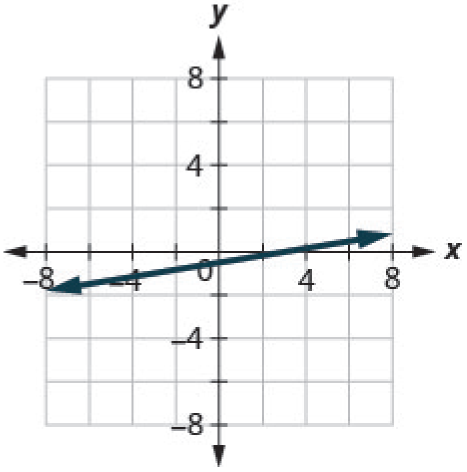
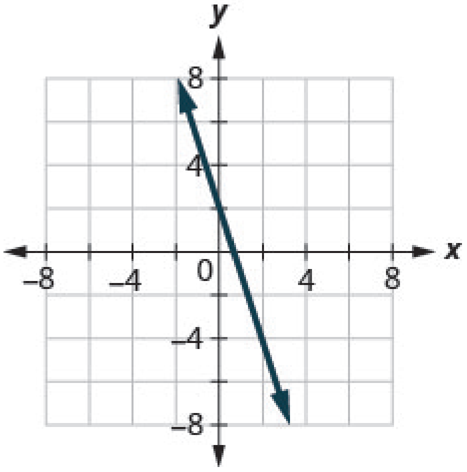
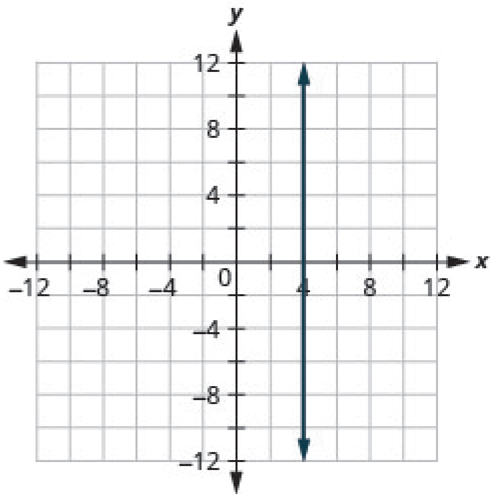
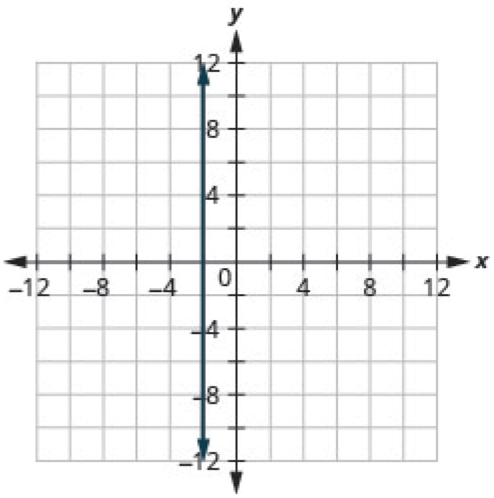
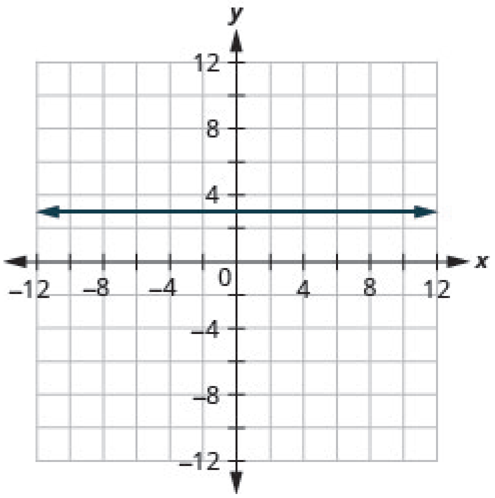
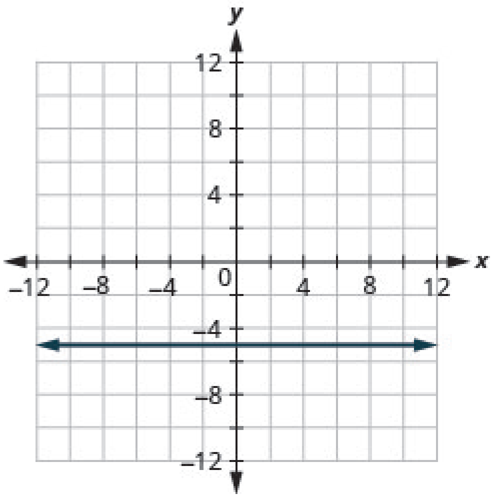


37. 	39. *ANSWER GRAPH LOOKS OFF; ie. graph should have m=2/5, not (-2/5).
41. 	43. 
45. 	47. 

49. 	51. 
53. 	55. 

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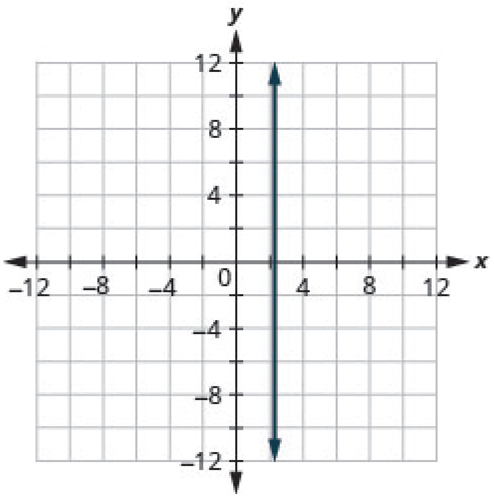

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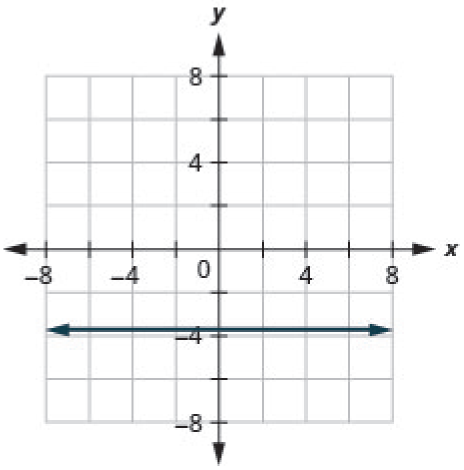

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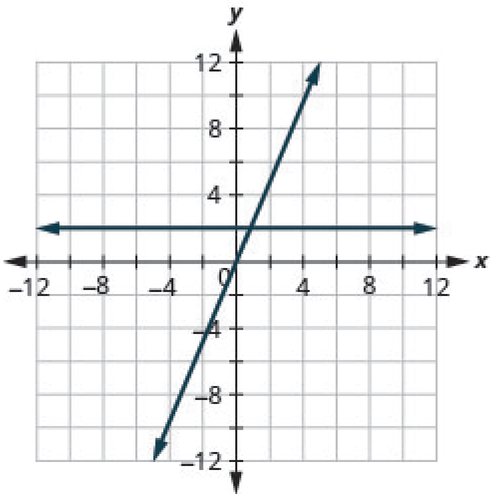

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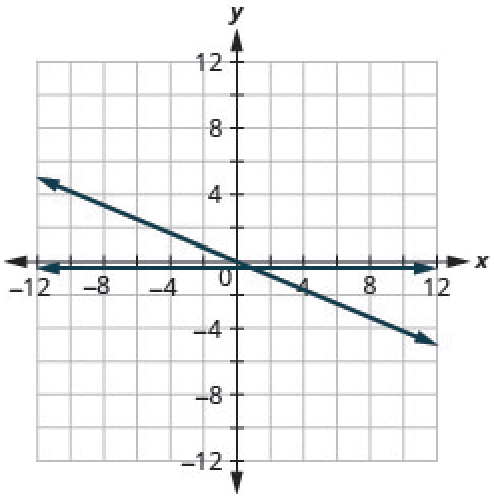

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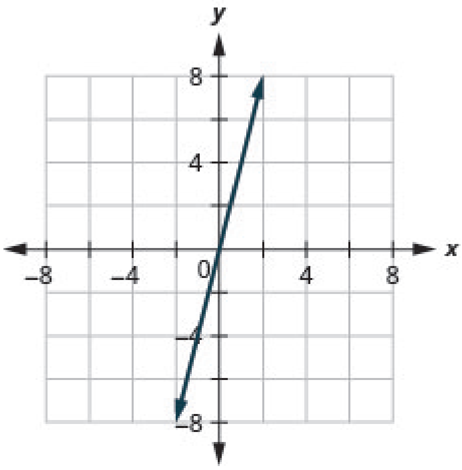

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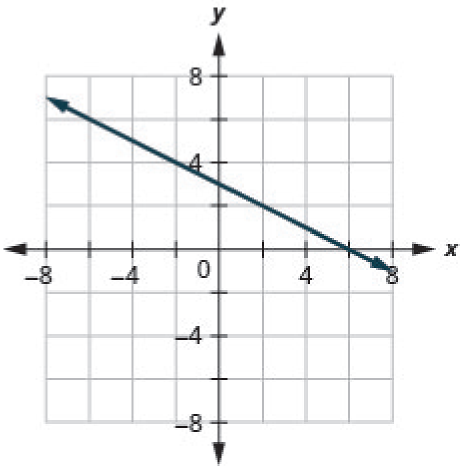

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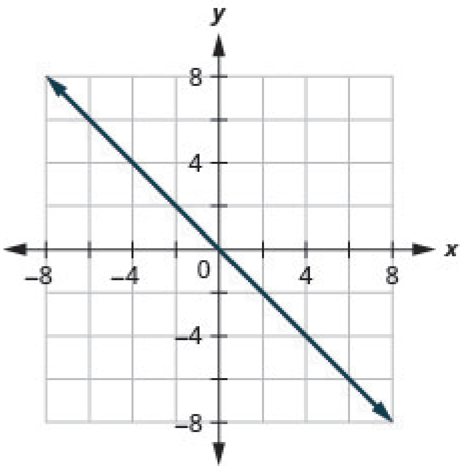

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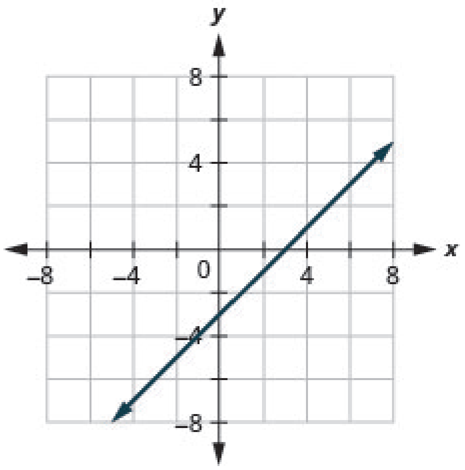

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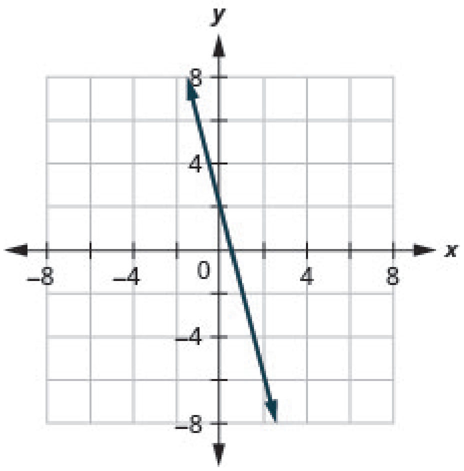

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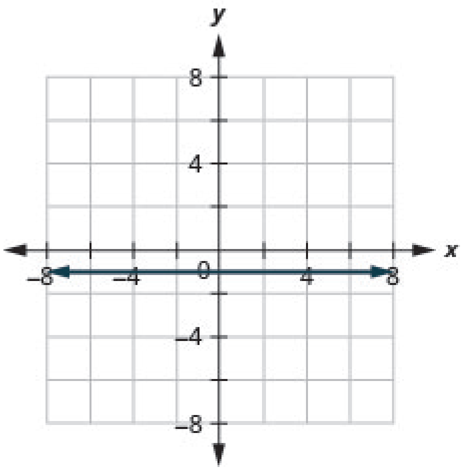

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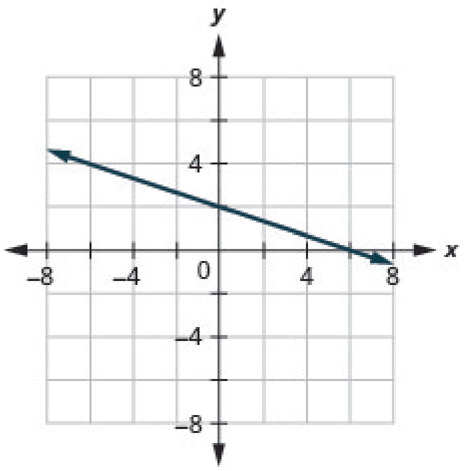

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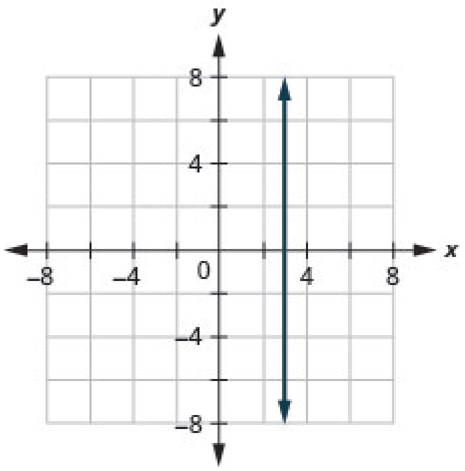
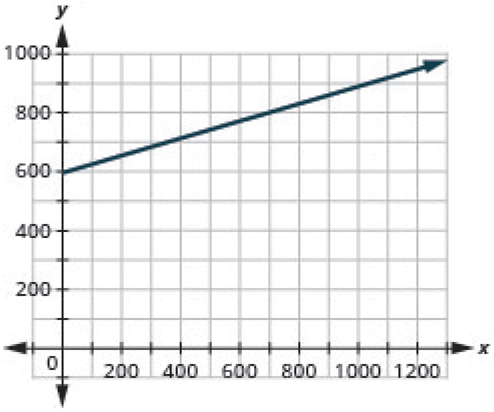


81. \$722, \$850, \$978  	83. Answers will vary.

Attributions

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6.3 Graph with Intercepts

Learning Objectives

By the end of this section, you will be able to:

- Identify the x– and y– intercepts on a graph
- Find the x– and y– intercepts from an equation of a line
- Graph a line using the intercepts

Identify the *x*– and *y*– Intercepts on a Graph

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

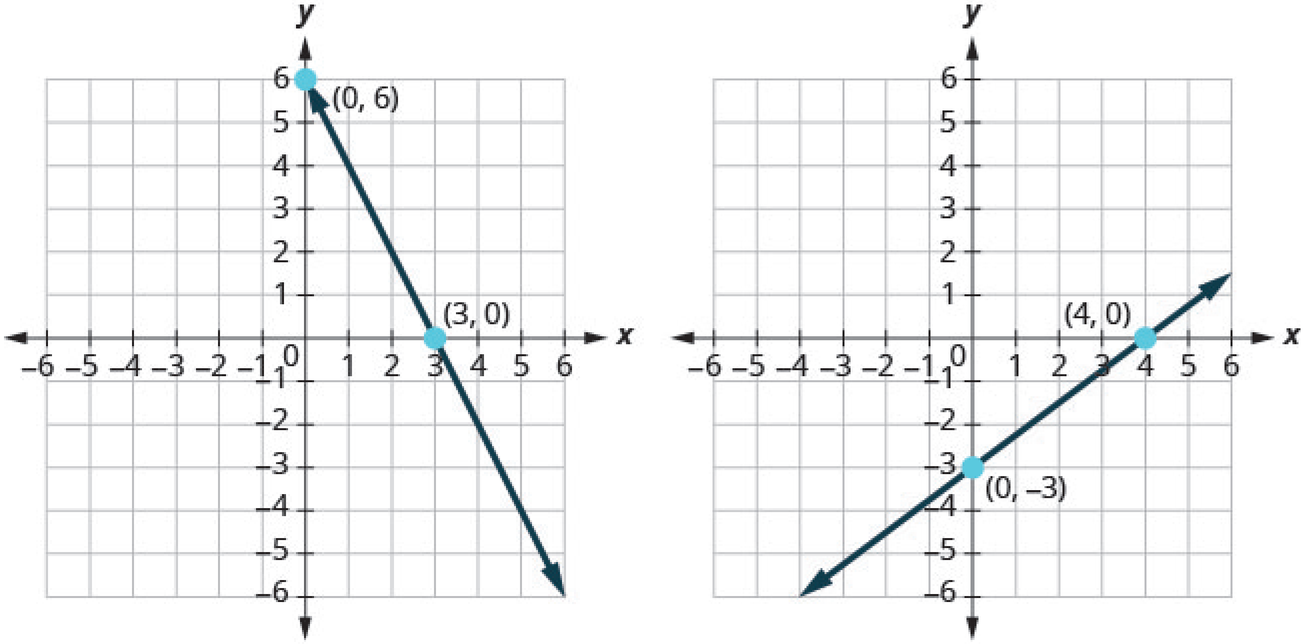
At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the *x*– axis and the *y*– axis. These points are called the *intercepts* of the line.

Intercepts of a line

The points where a line crosses the *x*– axis and the *y*– axis are called the intercepts of a line.

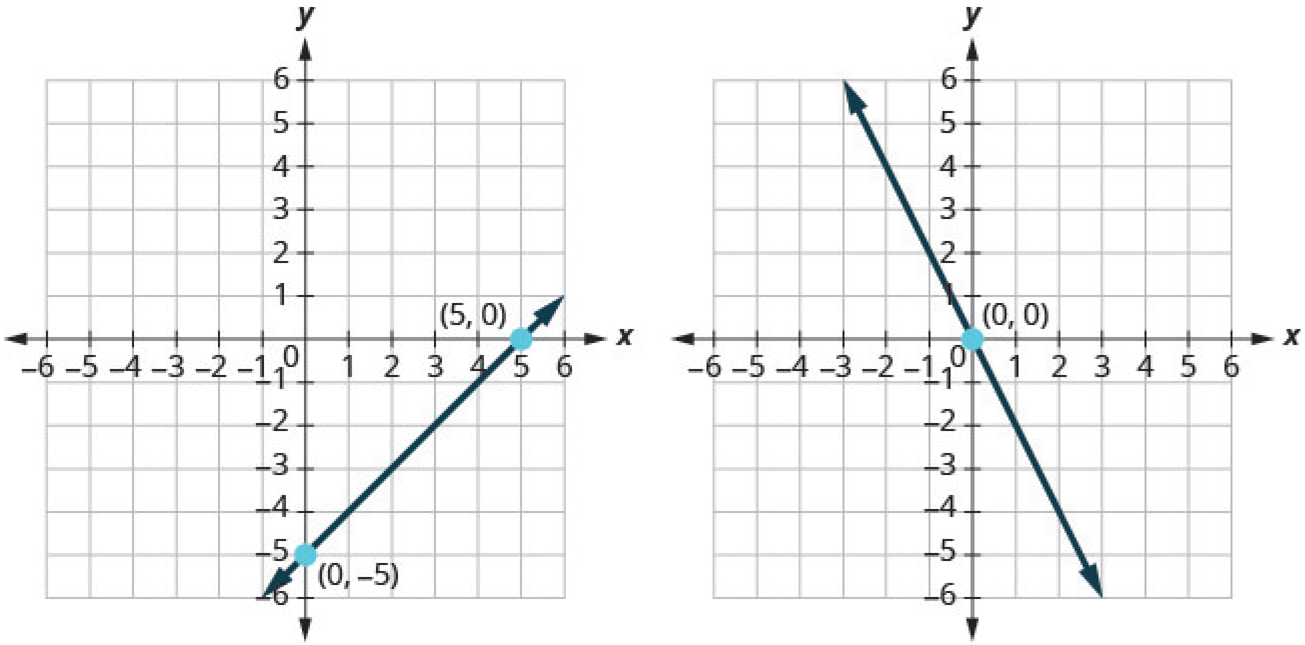
Let's look at the graphs of the lines in (Figure 1).

Examples of graphs crossing the x-negative axis.



(a) $2x + y = 6$

(b) $3x - 4y = 12$



(c) $x - y = 5$

(d) $y = -2x$

Figure .1

First, notice where each of these lines crosses the x negative axis. See (Figure 1).

Figure	**The line crosses the *x*– axis at:**	**Ordered pair of this point**
Figure (a)	3	$(3, 0)$
Figure (b)	4	$(4, 0)$
Figure (c)	5	$(5, 0)$
Figure (d)	0	$(0, 0)$

Do you see a pattern?

For each row, the *y*– coordinate of the point where the line crosses the *x*– axis is zero. The point where the line crosses the *x*– axis has the form $(a, 0)$ and is called the *x*– intercept of a line. The *x*– intercept occurs when y is zero.

Now, let's look at the points where these lines cross the *y*– axis. See the table below.

Figure	**The line crosses the *y*-axis at:**	**Ordered pair for this point**
Figure (a)	6	$(0, 6)$
Figure (b)	-3	$(0, -3)$
Figure (c)	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$

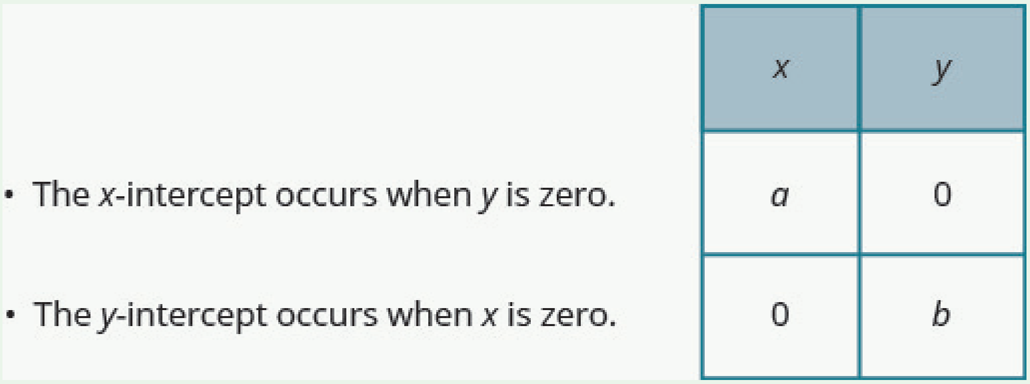
What is the pattern here?

In each row, the *x*– coordinate of the point where the line crosses the *y*– axis is zero. The point where the line crosses the *y*– axis has the form $(0, b)$ and is called the *y*- *intercept* of the line. The *y*– intercept occurs when x is zero.

_x_– intercept and _y_– intercept of a line

The *x*– intercept is the point $(a, 0)$ where the line crosses the *x*– axis.

The *y*– intercept is the point $(0, b)$ where the line crosses the *y*– axis.

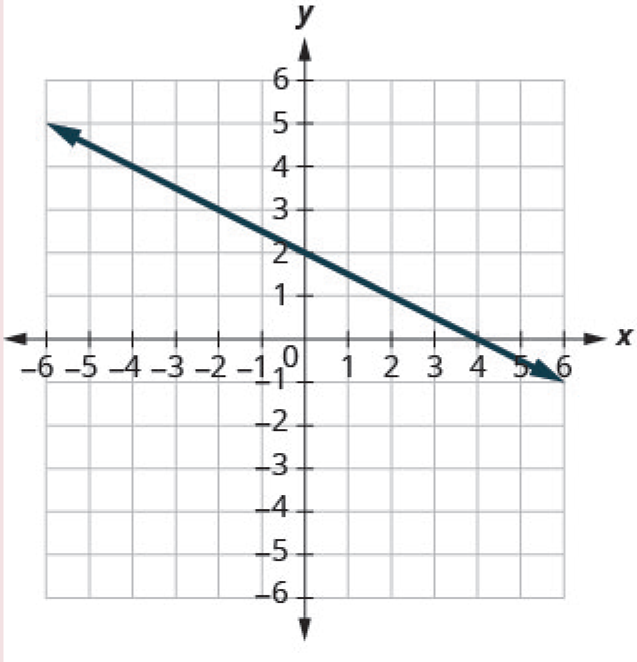


- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

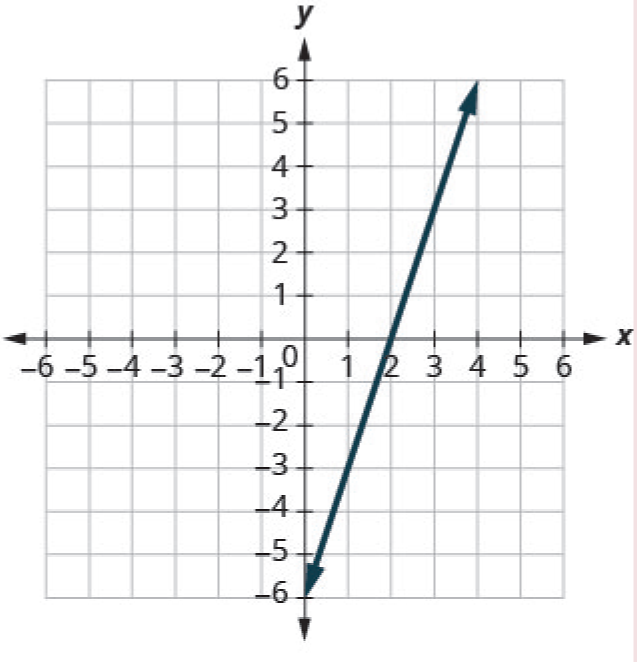
x	y
a	0
0	b

EXAMPLE 1

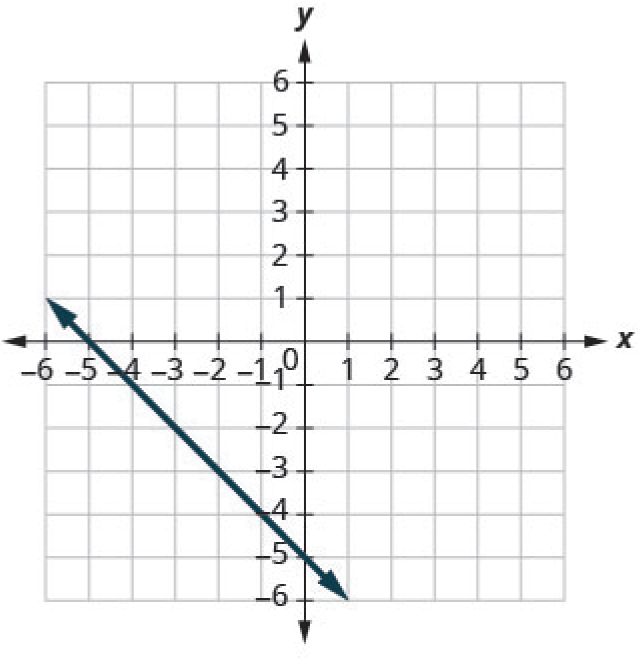
Find the *x*– and *y*– intercepts on each graph.



(a)



(b)



(c)

Solution

a. The graph crosses the *x*– axis at the point $(4, 0)$. The *x*– intercept is $(4, 0)$.
The graph crosses the *y*– axis at the point $(0, 2)$. The *y*– intercept is $(0, 2)$.

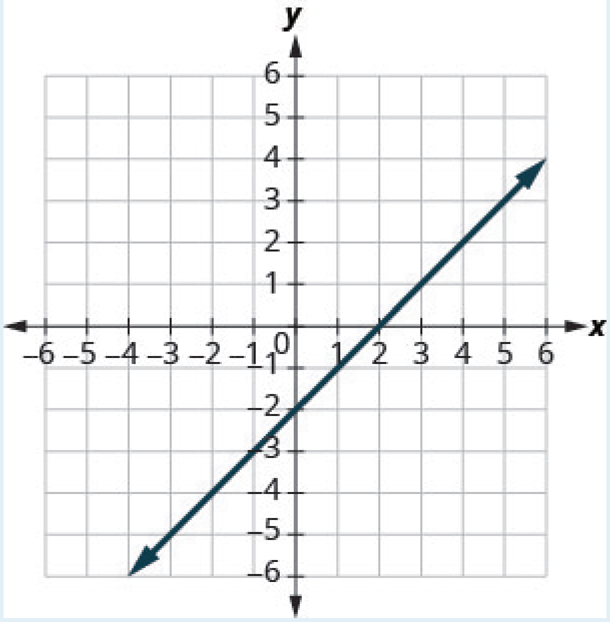
b. The graph crosses the *x*– axis at the point $(2, 0)$. The *x*– intercept is $(2, 0)$

The graph crosses the *y*– axis at the point $(0, -6)$. The *y*– intercept is $(0, -6)$.

c. The graph crosses the *x*– axis at the point $(-5, 0)$. The *x*– intercept is $(-5, 0)$.
The graph crosses the *y*– axis at the point $(0, -5)$. The *y*– intercept is $(0, -5)$.

TRY IT 1.1

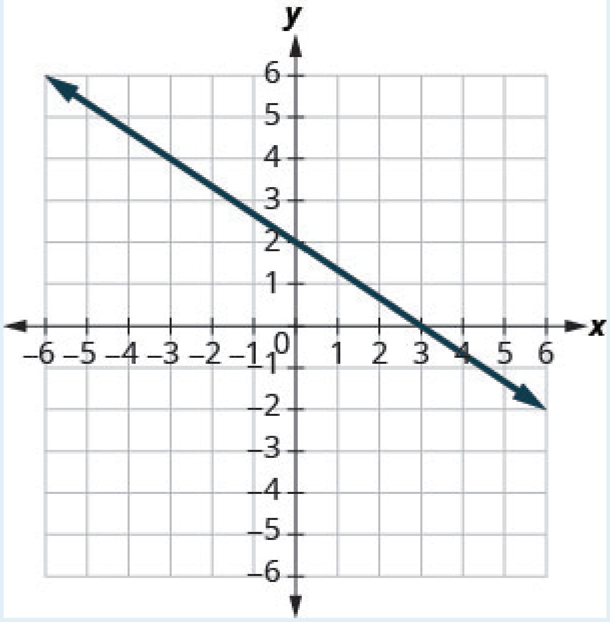
Find the *x*– and *y*– intercepts on the graph.



Show answer
x– intercept: $(2, 0)$; *y*– intercept: $(0, -2)$

TRY IT 1.2

Find the *x*– and *y*– intercepts on the graph.



Show answer
x– intercept: $(3, 0)$, *y*– intercept: $(0, 2)$

Find the *x*– and *y*– Intercepts from an Equation of a Line

Recognizing that the *x*– intercept occurs when *y* is zero and that the *y*– intercept occurs when *x* is zero, gives us a method to find the intercepts of a line from its equation. To find the *x*– intercept, let $y = 0$ and solve for *x*. To find the *y*– intercept, let $x = 0$ and solve for *y*.

Find the *x*– and *y*– intercepts from the equation of a line

Use the equation of the line. To find:

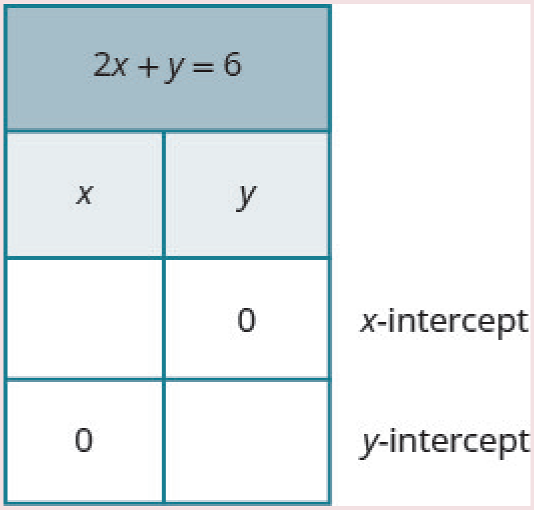
- the *x*– intercept of the line, let $y = 0$ and solve for x.
- the *y*– intercept of the line, let $x = 0$ and solve for y.

EXAMPLE 2

Find the intercepts of $2x + y = 6$.

Solution

We will let $y = 0$ to find the *x*– intercept, and let $x = 0$ to find the *y*– intercept. We will fill in the table, which reminds us of what we need to find.



$2x + y = 6$		
x	y	
	0	x-intercept
0		y-intercept

To find the *x*– intercept, let $y = 0$.

	$2x + y = 6$
Let $y = 0$.	$2x + 0 = 6$
Simplify.	$2x = 6$
	$x = 3$
The x-intercept is	(3, 0)
To find the y-intercept, let $x = 0$.	
	$2x + y = 6$
Let $x = 0$.	$2 \cdot 0 + y = 6$
Simplify.	$0 + y = 6$
	$y = 6$
The y-intercept is	(0, 6)

The intercepts are the points $(3, 0)$ and $(0, 6)$ as shown in the following table.

$2x + y = 6$	
x	y
3	0
0	6

TRY 2.1

Find the intercepts of $3x + y = 12$.

Show answer

x– intercept: $(4, 0)$, *y*– intercept: $(0, 12)$

TRY IT 2.2

Find the intercepts of $x + 4y = 8$.

Show answer

x– intercept: $(8, 0)$, *y*– intercept: $(0, 2)$

EXAMPLE 3

Find the intercepts of $4x - 3y = 12$.

Solution

To find the *x*-intercept, let *y* = 0.	
	$4x - 3y = 12$
Let *y* = 0.	$4x - 3 \cdot 0 = 12$
Simplify.	$4x - 0 = 12$
	$4x = 12$
	$x = 3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let *x* = 0.	
	$4x - 3y = 12$
Let *x* = 0.	$4 \cdot 0 - 3y = 12$
Simplify.	$0 - 3y = 12$
	$-3y = 12$
	$y = -4$
The *y*-intercept is	(0, −4)

The intercepts are the points (3, 0) and (0, −4) as shown in the following table.

$4x - 3y = 12$	
x	y
3	0
0	-4

TRY IT 3.1

Find the intercepts of $3x - 4y = 12$.

Show answer
x– intercept: $(4, 0)$, *y*– intercept: $(0, -3)$

TRY IT 3.2

Find the intercepts of $2x - 4y = 8$.

Show answer
x– intercept: $(4, 0)$, *y*– intercept: $(0, -2)$

Graph a Line Using the Intercepts

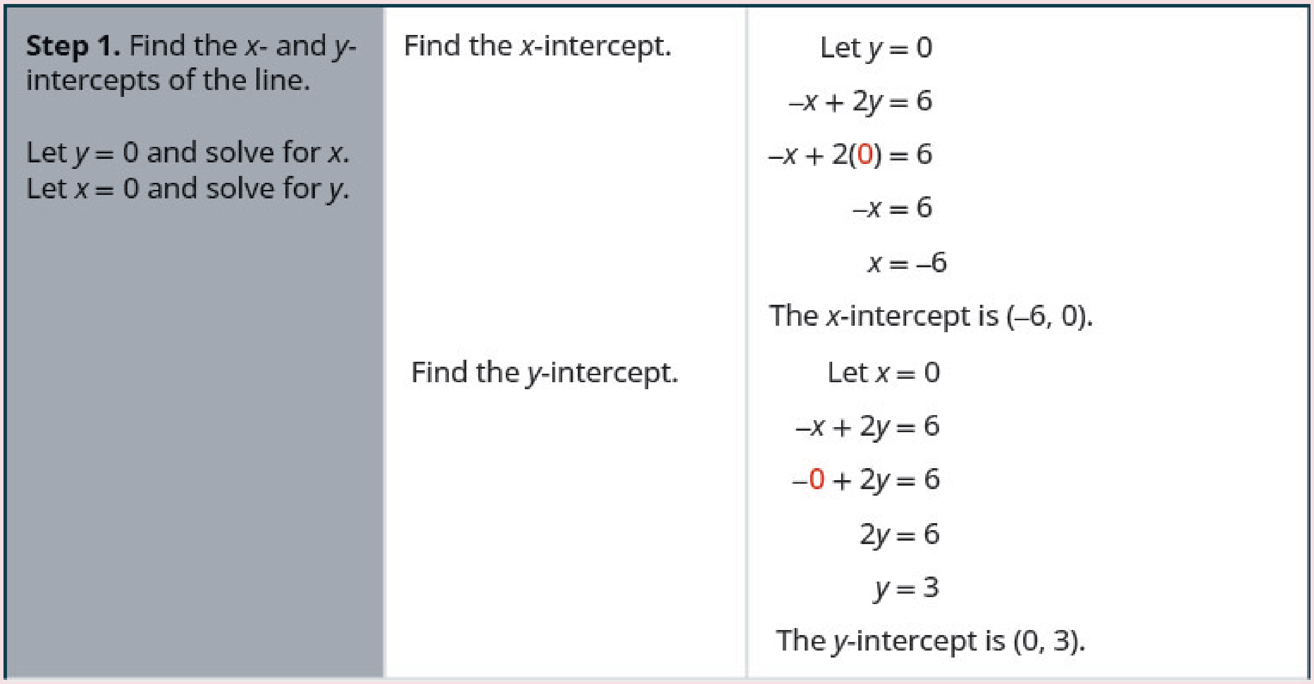
To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*– and *y*– intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

EXAMPLE 4

How to Graph a Line Using Intercepts

Graph $-x + 2y = 6$ using the intercepts.

Solution



Step 1. Find the *x*- and *y*-intercepts of the line. Let $y = 0$ and solve for *x*. Let $x = 0$ and solve for *y*.	Find the *x*-intercept.	Let $y = 0$ $-x + 2y = 6$ $-x + 2(0) = 6$ $-x = 6$ $x = -6$ The *x*-intercept is (–6, 0).
	Find the *y*-intercept.	Let $x = 0$ $-x + 2y = 6$ $-0 + 2y = 6$ $2y = 6$ $y = 3$ The *y*-intercept is (0, 3).

Step 2. Find another solution to the equation.

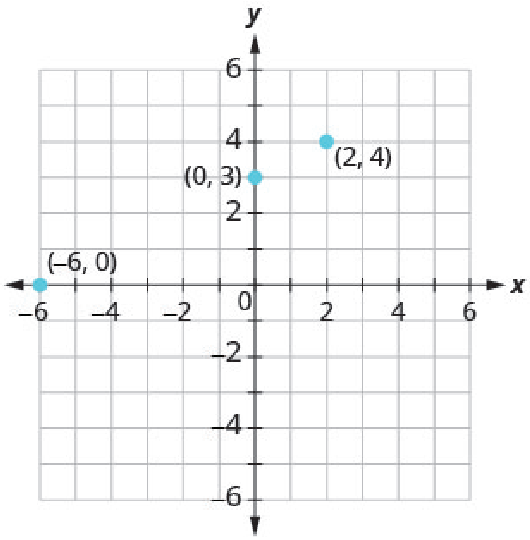
We'll use $x = 2$.

$$\begin{aligned} \text{Let } x &= 2 \\ -x + 2y &= 6 \\ -2 + 2y &= 6 \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

A third point is (2, 4).

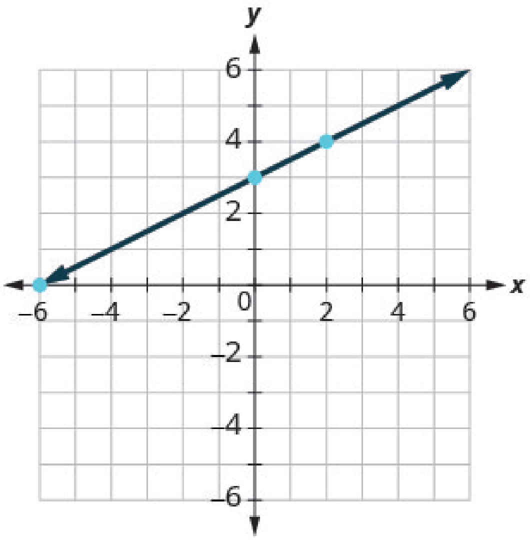
Step 3. Plot the three points. Check that the points line up.

x	y	(x, y)
–6	0	(–6, 0)
0	3	(0, 3)
2	4	(2, 4)



Step 4. Draw the line.

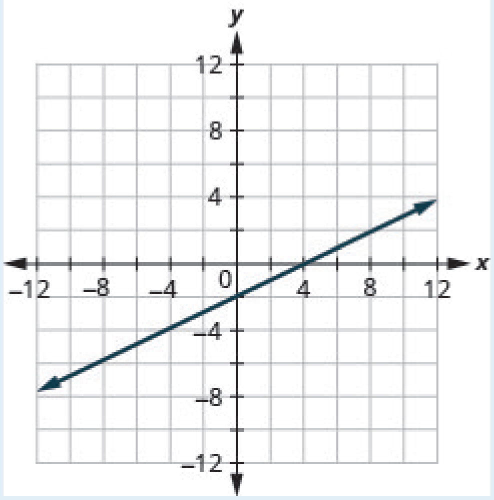
See the graph.



TRY IT 4.1

Graph $x - 2y = 4$ using the intercepts.

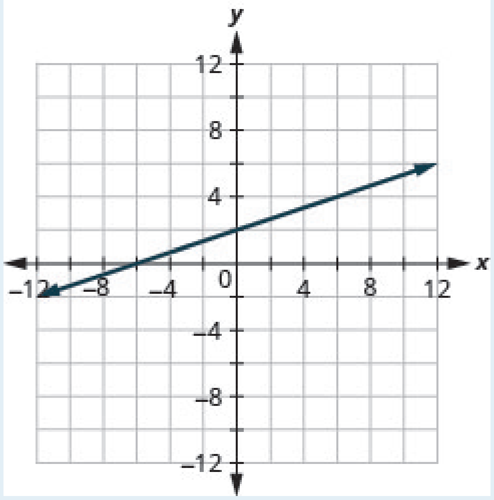
Show answer



TRY IT 4.2

Graph $-x + 3y = 6$ using the intercepts.

Show answer



HOW TO: Graph a linear equation using the intercepts

The steps to graph a linear equation using the intercepts are summarized below.

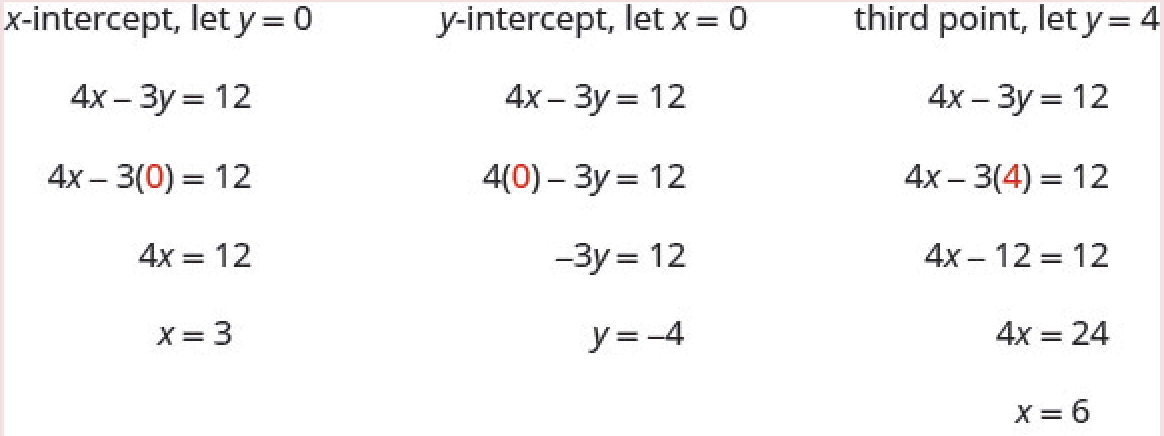
1. Find the *x*– and *y*– intercepts of the line.
   - Let $y = 0$ and solve for x
   - Let $x = 0$ and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line.

EXAMPLE 5

Graph $4x - 3y = 12$ using the intercepts.

Solution

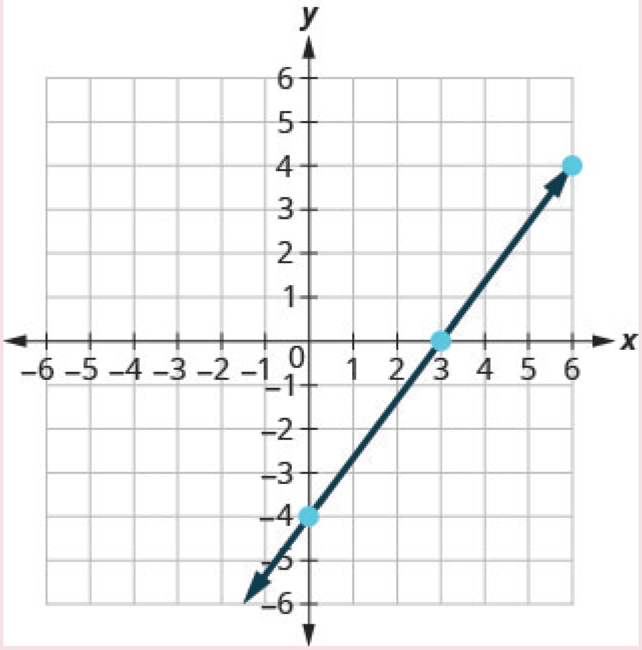
Find the intercepts and a third point.



x-intercept, let $y = 0$	*y*-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in following table and show the graph below.

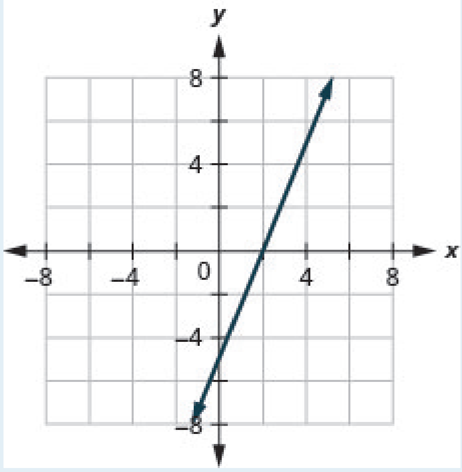
$4x - 3y = 12$		
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



TRY IT 5.1

Graph $5x - 2y = 10$ using the intercepts.

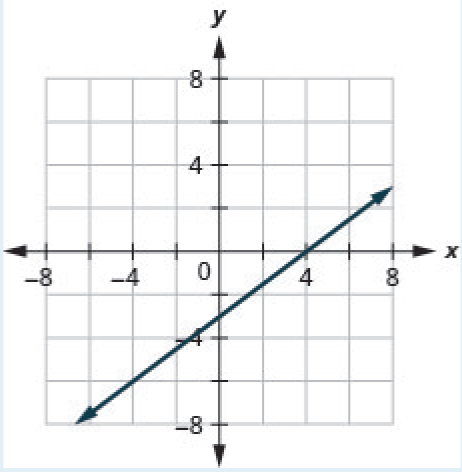
Show answer



TRY IT 5.2

Graph $3x - 4y = 12$ using the intercepts.

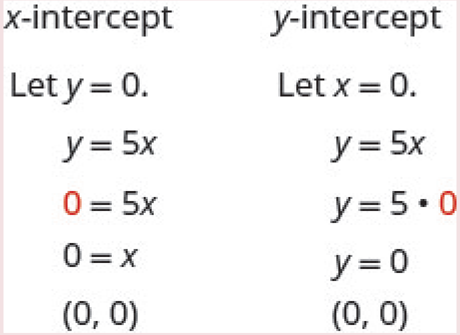
Show answer



EXAMPLE 6

Graph $y = 5x$ using the intercepts.

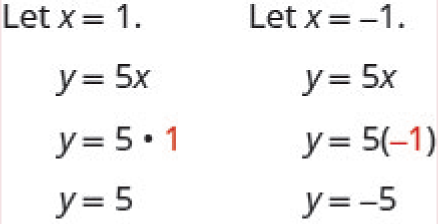
Solution



x-intercept	y-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
$(0, 0)$	$(0, 0)$

This line has only one intercept. It is the point $(0, 0)$.

To ensure accuracy we need to plot three points. Since the *x*– and *y*– intercepts are the same point, we need *two* more points to graph the line.

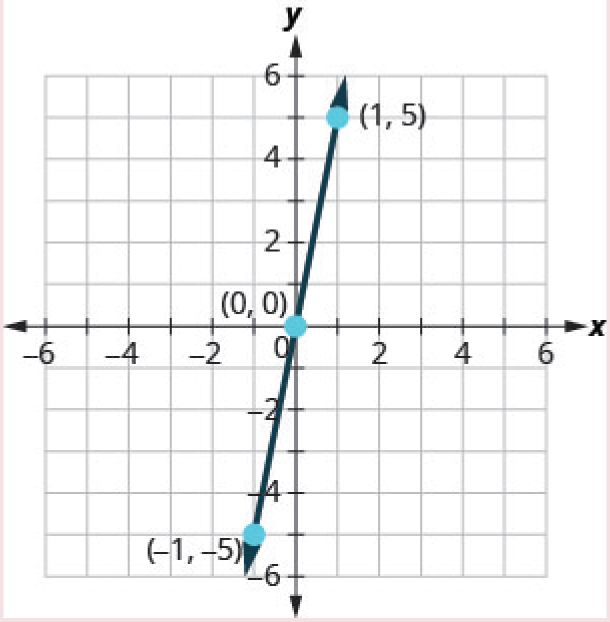


Let $x = 1$.	Let $x = -1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(-1)$
$y = 5$	$y = -5$

See following table..

$y=5x$		
x	y	(x,y)
0	0	$(0,0)$
1	5	$(1,5)$
-1	-5	$(-1,-5)$

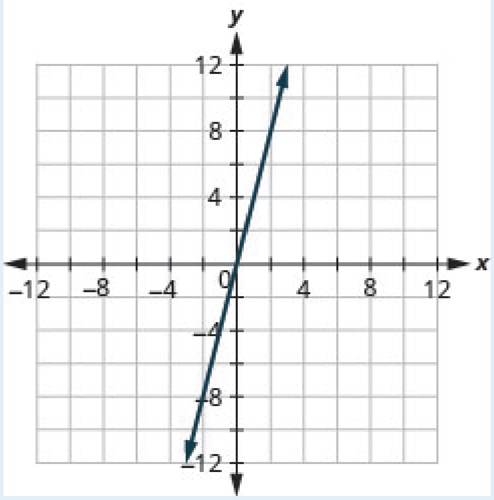
Plot the three points, check that they line up, and draw the line.



TRY IT 6.1

Graph $y=4x$ using the intercepts.

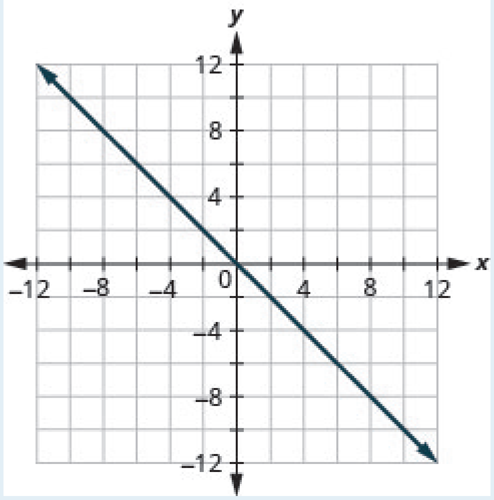
Show answer



TRY IT 6.2

Graph $y = -x$ the intercepts.

Show answer



Key Concepts

- **Find the *x*– and *y*– Intercepts from the Equation of a Line**
  - Use the equation of the line to find the *x*– intercept of the line, let $y = 0$ and solve for *x*.
  - Use the equation of the line to find the *y*– intercept of the line, let $x = 0$ and solve for

y.

- **Graph a Linear Equation using the Intercepts**
    1. Find the *x*– and *y*– intercepts of the line.
       Let $y = 0$ and solve for *x*.
       Let $x = 0$ and solve for *y*.
    2. Find a third solution to the equation.
    3. Plot the three points and then check that they line up.
    4. Draw the line.
- **Strategy for Choosing the Most Convenient Method to Graph a Line:**
    - Consider the form of the equation.
    - If it only has one variable, it is a vertical or horizontal line.
      $x = a$ is a vertical line passing through the *x*– axis at a
      $y = b$ is a horizontal line passing through the *y*– axis at b.
    - If *y* is isolated on one side of the equation, graph by plotting points.
    - Choose any three values for *x* and then solve for the corresponding *y*– values.
    - If the equation is of the form $ax + by = c$, find the intercepts. Find the *x*– and *y*– intercepts and then a third point.

Glossary

intercepts of a line
The points where a line crosses the *x*– axis and the *y*– axis are called the intercepts of the line.

***x*– intercept**
The point $(a, 0)$ where the line crosses the *x*– axis; the *x*– intercept occurs when y is zero.

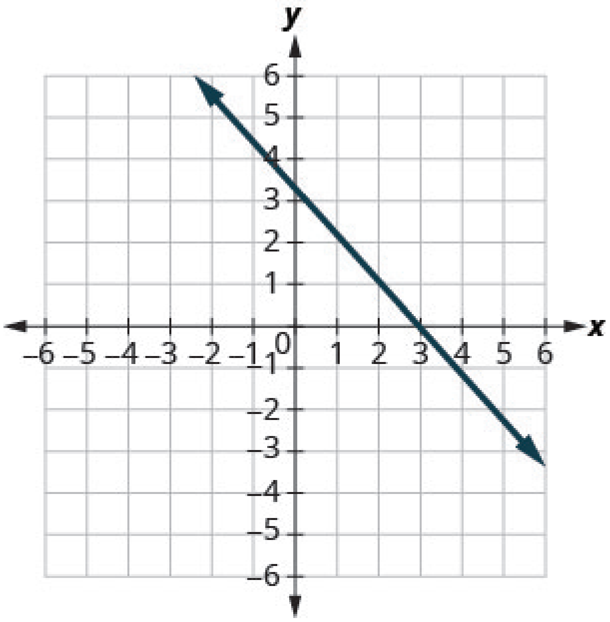
y-intercept
The point $(0, b)$ where the line crosses the *y*– axis; the *y*– intercept occurs when x is zero.

Practice Makes Perfect

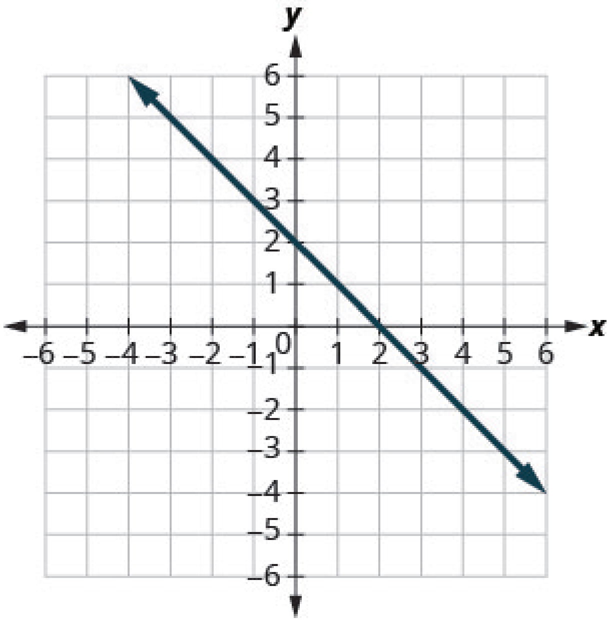
Identify the *x*– and *y*– Intercepts on a Graph

In the following exercises, find the *x*– and *y*– intercepts on each graph.

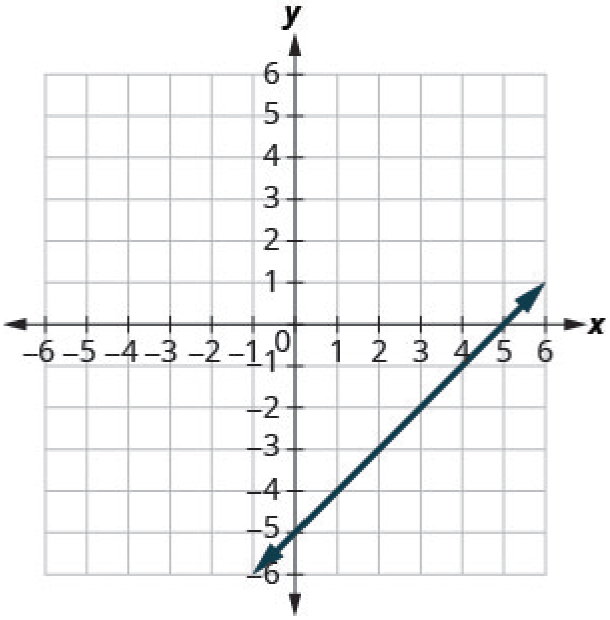
1.


y
x
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

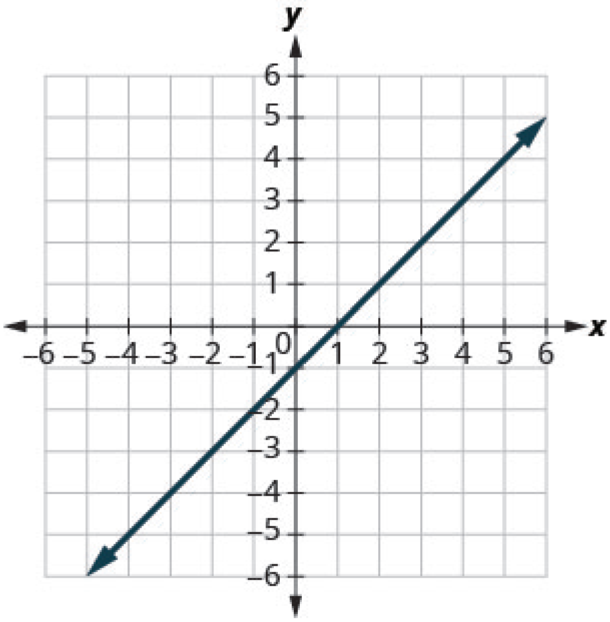
2.


y
x
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

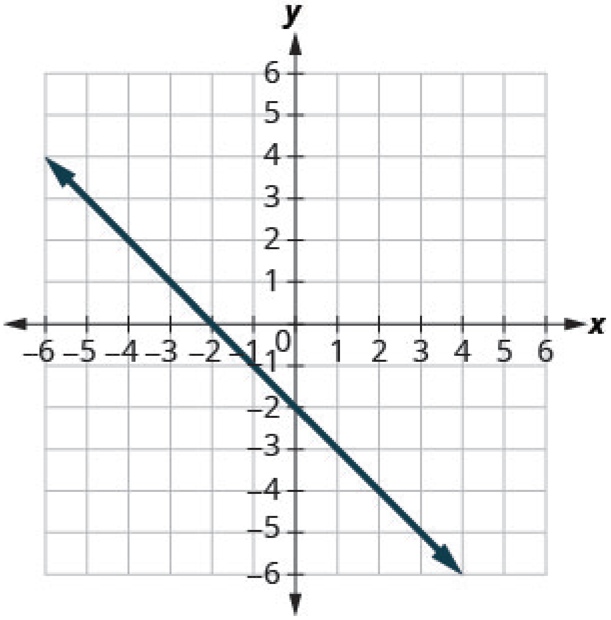
3.


y
x
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

4.


y
x
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

5.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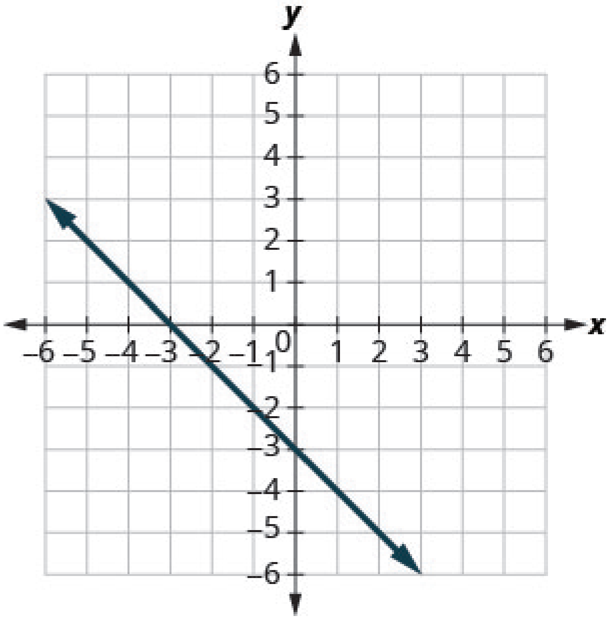
–6 –5 –4 –3 –2 –1
1 2 3 4 5 6

6.


y
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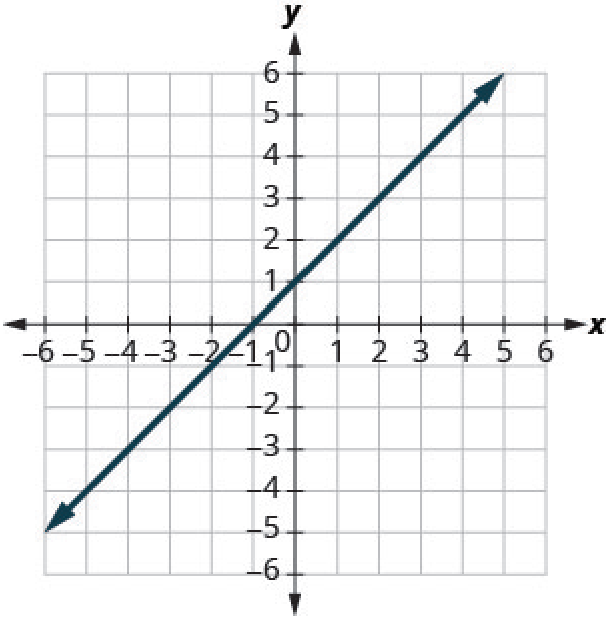
–5
–6
–6 –5 –4 –3 –2 –1
1 2 3 4 5 6

7.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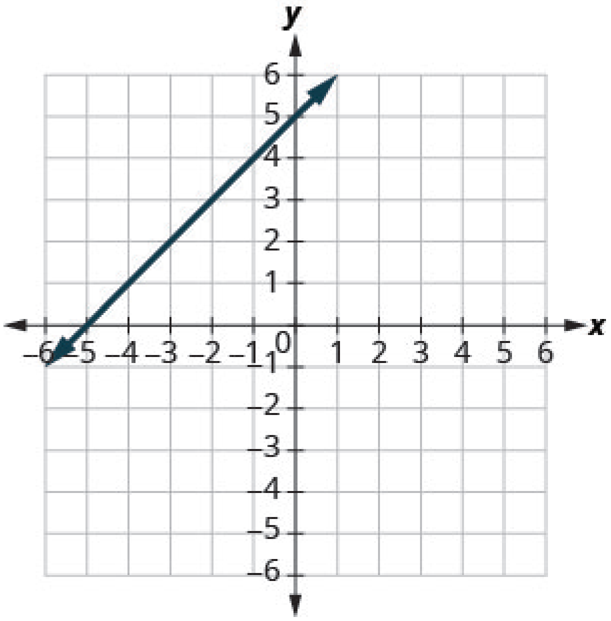
–6 –5 –4 –3 –2 –1
1 2 3 4 5 6

8.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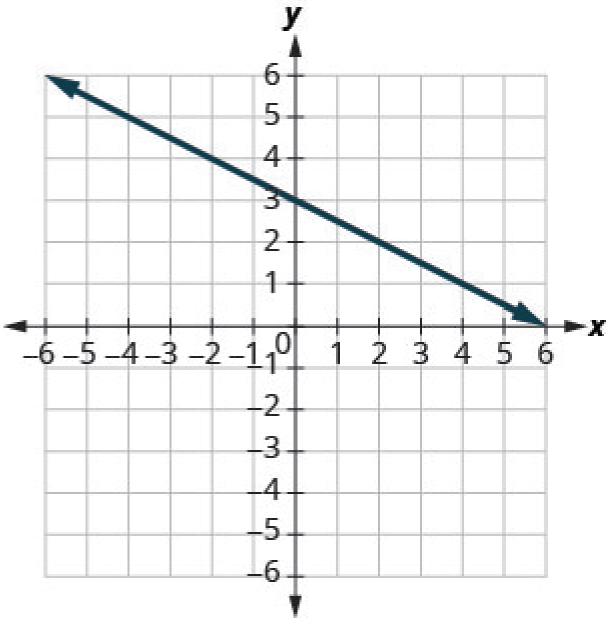
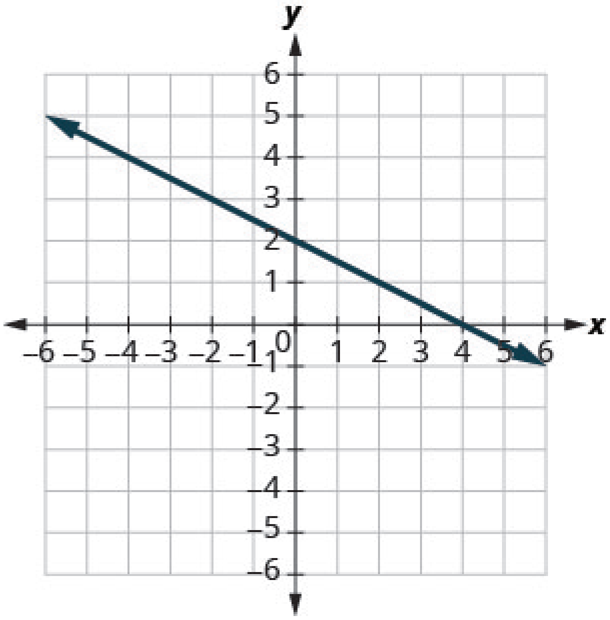
–3
–4
–5
–6
–6 –5 –4 –3 –2 –1
1 2 3 4 5 6

9.



10.



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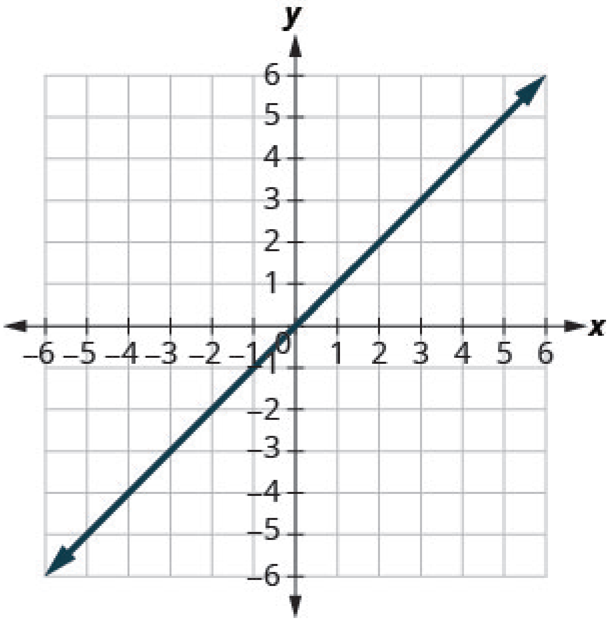
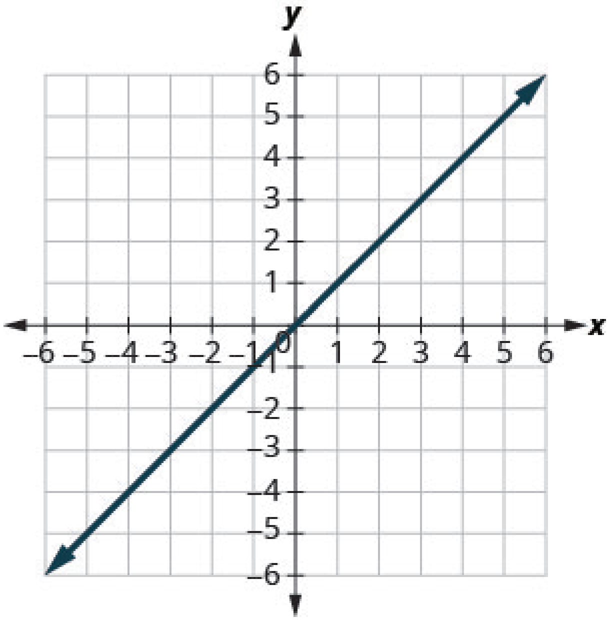


12.



Find the *x*– and *y*– Intercepts from an Equation of a Line

In the following exercises, find the intercepts for each equation.

13. $x+y=4$	14. $x+y=3$
15. $x+y=-2$	17. $x-y=5$
18. $x-y=1$	19. $x-y=-3$
20. $x-y=-4$	21. $x+2y=8$
22. $x+2y=10$	23. $3x+y=6$
24. $3x+y=9$	25. $x-3y=12$
25. $x-3y=12$	27. $4x-y=8$
28. $5x-y=5$	28. $5x-y=5$
30. $2x+3y=6$	31. $3x-2y=12$
32. $3x-5y=30$	33. $y=\frac{1}{3}x+1$
34. $y=\frac{1}{4}x-1$	35. $y=\frac{1}{5}x+2$
36. $y=\frac{1}{3}x+4$	37. $y=3x$
38. $y=-2x$	39. $y=-4x$
40. $y=5x$	

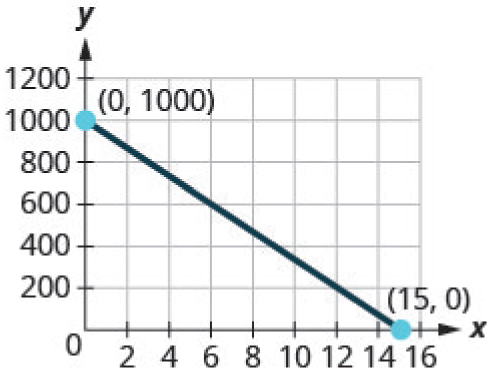
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

41. $-x+5y=10$	42. $-x+4y=8$
43. $x+2y=4$	44. $x+2y=6$
45. $x+y=2$	46. $x+y=5$
47. $x+y=-3$	48. $x+y=-1$
49. $x-y=1$	49. $x-y=1$
51. $x-y=-4$	52. $x-y=-3$
53. $4x+y=4$	54. $3x+y=3$
55. $2x+4y=12$	56. $3x+2y=12$
57. $3x-2y=6$	58. $5x-2y=10$
59. $2x-5y=-20$	60. $3x-4y=-12$
61. $3x-y=-6$	62. $2x-y=-8$
63. $y=\frac{3}{2}x$	64. $y=-4x$
65. $y=x$	66. $y=3x$

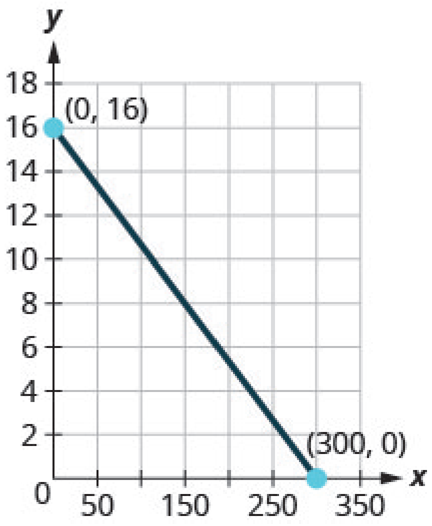
Everyday Math

67. **Road trip.** Damien is driving from Thunder Bay to Montreal, a distance of 1000 miles. The *x*– axis on the graph below shows the time in hours since Damien left Thunder Bay. The *y*– axis represents the distance he has left to drive.



1. a) Find the *x*– and *y*– intercepts.
2. b) Explain what the *x*– and *y*– intercepts mean for Damien.

68. **Road trip.** Jenna filled up the gas tank of her truck and headed out on a road trip. The *x*– axis on the graph below shows the number of miles Jenna drove since filling up. The *y*– axis represents the number of gallons of gas in the truck's gas tank.



1. a) Find the *x*– and *y*– intercepts.
2. b) Explain what the *x*– and *y*– intercepts mean for Ozzie.

Writing Exercises

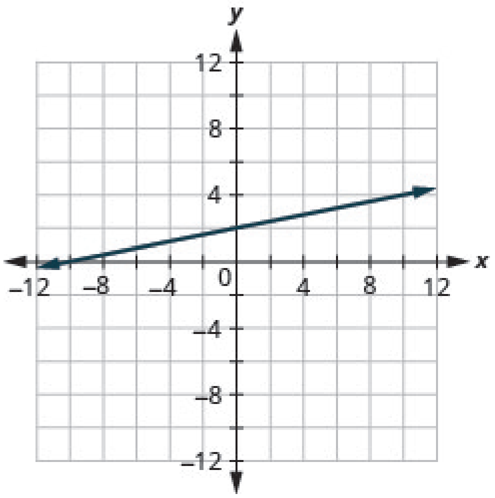
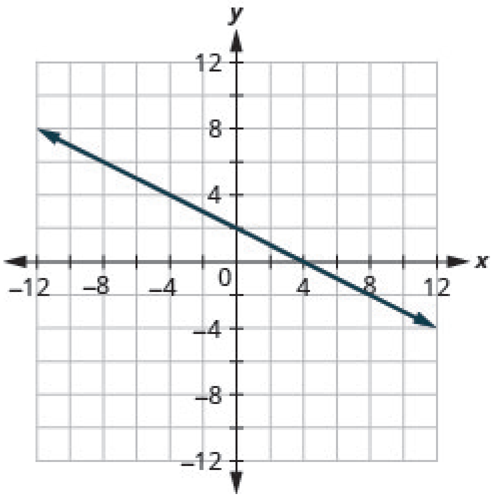
69. How do you find the *x*– intercept of the graph of $3x - 2y = 6$?

70. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x + y = -4$? Why?

71. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = \frac{2}{3}x - 2$? Why?

72. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = 6$? Why?

Answers

1. $(3,0)$, $(0,3)$	3. $(5,0)$, $(0,-5)$
5. $(-2,0)$, $(0,-2)$	7. $(-1,0)$, $(0,1)$
9. $(6,0)$, $(0,3)$	11. $(0,0)$
13. $(4,0)$, $(0,4)$	15. $(-2,0)$, $(0,-2)$
17. $(5,0)$, $(0,-5)$	19. $(-3,0)$, $(0,3)$
21. $(8,0)$, $(0,4)$	23. $(2,0)$, $(0,6)$
25. $(12,0)$, $(0,-4)$	27. $(2,0)$, $(0,-8)$
29. $(5,0)$, $(0,2)$	31. $(4,0)$, $(0,-6)$
33. $(-3,0)$, $(0,1)$	35. $(-10,0)$, $(0,2)$
37. $(0,0)$	39. $(0,0)$
41. 	43. 

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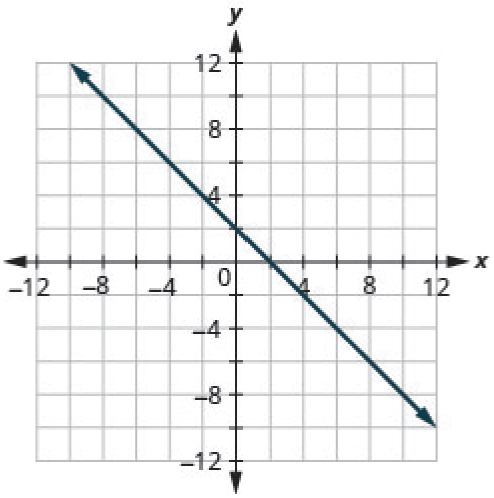
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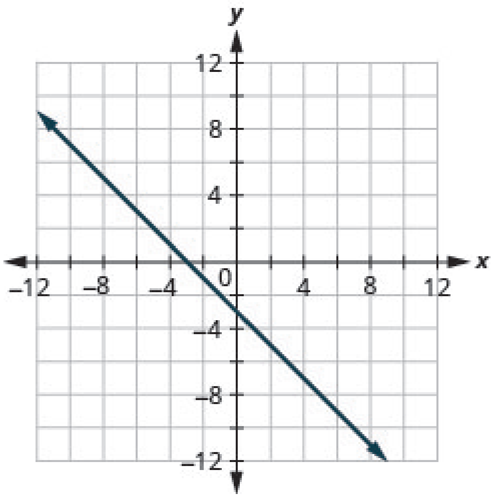
47.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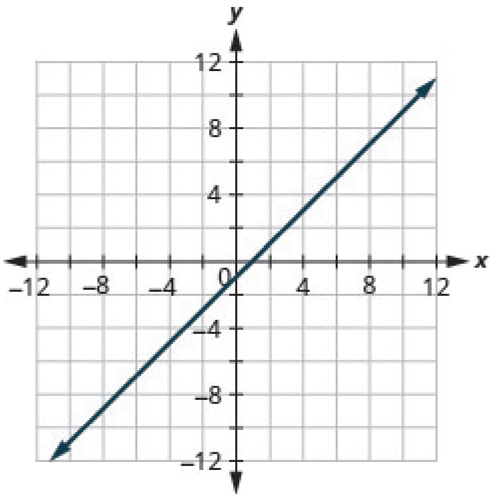
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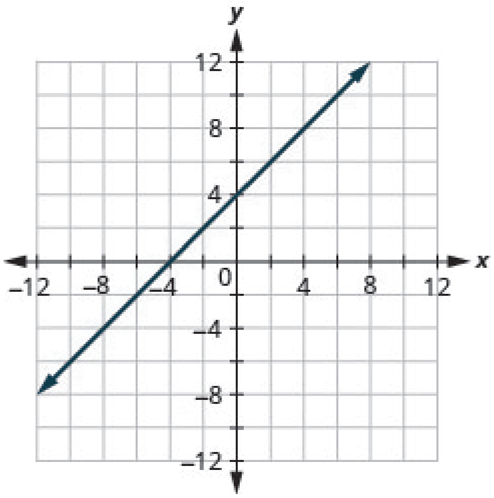
51.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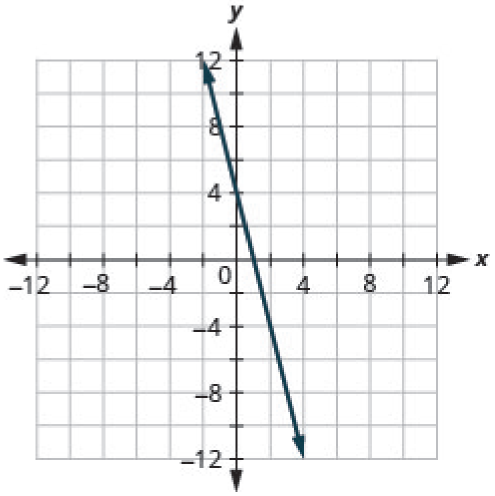
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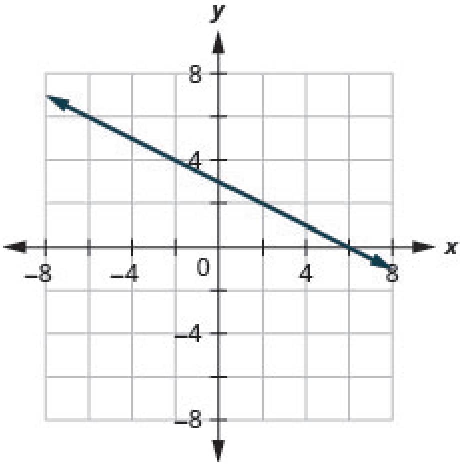
55.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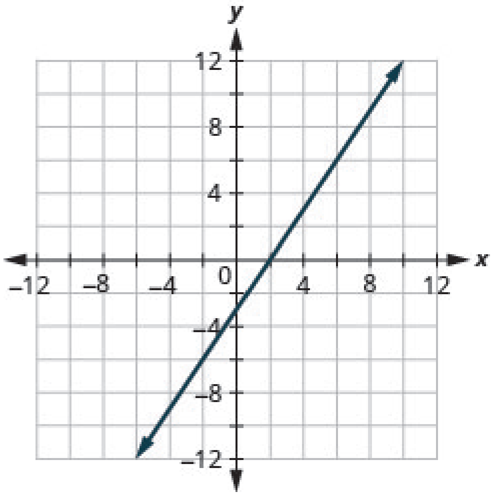
–4
–8
–8
–4
4
8

57.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–4
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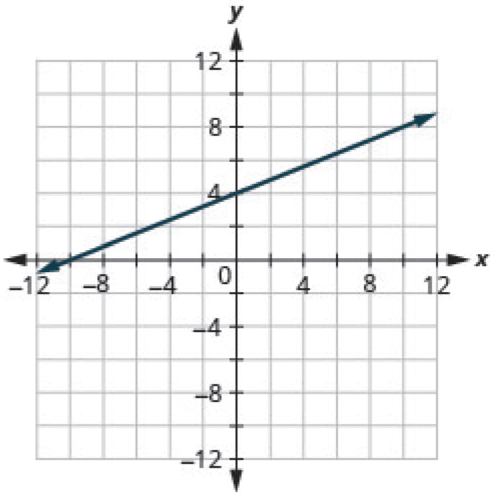
–12
–8
–4
4
8
12

59.


y
x
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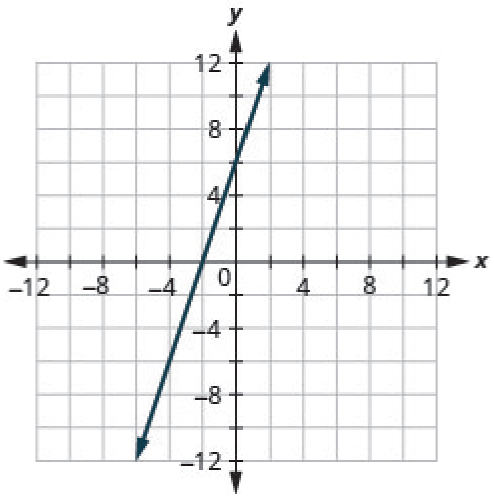
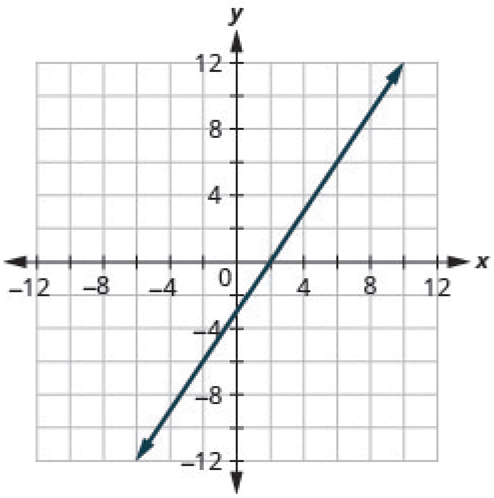
0
–4
–8
–12
–12
–8
–4
4
8
12

61.



63.



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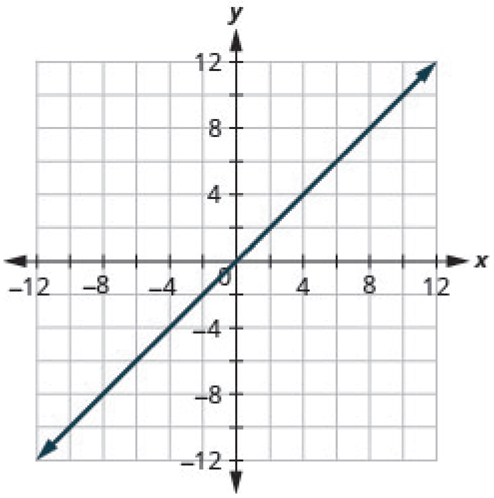

67.

a)$(0, 1000)$, $(15, 0)$
b) At $(0, 1000)$, he has been gone 0 hours and has 1000 miles left. At $(15, 0)$, he has been gone 15 hours and has 0 miles left to go.

69. Answers will vary.

71. Answers will vary.

Attributions

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6.4 Understand Slope of a Line

Learning Objectives

By the end of this section, you will be able to:

- Use geoboards to model slope
- Use $m = \frac{rise}{run}$ to find the slope of a line from its graph
- Find the slope of horizontal and vertical lines
- Use the slope formula to find the slope of a line between two points
- Graph a line given a point and the slope
- Solve slope applications

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. What determines whether a line tilts up or down or if it is steep or flat?

In mathematics, the ‘tilt’ of a line is called the *slope* of the line. The concept of slope has many applications in the real world. The pitch of a roof, grade of a highway, and a ramp for a wheelchair are some examples where you literally see slopes. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

In this section, we will explore the concept of slope.

Use Geoboards to Model Slope

A geoboard is a board with a grid of pegs on it. Using rubber bands on a geoboard gives us a concrete way to model lines on a coordinate grid. By stretching a rubber band between two pegs on a geoboard, we can discover how to find the slope of a line.

Doing the Manipulative Mathematics activity “Exploring Slope” will help you develop a better understanding of the slope of a line. (Graph paper can be used instead of a geoboard, if needed.)

We’ll start by stretching a rubber band between two pegs as shown in (Figure 1).

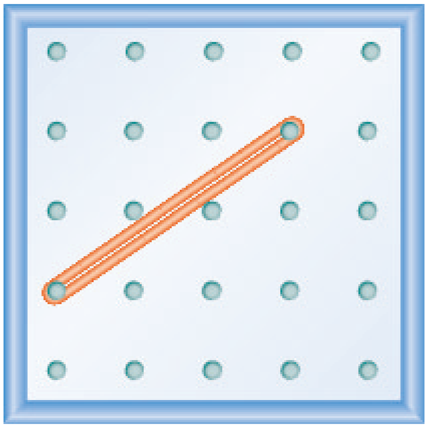


Figure .1

Doesn’t it look like a line?

Now we stretch one part of the rubber band straight up from the left peg and around a third peg to make the sides of a right triangle, as shown in (Figure 2)

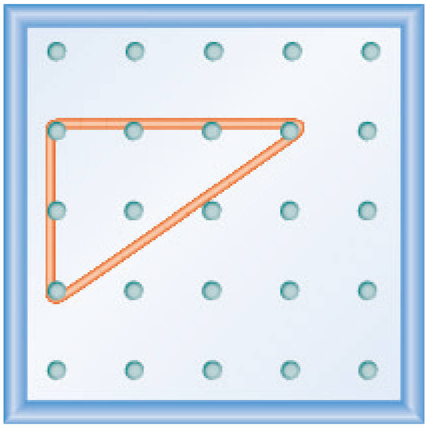


Figure .2

We carefully make a 90º angle around the third peg, so one of the newly formed lines is vertical and the other is horizontal.

To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the rise and the horizontal distance is called the run, as shown in (Figure 3).

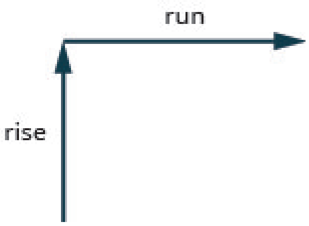


Figure .3

If our geoboard and rubber band look just like the one shown in (Figure 4), the rise is 2. The rubber band goes up 2 units. (Each space is one unit.)

The rise on this geoboard is 2, as the rubber band goes up two units.

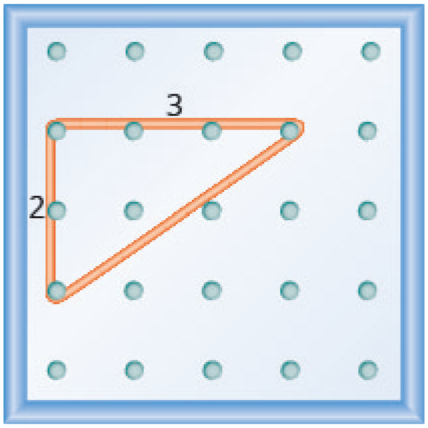


Figure .4

What is the run?

The rubber band goes across 3 units. The run is 3 (see (Figure 4)).

The slope of a line is the ratio of the rise to the run. In mathematics, it is always referred to with the letter m.

Slope of a line

The slope of a line of a line is $m = \frac{\text{rise}}{\text{run}}$.

The rise measures the vertical change and the run measures the horizontal change between two points on the line.

What is the slope of the line on the geoboard in (Figure 4)?

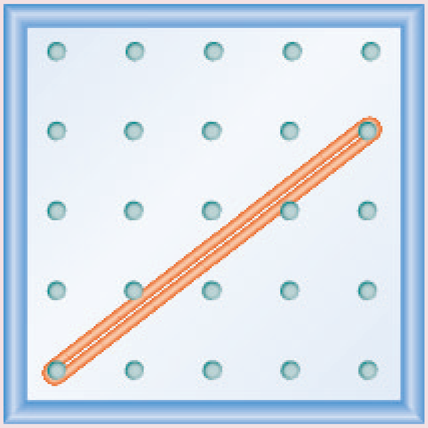
$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ m &= \frac{2}{3} \end{aligned}$$

The line has slope $\frac{2}{3}$. This means that the line rises 2 units for every 3 units of run.

When we work with geoboards, it is a good idea to get in the habit of starting at a peg on the left and connecting to a peg to the right. If the rise goes up it is positive and if it goes down it is negative. The run will go from left to right and be positive.

EXAMPLE 1

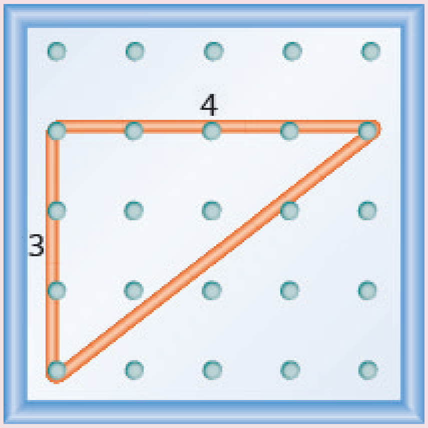
What is the slope of the line on the geoboard shown?



Solution

Use the definition of slope: $m = \dfrac{\text{rise}}{\text{run}}$.

Start at the left peg and count the spaces up and to the right to reach the second peg.

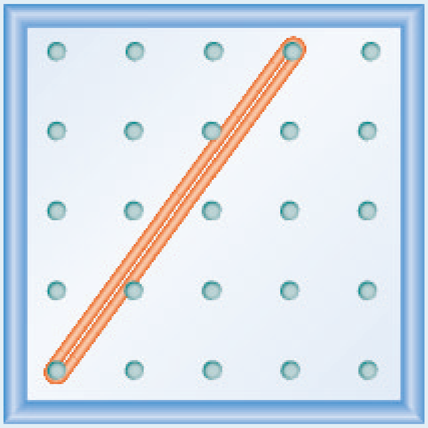


The rise is 3.	$m = \dfrac{3}{\text{run}}$
The run is 4.	$m = \dfrac{3}{4}$
	The slope is $\dfrac{3}{4}$.

This means that the line rises 3 units for every 4 units of run.

TRY IT 1.1

What is the slope of the line on the geoboard shown?

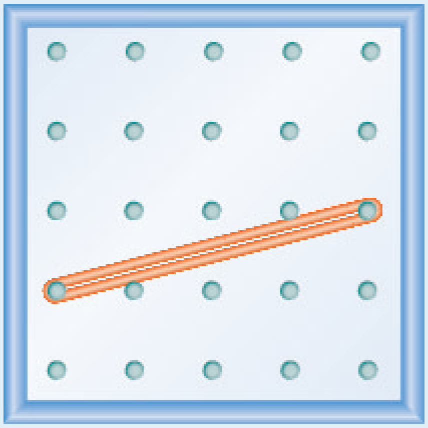


Show answer

$\frac{4}{3}$

TRY IT 1.2

What is the slope of the line on the geoboard shown?

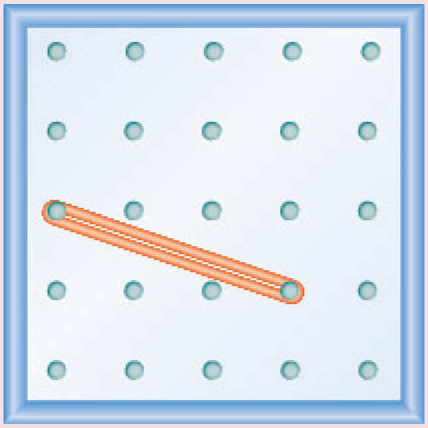


Show answer

$\frac{1}{4}$

EXAMPLE 2

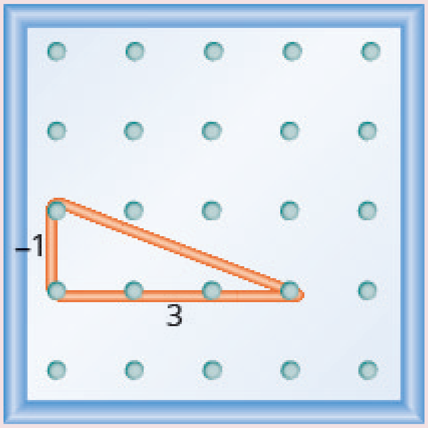
What is the slope of the line on the geoboard shown?



Solution

Use the definition of slope: $m = \frac{\text{rise}}{\text{run}}$.

Start at the left peg and count the units down and to the right to reach the second peg.

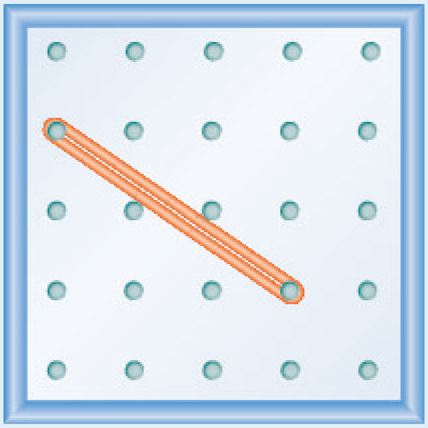


The rise is −1.	$= \frac{-1}{\text{run}}$
The run is 3.	$m = \frac{-1}{3}$ $m = -\frac{1}{3}$
	The slope is $-\frac{1}{3}$.

This means that the line drops 1 unit for every 3 units of run.

TRY IT 2.1

What is the slope of the line on the geoboard?

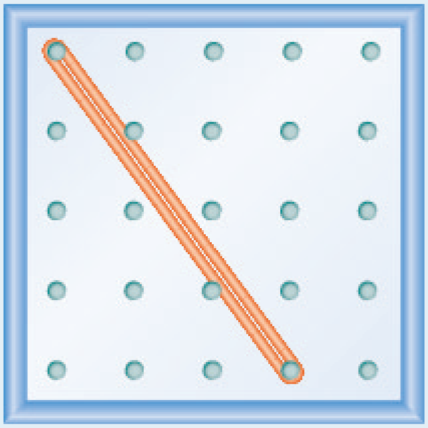


Show answer

$-\frac{2}{3}$

TRY IT 2.2

What is the slope of the line on the geoboard?



Show answer

$-\frac{4}{3}$

Notice that in (Example 1) the slope is positive and in (Example 2) the slope is negative. Do you notice any difference in the two lines shown in (Figure 5a) and (Figure 5b)?

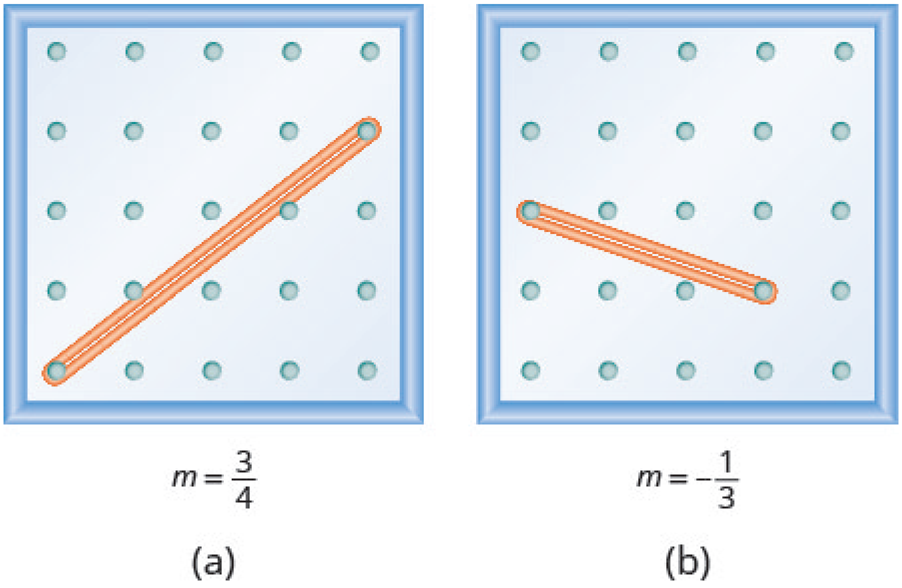
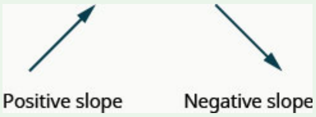


Figure .5 (a) (b)

Positive and negative slopes

We ‘read’ a line from left to right just like we read words in English. As you read from left to right, the line in (Figure 5a) is going up; it has positive slope. The line in (Figure 5b) is going down; it has negative slope.



EXAMPLE 3

Use a geoboard to model a line with slope $\frac{1}{2}$.

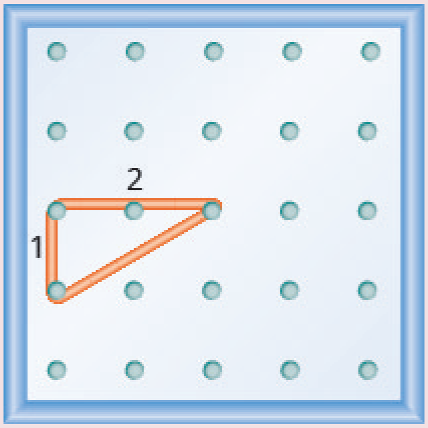
Solution
To model a line on a geoboard, we need the rise and the run.

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Replace m with $\frac{1}{2}$.	$\frac{1}{2} = \frac{\text{rise}}{\text{run}}$

So, the rise is 1 and the run is 2

Start at a peg in the lower left of the geoboard.

Stretch the rubber band up 1 unit, and then right 2 units.

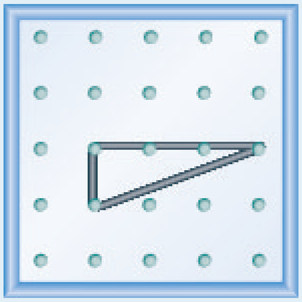


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $\frac{1}{2}$.

TRY IT 3.1

Model the slope $m = \frac{1}{3}$. Draw a picture to show your results.

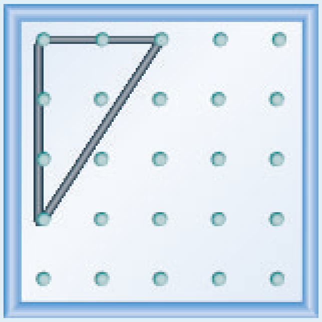
Show answer



TRY IT 3.2

Model the slope $m = \frac{3}{2}$. Draw a picture to show your results.

Show answer



EXAMPLE 4

Use a geoboard to model a line with slope $\frac{-1}{4}$.

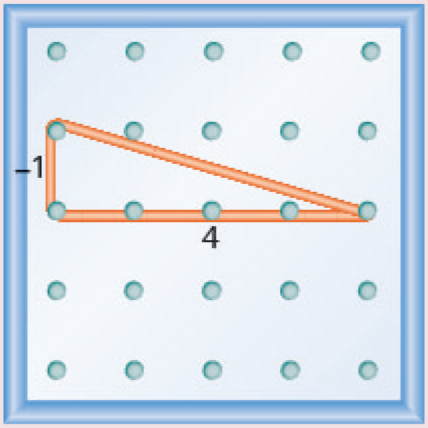
Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Replace m with $\frac{-1}{4}$.	$\frac{-1}{4} = \frac{\text{rise}}{\text{run}}$

So, the rise is -1 and the run is 4

Since the rise is negative, we choose a starting peg on the upper left that will give us room to count down.

We stretch the rubber band down 1 unit, then go to the right 4 units, as shown.

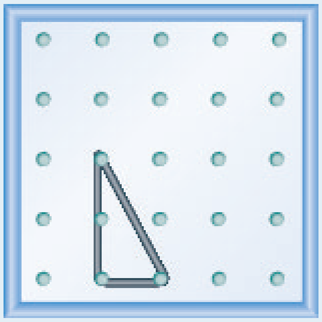


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $\frac{-1}{4}$.

TRY IT 4.1

Model the slope $m = \frac{-2}{3}$. Draw a picture to show your results.

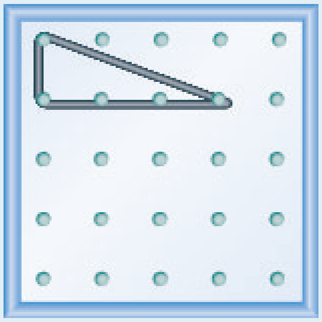
Show answer



TRY IT 4.2

Model the slope $m = \frac{-1}{3}$. Draw a picture to show your results.

Show answer



Use $m = \frac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph

Now, we'll look at some graphs on the xy-coordinate plane and see how to find their slopes. The method will be very similar to what we just modeled on our geoboards.

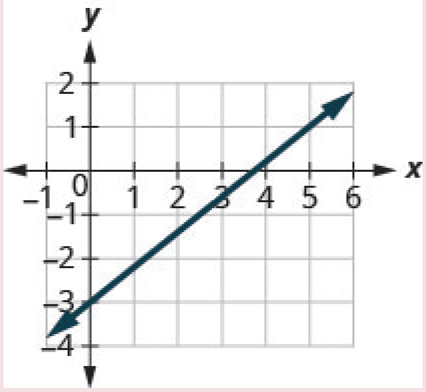
To find the slope, we must count out the rise and the run. But where do we start?

We locate two points on the line whose coordinates are integers. We then start with the point on the left and sketch a right triangle, so we can count the rise and run.

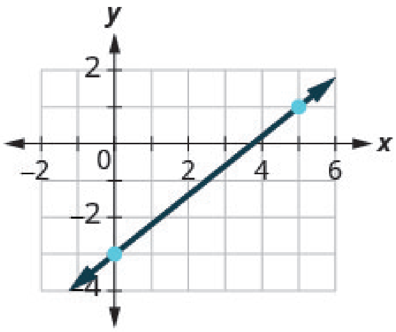
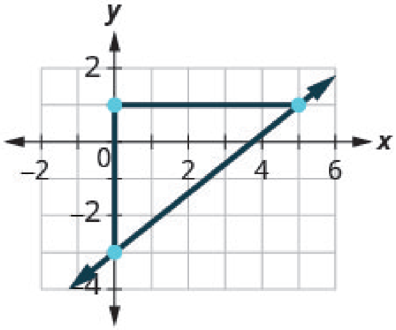
EXAMPLE 5

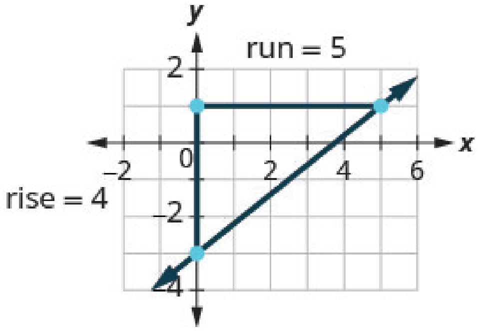
How to Use $m = \dfrac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph

Find the slope of the line shown.



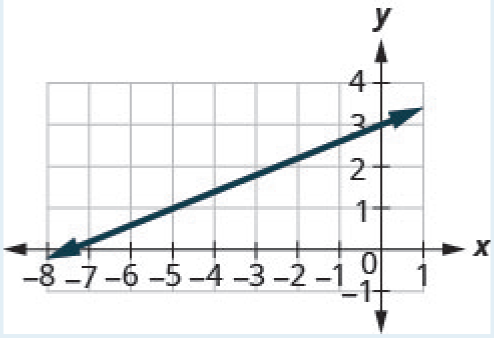
Solution

Step 1. Locate two points on the graph whose coordinates are integers.	Mark (0, –3) and (5, 1).	
Step 2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.	Starting at (0, –3), sketch a right triangle to (5, 1).	

Step 3. Count the rise and the run on the legs of the triangle.	Count the rise. Count the run.	 The rise is 4. The run is 5.
Step 4. Take the ratio of rise to run to find the slope. $m = \frac{\text{rise}}{\text{run}}$	Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
	Substitute the values of the rise and run.	$m = \frac{4}{5}$ The slope of the line is $\frac{4}{5}$. This means that y increases 4 units as x increases 5 units.

TRY IT 5.1

Find the slope of the line shown.

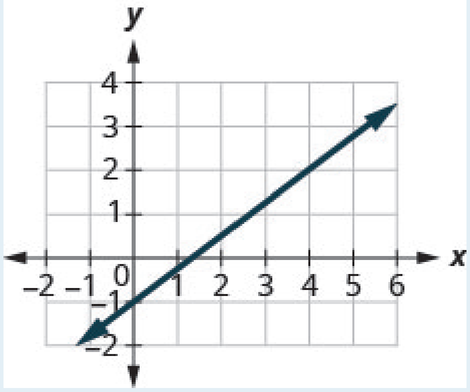


Show answer

$\frac{2}{5}$

TRY IT 5.2

Find the slope of the line shown.



Show answer

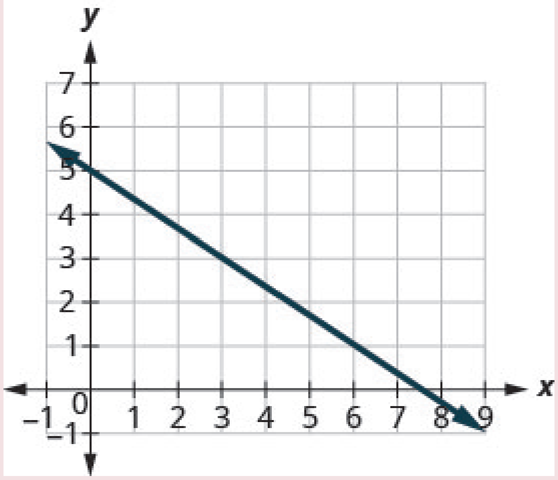
$\frac{3}{4}$

HOW TO: Find the slope of a line from its graph using $whitem = \frac{\text{rise}}{\text{run}}$.

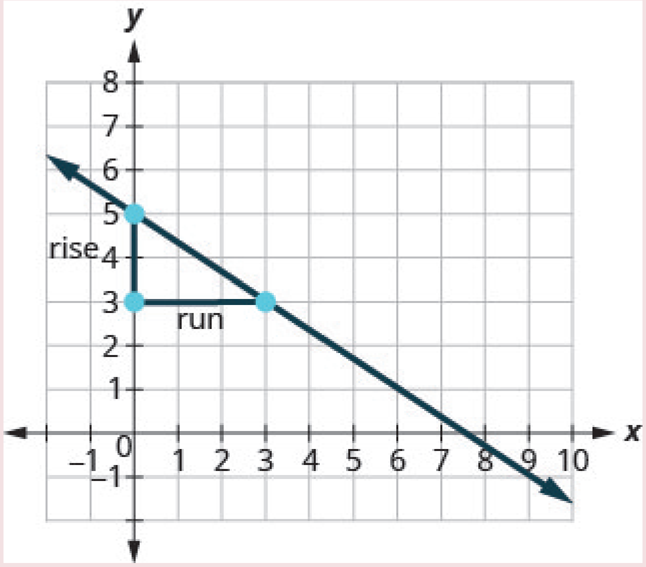
1. Locate two points on the line whose coordinates are integers.
2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope, $m = \frac{\text{rise}}{\text{run}}$.

EXAMPLE 6

Find the slope of the line shown.

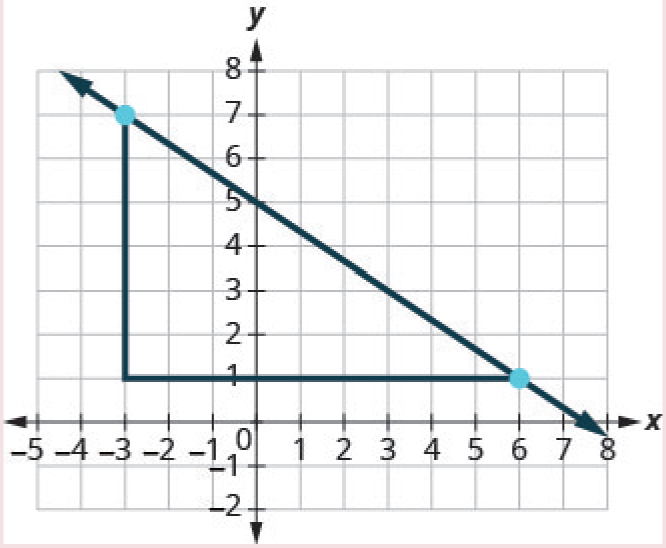


Solution

Locate two points on the graph whose coordinates are integers.	$(0,5)$ and $(3,3)$
Which point is on the left?	$(0,5)$
Starting at $(0,5)$, sketch a right triangle to $(3,3)$.	
Count the rise—it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m=\frac{-2}{3}$
Simplify.	$m=-\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.

So y increases by 3 units as x decreases by 2 units.

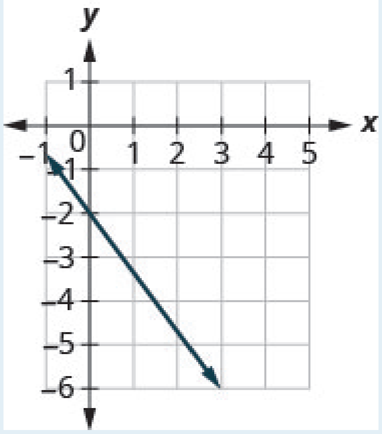
What if we used the points $(-3,7)$ and $(6,1)$ to find the slope of the line?



The rise would be -6 and the run would be 9. Then $m = \frac{-6}{9}$, and that simplifies to $m = -\frac{2}{3}$. Remember, it does not matter which points you use—the slope of the line is always the same.

TRY IT 6.1

Find the slope of the line shown.

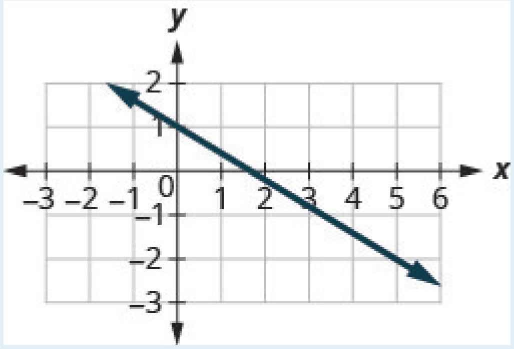


Show answer

$-\frac{4}{3}$

TRY IT 6.2

Find the slope of the line shown.



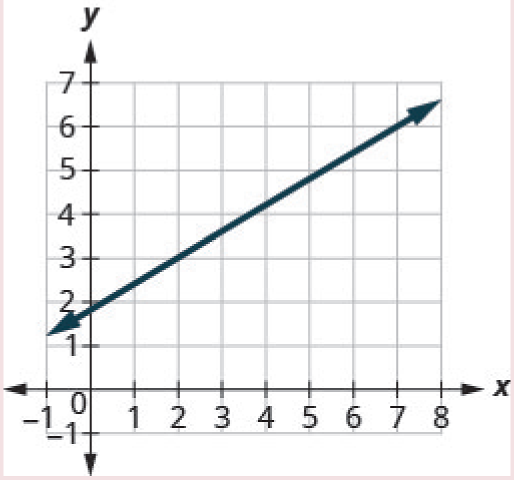
Show answer

$-\frac{3}{5}$

In the last two examples, the lines had *y*-intercepts with integer values, so it was convenient to use the *y*-intercept as one of the points to find the slope. In the next example, the *y*-intercept is a fraction. Instead of using that point, we'll look for two other points whose coordinates are integers. This will make the slope calculations easier.

EXAMPLE 7

Find the slope of the line shown.



Solution

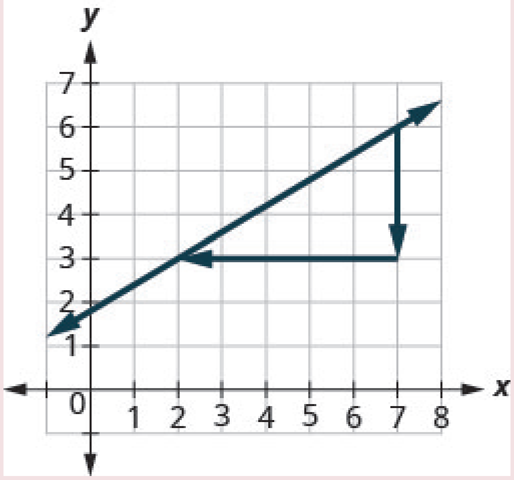
Locate two points on the graph whose coordinates are integers.	$(2, 3)$ and $(7, 6)$
Which point is on the left?	$(2, 3)$
Starting at $(2, 3)$, sketch a right triangle to $(7, 6)$.	
Count the rise.	The rise is 3.
Count the run.	The run is 5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{3}{5}$
	The slope of the line is $\frac{3}{5}$.

This means that y increases 5 units as x increases 3 units.

When we used geoboards to introduce the concept of slope, we said that we would always start with the point on the left and count the rise and the run to get to the point on the right. That way the run was always positive and the rise determined whether the slope was positive or negative.

What would happen if we started with the point on the right?

Let's use the points $(2, 3)$ and $(7, 6)$ again, but now we'll start at $(7, 6)$.

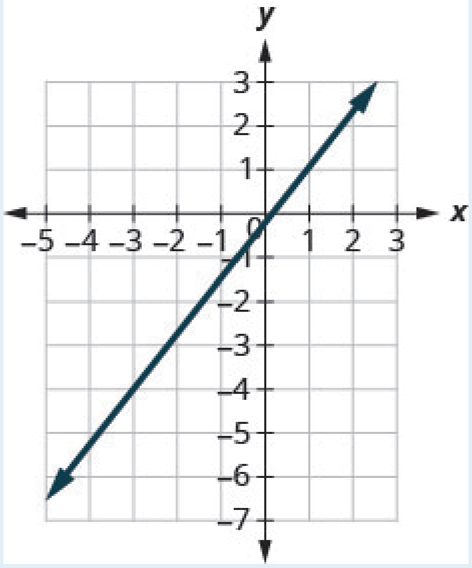


Count the rise.	The rise is -3.
Count the run. It goes from right to left, so it is negative.	The run is -5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-3}{-5}$
	The slope of the line is $\frac{-3}{-5}$.

It does not matter where you start—the slope of the line is always the same.

TRY IT 7.1

Find the slope of the line shown.

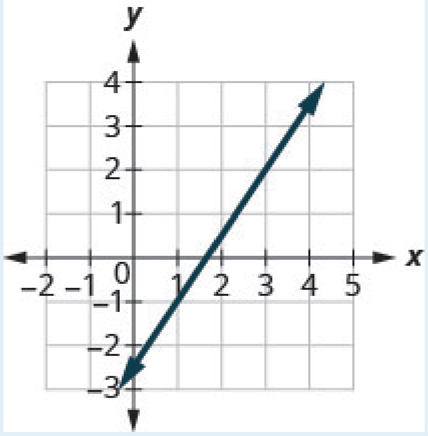


Show answer

$\frac{5}{4}$

EXAMPLE 7.2

Find the slope of the line shown.



Show answer

$\frac{3}{2}$

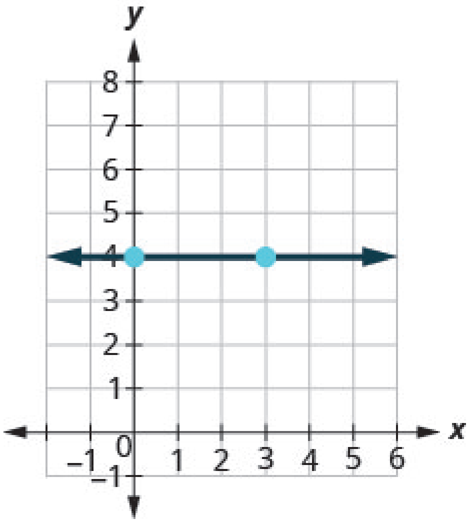
Find the Slope of Horizontal and Vertical Lines

Do you remember what was special about horizontal and vertical lines? Their equations had just one variable.

Horizontal line	$\mathbf{y = b}$	**Vertical line**	$\mathbf{x = a}$

y-coordinates are the same. x-coordinates are the same.

So how do we find the slope of the horizontal line $y = 4$? One approach would be to graph the horizontal line, find two points on it, and count the rise and the run. Let’s see what happens when we do this.



What is the rise?	The rise is 0.
Count the run.	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{0}{3}$ $m = 0$
	The slope of the horizontal line $y = 4$ is 0.

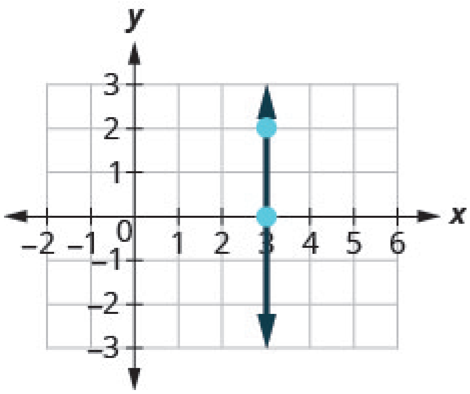
All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

Slope of a horizontal line

The slope of a horizontal line, $y = b$, is 0.

The floor of your room is horizontal. Its slope is 0. If you carefully placed a ball on the floor, it would not roll away.

Now, we'll consider a vertical line, the line.



What is the rise?	The rise is 2.
Count the run.	The run is 0.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{2}{0}$

But we can't divide by 0. Division by 0 is not defined. So we say that the slope of the vertical line $x = 3$ is undefined.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Slope of a vertical line

The slope of a vertical line, $x = a$, is undefined.

EXAMPLE 8

Find the slope of each line:

a) $x = 8$ b) $y = -5$.

Solution
a) $x = 8$

This is a vertical line.
Its slope is undefined.

b) $y = -5$
This is a horizontal line.
It has slope 0.

TRY IT 8.1

Find the slope of the line: $x = -4$.

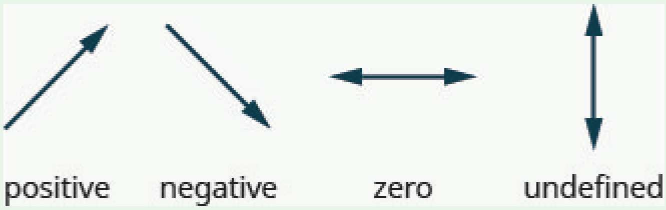
Show answer
undefined

TRY 8.2

Find the slope of the line: $y = 7$.

Show answer
0

Quick guide to the slopes of lines



Remember, we 'read' a line from left to right, just like we read written words in English.

Use the Slope Formula to find the Slope of a Line Between Two Points

Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

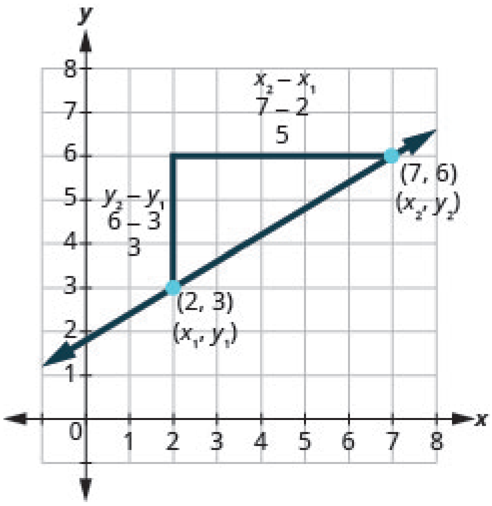
$$\begin{array}{ll} (x_1, y_1) & \text{read ' } x \text{ sub 1, } y \text{ sub 1'} \\ (x_2, y_2) & \text{read ' } x \text{ sub 2, } y \text{ sub 2'} \end{array}$$

The use of subscripts in math is very much like the use of last name initials in elementary school. Maybe you remember Laura C. and Laura M. in your third grade class?

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$.



Since we have two points, we will use subscript notation, $\begin{pmatrix} x_1, & y_1 \\ 2, & 3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ 7, & 6 \end{pmatrix}$.

On the graph, we counted the rise of 3 and the run of 5

Notice that the rise of 3 can be found by subtracting the *y*-coordinates 6 and 3

$$3 = 6 - 3$$

And the run of 5 can be found by subtracting the *x*-coordinates 7 and 2

$$5 = 7 - 2$$

We know $m = \dfrac{\text{rise}}{\text{run}}$. So $m = \dfrac{3}{5}$.

We rewrite the rise and run by putting in the coordinates $m = \dfrac{6 - 3}{7 - 2}$.

But 6 is y_2, the *y*-coordinate of the second point and 3 is y_1, the *y*-coordinate of the first point.

So we can rewrite the slope using subscript notation. $m = \dfrac{y_2 - y_1}{7 - 2}$

Also, 7 is x_2, the *x*-coordinate of the second point and 2 is x_1, the *x*-coordinate of the first point.

So, again, we rewrite the slope using subscript notation. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

We've shown that $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \dfrac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Slope formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the slope formula.

The slope is:

$$\begin{array}{c}\text{y of the second point minus y of the first point}\\ \text{over}\\ \text{x of the second point minus x of the first point.}\end{array}$$

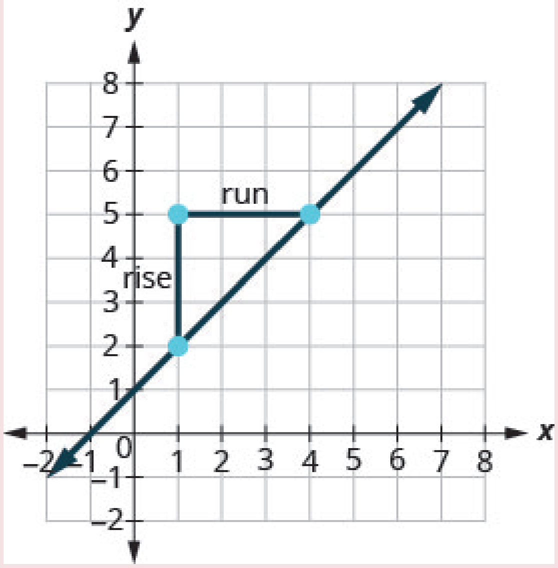
EXAMPLE 9

Use the slope formula to find the slope of the line between the points $(1, 2)$ and $(4, 5)$.

Solution

We'll call $(1, 2)$ point #1 and $(4, 5)$ point #2.	$\begin{pmatrix} x_1, & y_1 \\ 1, & 2 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ 4, & 5 \end{pmatrix}.$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}.$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{5 - 2}{x_2 - x_1}.$
x of the second point minus x of the first point	$m = \frac{5 - 2}{4 - 1}.$
Simplify the numerator and the denominator.	$m = \frac{3}{3}.$
Simplify.	$m = 1.$

Let's confirm this by counting out the slope on a graph using $m = \frac{\text{rise}}{\text{run}}$.



It doesn't matter which point you call point #1 and which one you call point #2. The slope will be the same. Try the calculation yourself.

TRY IT 9.1

Use the slope formula to find the slope of the line through the points: $(8, 5)$ and $(6, 3)$.

Show answer

1

TRY IT 9.2

Use the slope formula to find the slope of the line through the points: $(1, 5)$ and $(5, 9)$.

Show answer
1

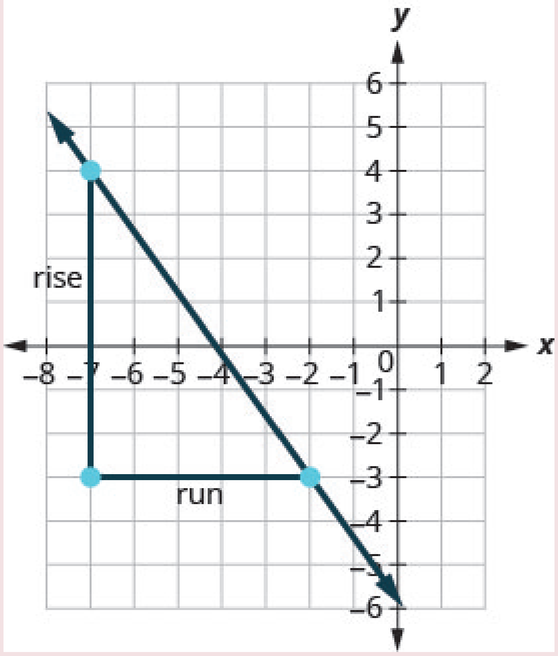
EXAMPLE 10

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

We'll call $(-2, -3)$ point #1 and $(-7, 4)$ point #2.	$\begin{pmatrix} x_1, & y_1 \\ -2, & -3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ -7, & 4 \end{pmatrix}.$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}.$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{4 - (-3)}{x_2 - x_1}.$
x of the second point minus x of the first point	$m = \frac{4 - (-3)}{-7 - (-2)}.$
Simplify.	$m = \frac{7}{-5}$ $m = -\frac{7}{5}$

Let's verify this slope on the graph shown.



$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ m &= \frac{-7}{5} \\ m &= -\frac{7}{5} \end{aligned}$$

TRY IT 10.1

Use the slope formula to find the slope of the line through the points: $(-3, 4)$ and $(2, -1)$.

Show answer
-1

TRY IT 10.2

Use the slope formula to find the slope of the line through the pair of points: $(-2, 6)$ and $(-3, -4)$.

Show answer
10

Graph a Line Given a Point and the Slope

Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

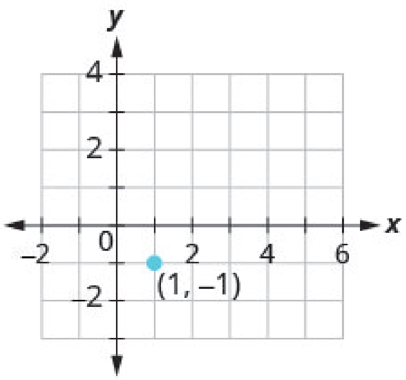
One other method we can use to graph lines is called the point–slope method. We will use this method when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

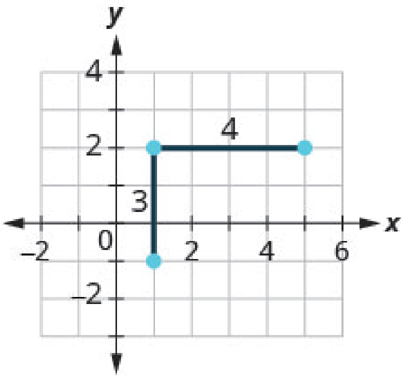
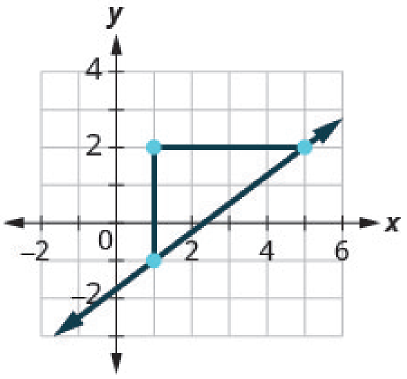
EXAMPLE 11

How To Graph a Line Given a Point and The Slope

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

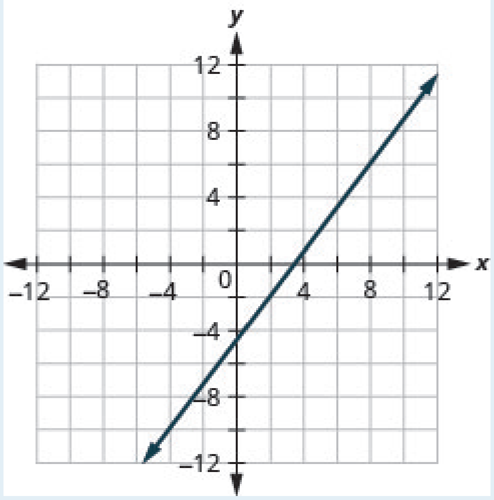
Step 1. Plot the given point.	Plot (1, –1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ rise = 3 run = 4

Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

EXAMPLE 11.1

Graph the line passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.

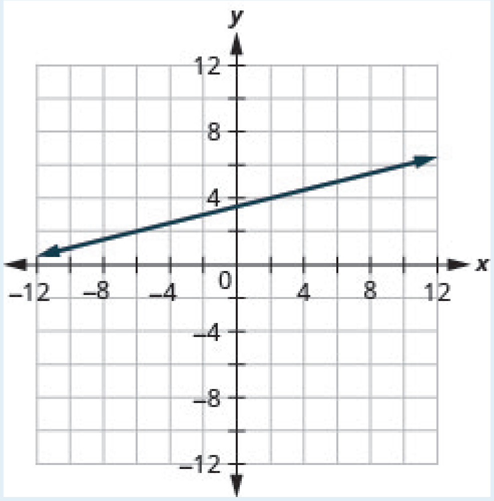
Show answer



TRY IT 11.2

Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Show answer



Graph a line given a point and the slope.

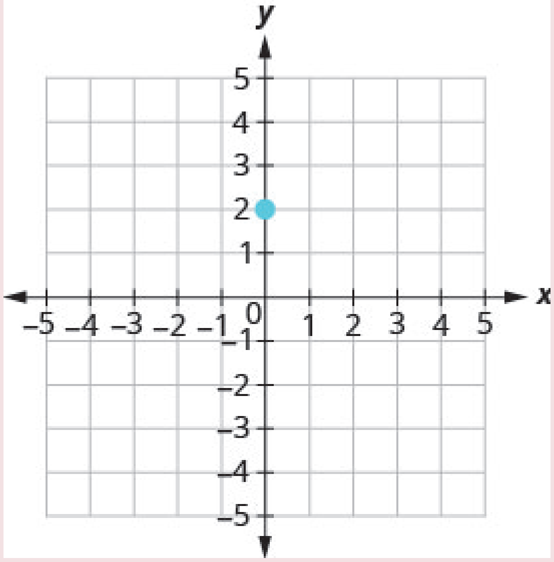
1. Plot the given point.
2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.

3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

EXAMPLE 12

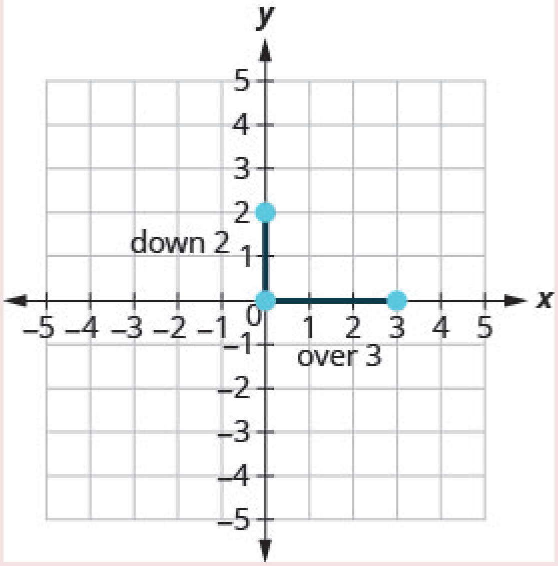
Graph the line with *y*-intercept 2 whose slope is $m = -\dfrac{2}{3}$.

Solution
Plot the given point, the *y*-intercept, $(0, 2)$.

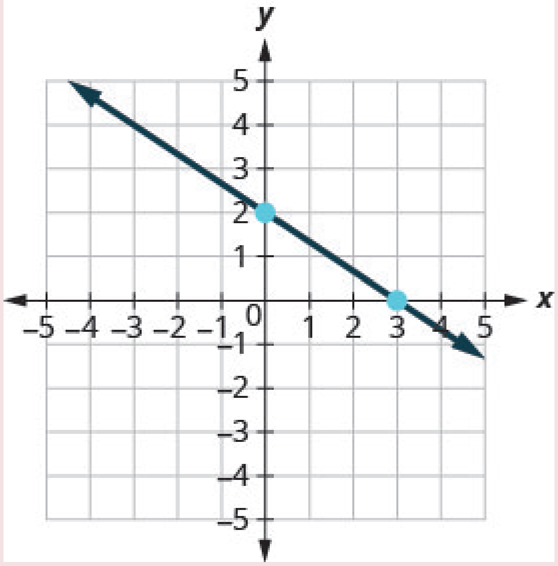


Identify the rise and the run.	$m = -\dfrac{2}{3}$
	$\dfrac{\text{rise}}{\text{run}} = \dfrac{-2}{3}$
	$\text{rise} = -2$
	$\text{run} = 3$

Count the rise and the run. Mark the second point.



Connect the two points with a line.

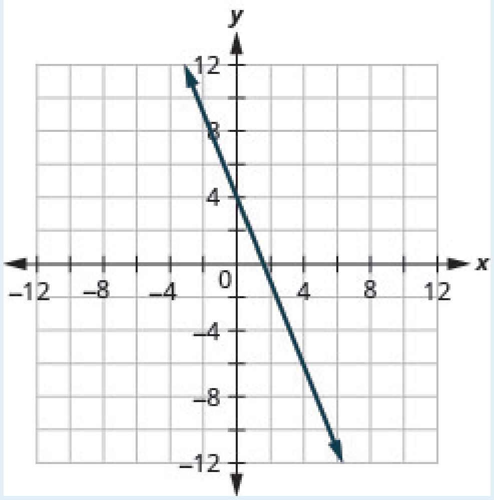


You can check your work by finding a third point. Since the slope is $m = -\frac{2}{3}$, it can be written as $m = \frac{2}{-3}$. Go back to $(0, 2)$ and count out the rise, 2, and the run, -3.

TRY IT 12.1

Graph the line with the *y*-intercept 4 and slope $m = -\frac{5}{2}$.

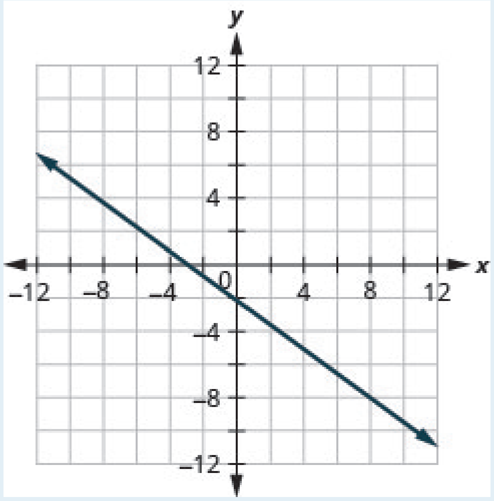
Show answer



TRY IT 12.2

Graph the line with the *x*-intercept -3 and slope $m=-\frac{3}{4}$.

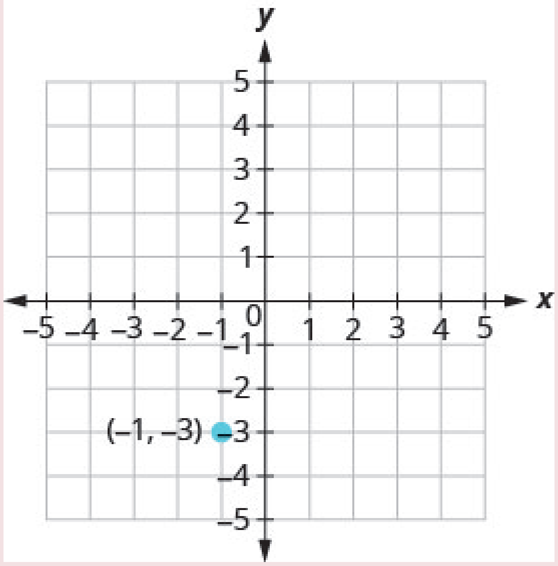
Show answer



EXAMPLE 13

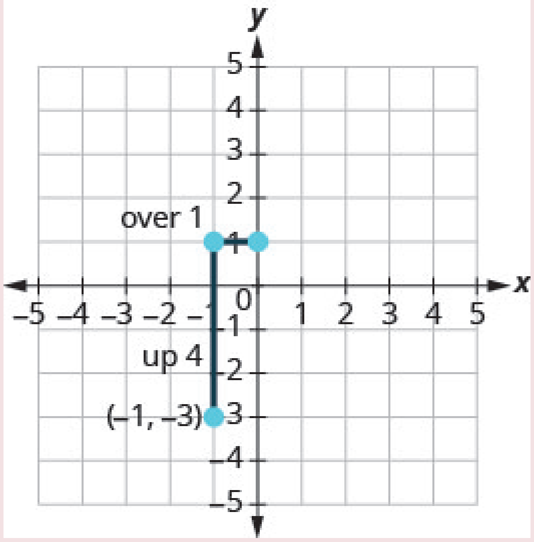
Graph the line passing through the point $(-1,-3)$ whose slope is $m=4$.

Solution
Plot the given point.

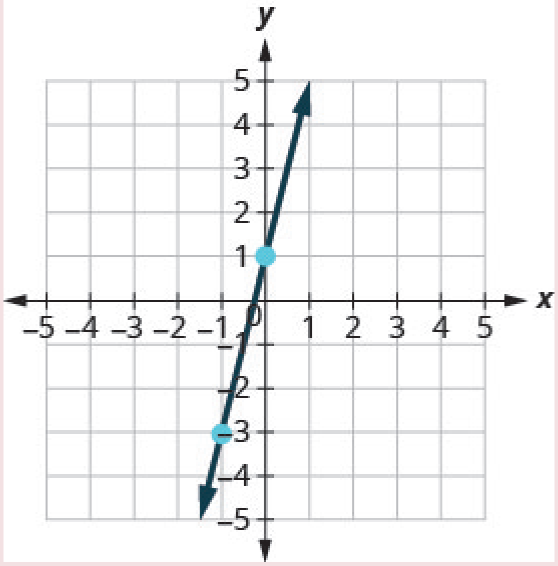


Identify the rise and the run.	$m = 4$
Write 4 as a fraction.	$\frac{\text{rise}}{\text{run}} = \frac{4}{1}$
	$\text{rise} = 4, \text{run} = 1$

Count the rise and run and mark the second point.



Connect the two points with a line.

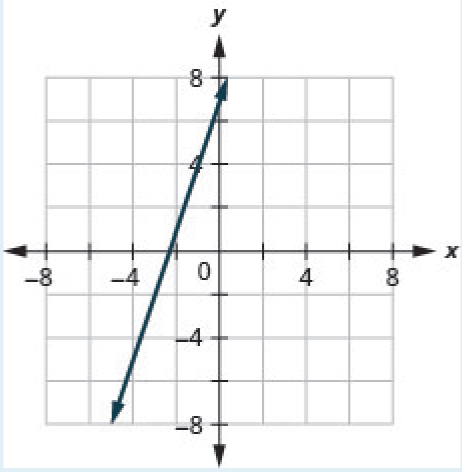


You can check your work by finding a third point. Since the slope is $m = 4$, it can be written as $m = \frac{-4}{-1}$. Go back to $(-1, -3)$ and count out the rise, -4, and the run, -1.

TRY IT 13.1

Graph the line with the point $(-2, 1)$ and slope $m = 3$.

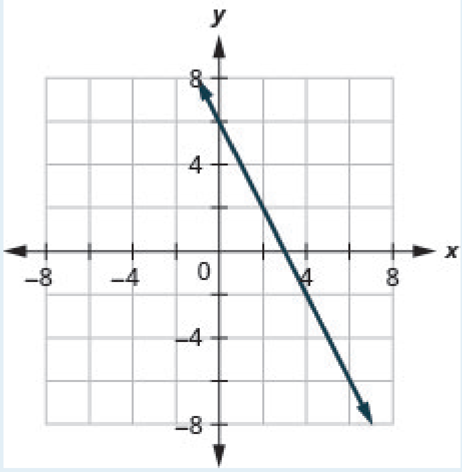
Show answer



EXAMPLE 13.2

Graph the line with the point $(4, -2)$ and slope $m = -2$.

Show answer

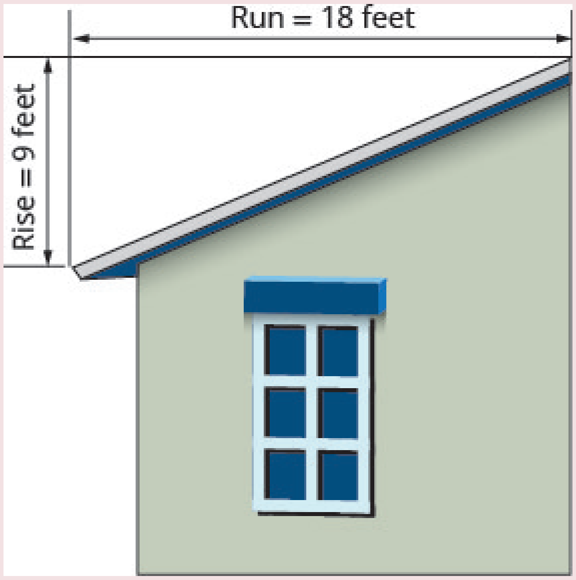


Solve Slope Applications

At the beginning of this section, we said there are many applications of slope in the real world. Let's look at a few now.

EXAMPLE 14

The 'pitch' of a building's roof is the slope of the roof. Knowing the pitch is important in climates where there is heavy snowfall. If the roof is too flat, the weight of the snow may cause it to collapse. What is the slope of the roof shown?



Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values for rise and run.	$m = \frac{9}{18}$
Simplify.	$m = \frac{1}{2}$
The slope of the roof is $\frac{1}{2}$.	
	The roof rises 1 foot for every 2 feet of horizontal run.

TRY IT 14.1

Use (Example 14), substituting the rise = 14 and run = 24

Show answer

$$\frac{7}{12}$$

TRY IT 14.2

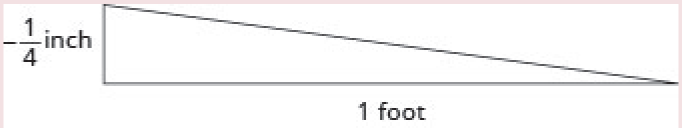
Use (Example 14), substituting rise = 15 and run = 36

Show answer

$$\frac{5}{12}$$

EXAMPLE 15

Have you ever thought about the sewage pipes going from your house to the street? They must slope down $\frac{1}{4}$ inch per foot in order to drain properly. What is the required slope?



Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{-\frac{1}{4}\text{inch}}{1\ \text{foot}}$ $m = \frac{-\frac{1}{4}\text{inch}}{12\ \text{inches}}$
Simplify.	$m = -\frac{1}{48}$
	The slope of the pipe is $-\frac{1}{48}$.

The pipe drops 1 inch for every 48 inches of horizontal run.

TRY IT 15.1

Find the slope of a pipe that slopes down $\frac{1}{3}$ inch per foot.

Show answer

$-\frac{1}{36}$

TRY IT 15.2

Find the slope of a pipe that slopes down $\frac{3}{4}$ inch per yard.

Show answer

$-\frac{1}{48}$

Access these online resources for additional instruction and practice with understanding slope of a line.

- Practice Slope with a Virtual Geoboard
- Explore Area and Perimeter with a Geoboard

Key Concepts

- **Find the Slope of a Line from its Graph using** $m = \frac{\text{rise}}{\text{run}}$
    1. Locate two points on the line whose coordinates are integers.
    2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
    3. Count the rise and the run on the legs of the triangle.
    4. Take the ratio of rise to run to find the slope.
- **Graph a Line Given a Point and the Slope**
    1. Plot the given point.
    2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
    3. Starting at the given point, count out the rise and run to mark the second point.
    4. Connect the points with a line.
- **Slope of a Horizontal Line**
    - The slope of a horizontal line, $y = b$, is 0.
- **Slope of a vertical line**

- The slope of a vertical line, $x = a$, is undefined

Glossary

geoboard
A geoboard is a board with a grid of pegs on it.

negative slope
A negative slope of a line goes down as you read from left to right.

positive slope
A positive slope of a line goes up as you read from left to right.

rise
The rise of a line is its vertical change.

run
The run of a line is its horizontal change.

slope formula
The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

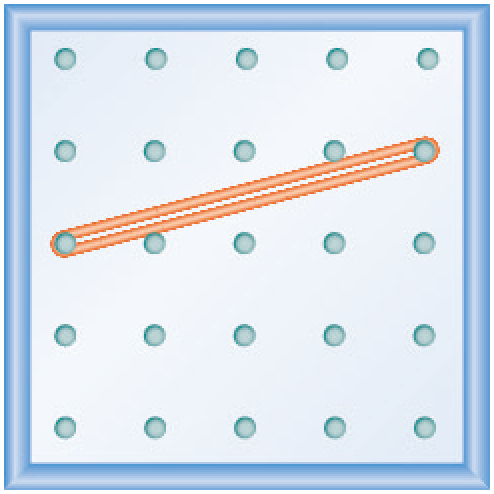
slope of a line
The slope of a line is $m = \dfrac{\text{rise}}{\text{run}}$. The rise measures the vertical change and the run measures the horizontal change.

Practice Makes Perfect

Use Geoboards to Model Slope

In the following exercises, find the slope modeled on each geoboard.

1.



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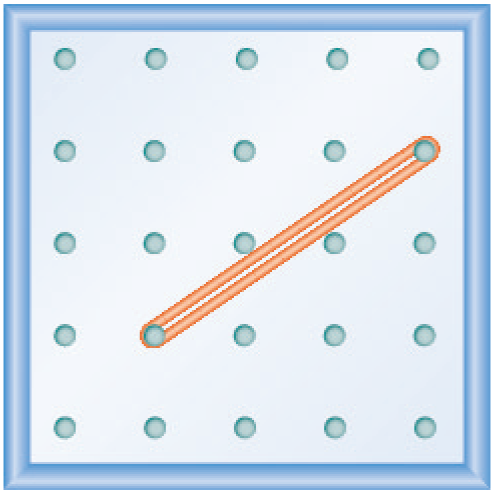
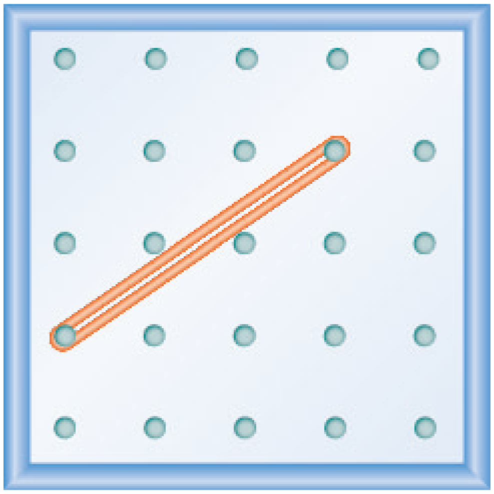


3.



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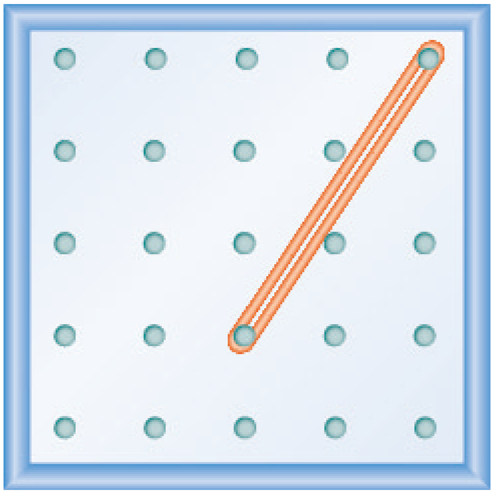
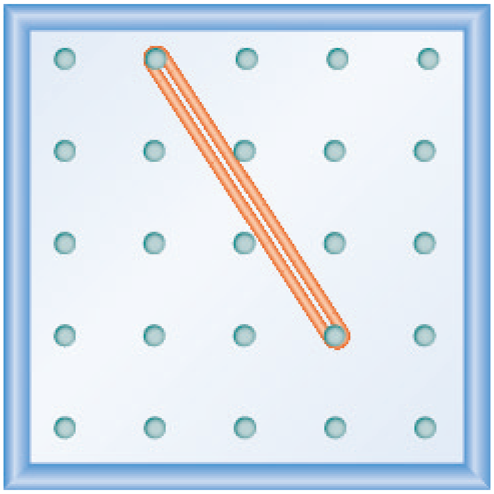


5.



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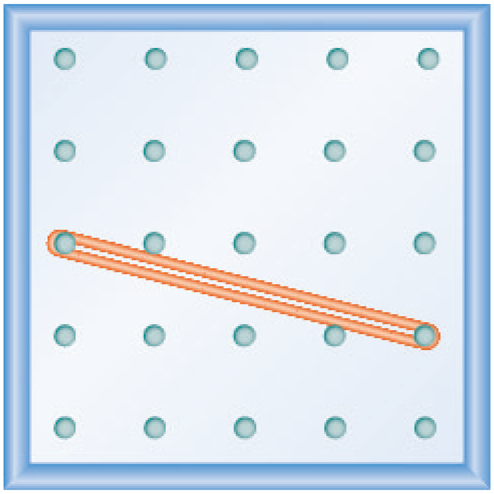
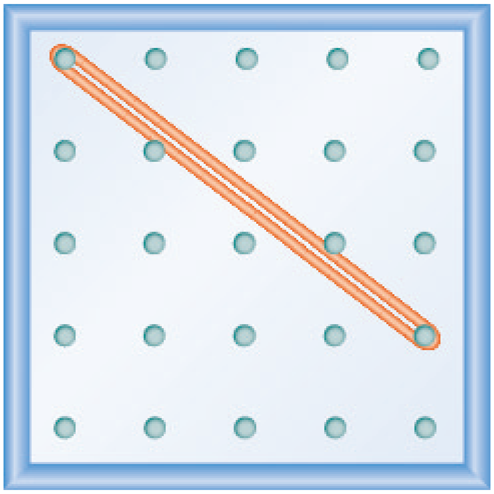
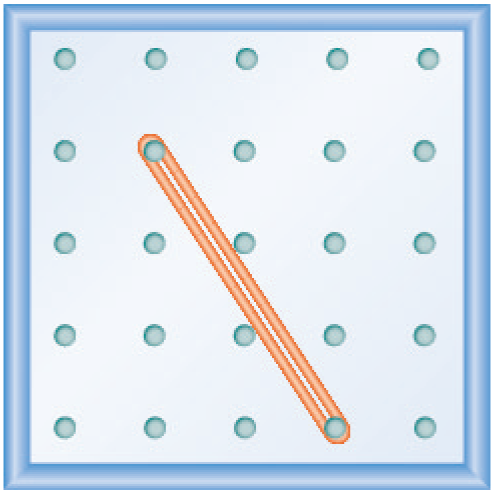


7. 	8. 

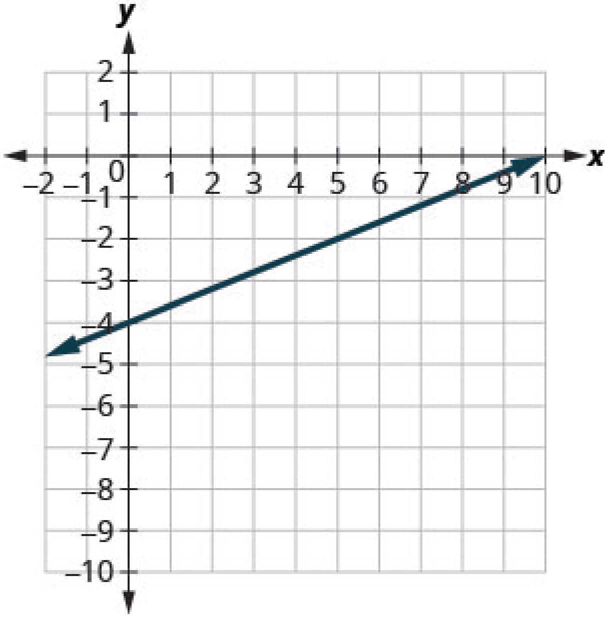
In the following exercises, model each slope. Draw a picture to show your results.

9. $\frac{2}{3}$	10. $\frac{3}{4}$
11. $\frac{1}{4}$	12. $\frac{4}{3}$
13. $-\frac{1}{2}$	14. $-\frac{3}{4}$
15. $-\frac{2}{3}$	16. $-\frac{3}{2}$

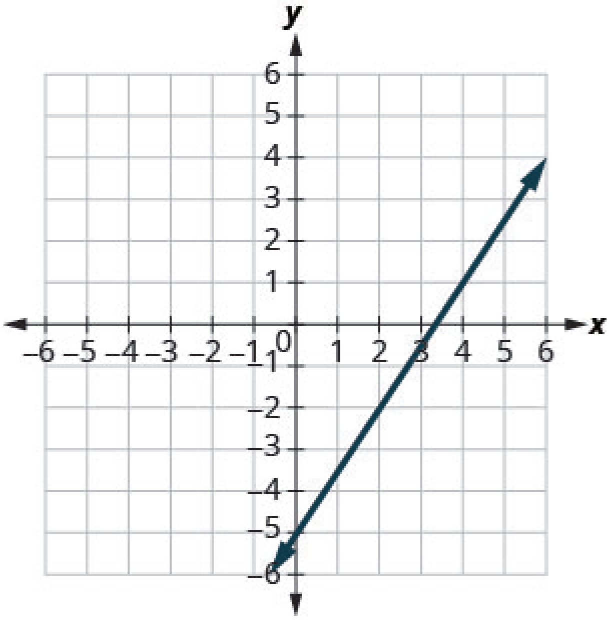
Use $m = \frac{\text{rise}}{\text{run}}$ to find the Slope of a Line from its Graph

In the following exercises, find the slope of each line shown.

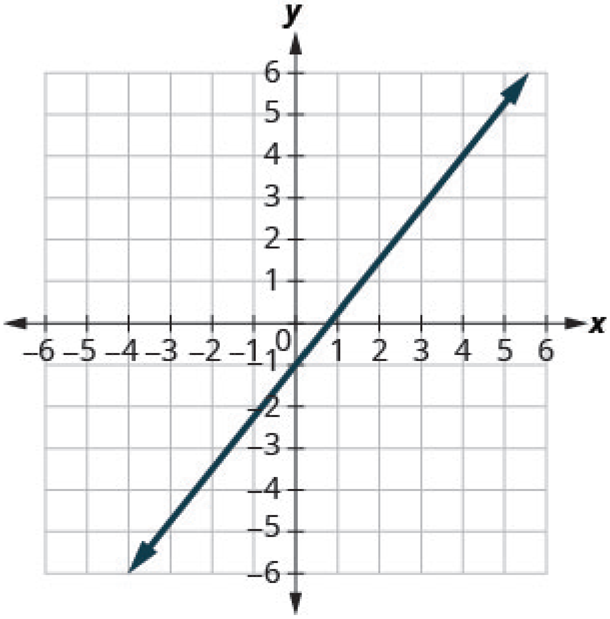
17.


y
x
2
1
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10
−2 −1
1 2 3 4 5 6 7 8 9 10

18.


y
x
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6

19.


y
x
6
5
4
3
2
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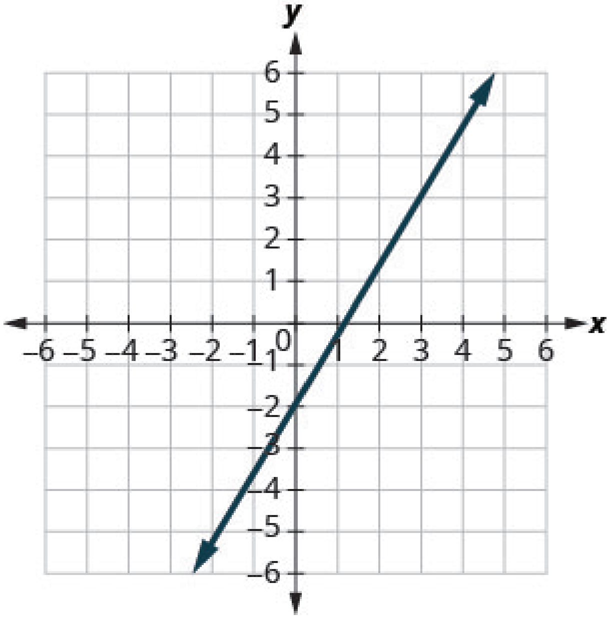
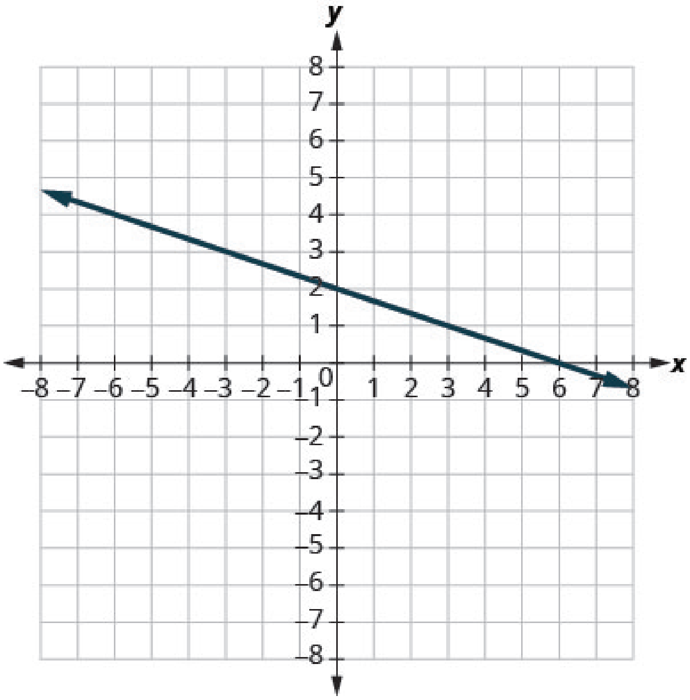
−1
−2
−3
−4
−5
−6
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6

20.


y
x
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6

21.


y
x
0
−8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8
8 7 6 5 4 3 2 1 −1 −2 −3 −4 −5 −6 −7 −8

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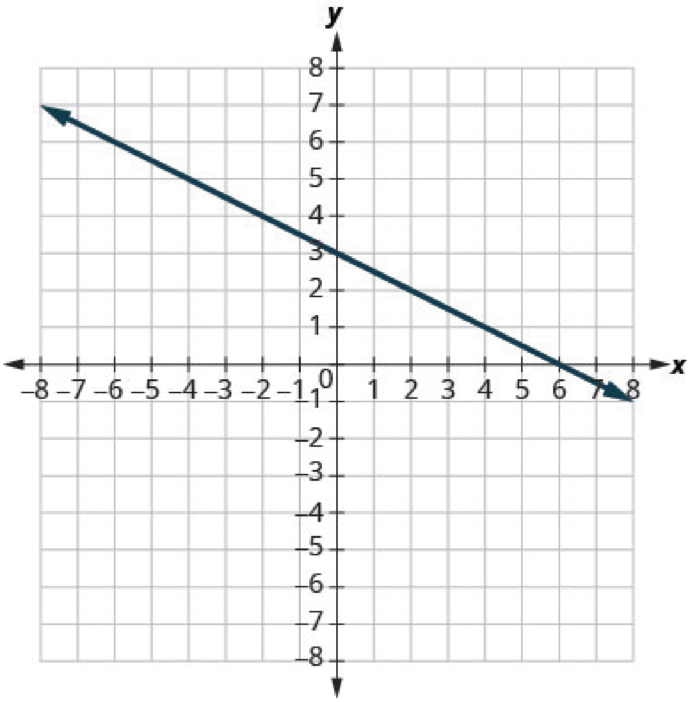
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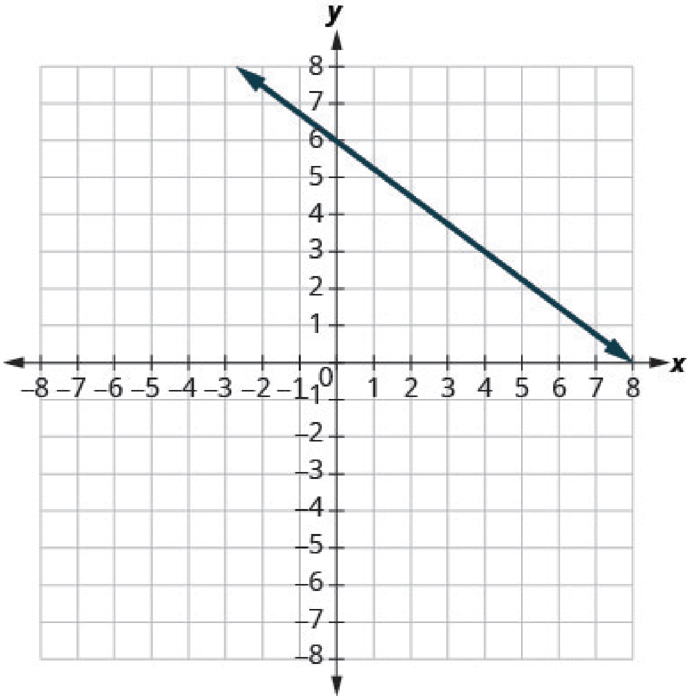
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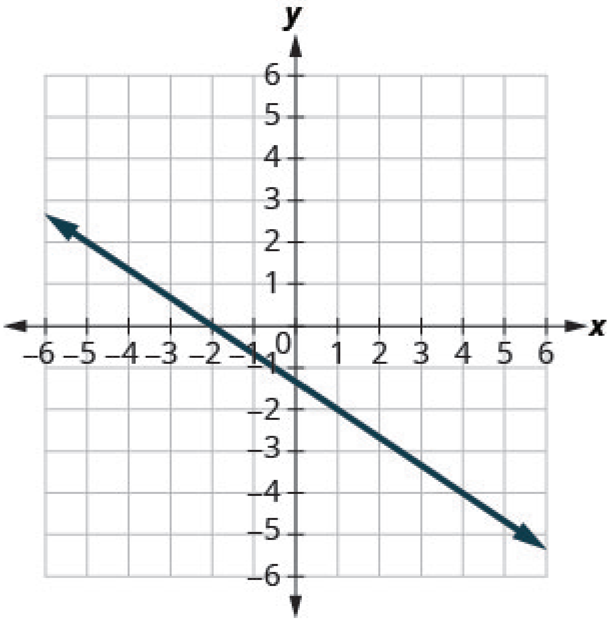
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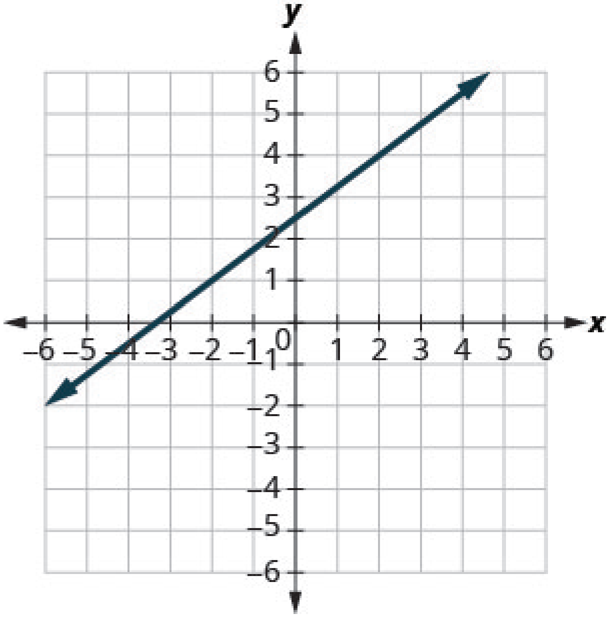
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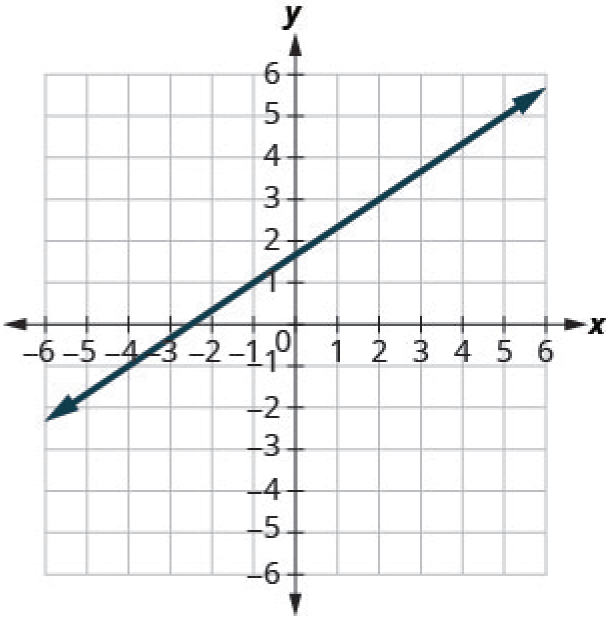
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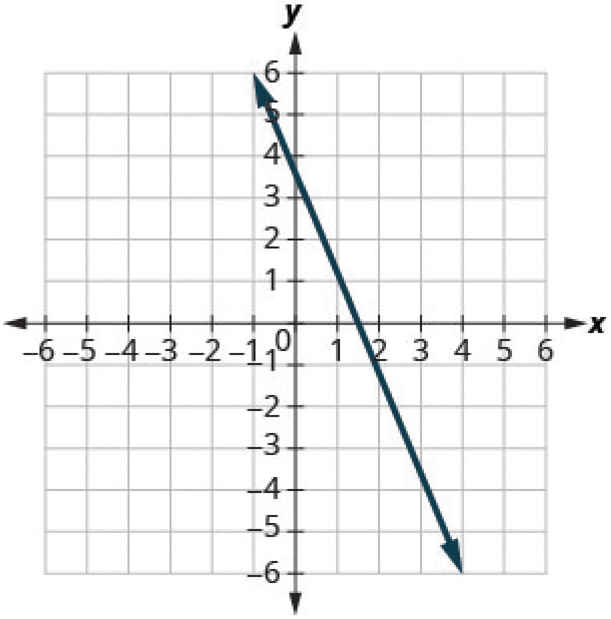
27.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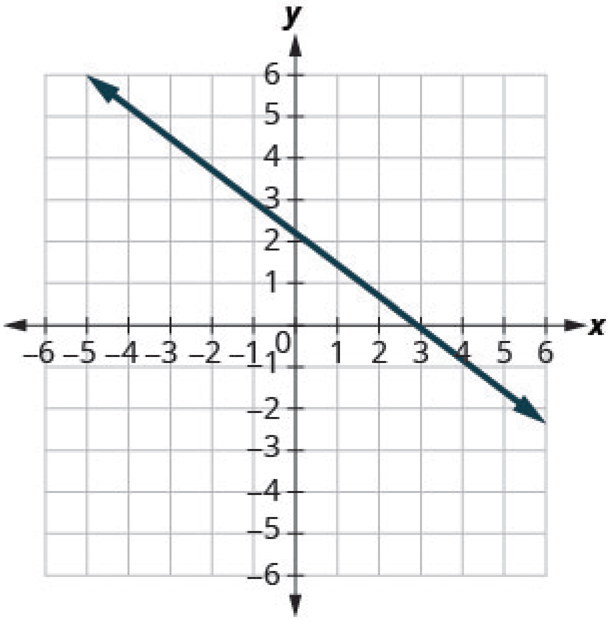
28.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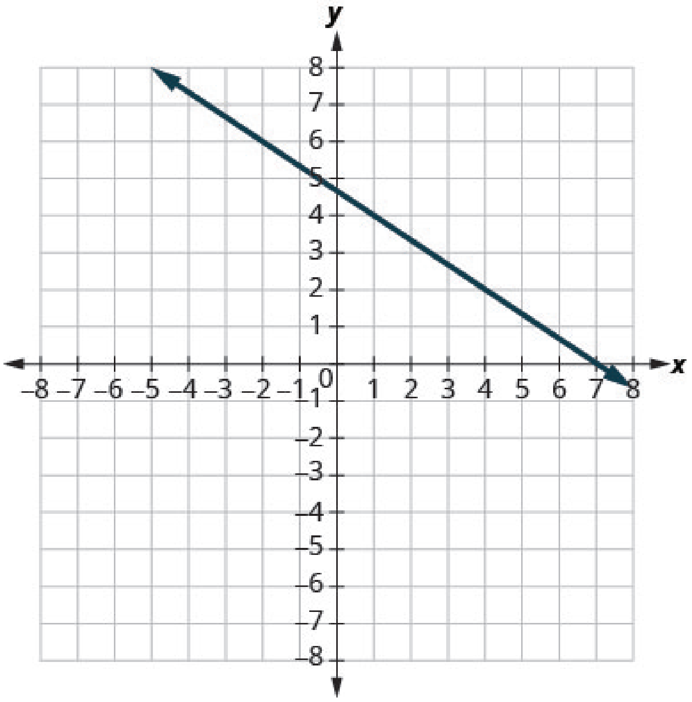

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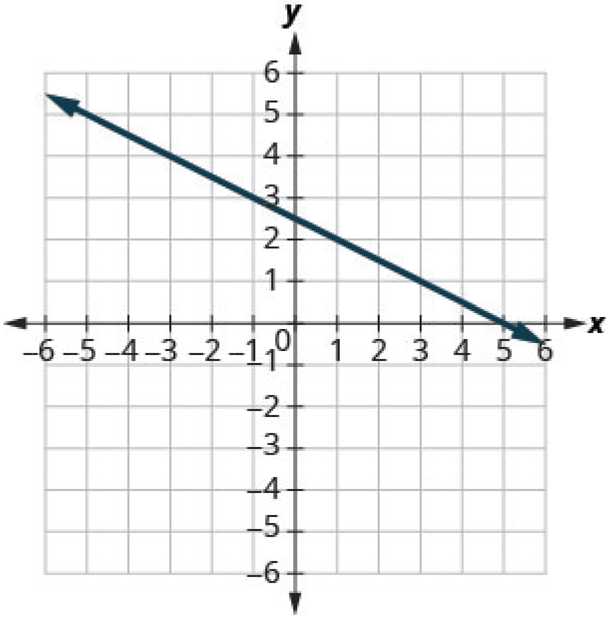

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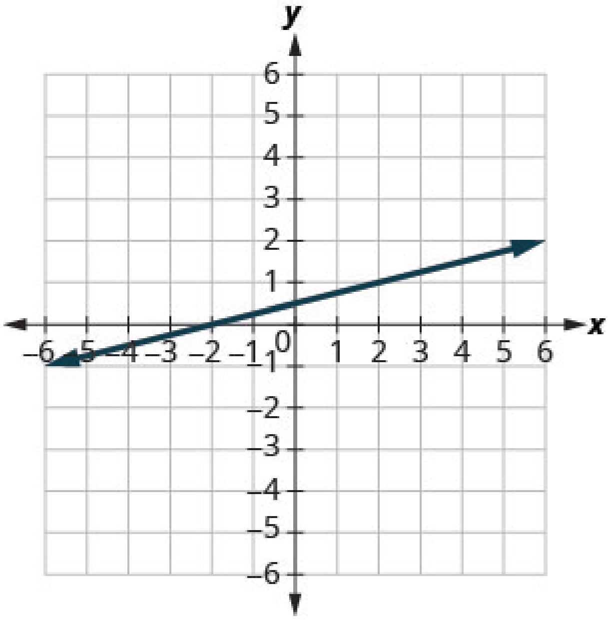

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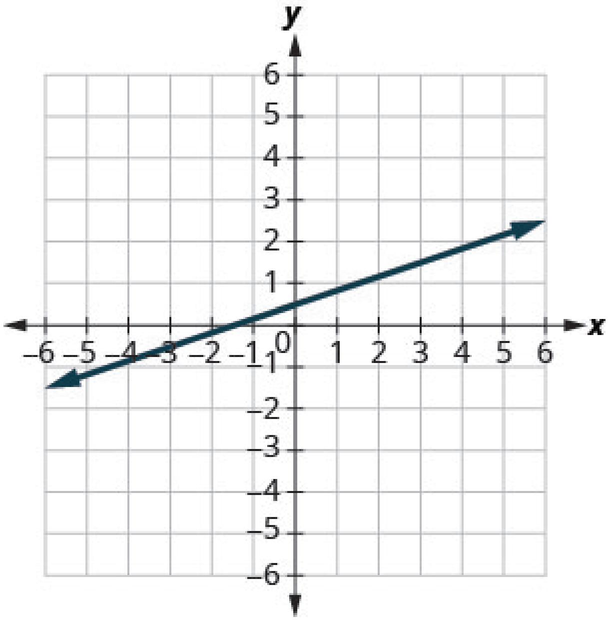


Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

33. $y=3$	34. $y=1$
35. $x=4$	36. $x=2$
37. $y=-2$	38. $y=-3$
39. $x=-5$	40. $x=-4$

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

41. $(1,4)\,,(3,9)$	42. $(2,3)\,,(5,7)$
43. $(0,3)\,,(4,6)$	44. $(0,1)\,,(5,4)$
45. $(2,5)\,,(4,0)$	46. $(3,6)\,,(8,0)$
47. $(-3,3)\,,(4,-5)$	48. $(-2,4)\,,(3,-1)$
49. $(-1,-2)\,,(2,5)$	50. $(-2,-1)\,,(6,5)$
51. $(4,-5)\,,(1,-2)$	52. $(3,-6)\,,(2,-2)$

Graph a Line Given a Point and the Slope

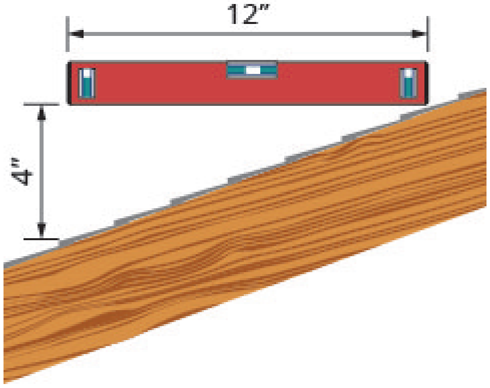
In the following exercises, graph each line with the given point and slope.

53. $(1,-2); m=\frac{3}{4}$	54. $(1,-1); m=\frac{2}{3}$
55. $(2,5); m=-\frac{1}{3}$	56. $(1,4); m=-\frac{1}{2}$
57. $(-3,4); m=-\frac{3}{2}$	58. $(-2,5); m=-\frac{5}{4}$
59. $(-1,-4); m=\frac{4}{3}$	60.$(-3,-5); m=\frac{3}{2}$
61. *y*-intercept 3; $m=-\frac{2}{5}$	62. *y*-intercept 5; $m=-\frac{4}{3}$
63. *x*-intercept -2; $m=\frac{3}{4}$	64. *x*-intercept -1; $m=\frac{1}{5}$
65. $(-3,3); m=2$	66. $(-4,2); m=4$
67. $(1,5); m=-3$	67. $(1,5); m=-3$

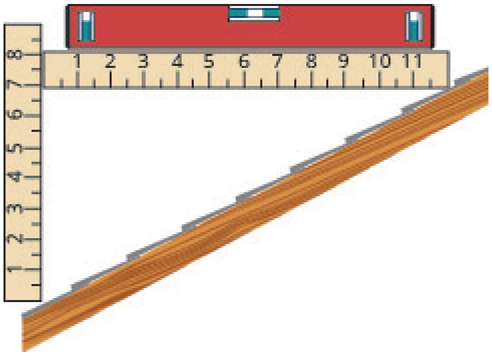
Everyday Math

69. **Slope of a roof.** An easy way to determine the slope of a roof is to set one end of a 12 inch level on the roof surface and hold it level. Then take a tape measure or ruler and measure from the other end of the level down to the roof surface. This will give you the slope of the roof. Builders, sometimes, refer to this as pitch and state it as an “x 12 pitch” meaning $\frac{x}{12}$, where x is the measurement from the roof to the level—the rise. It is also sometimes stated as an “x-in-12 pitch”.

1. a) What is the slope of the roof in this picture?
2. b) What is the pitch in construction terms?



70. The slope of the roof shown here is measured with a 12” level and a ruler. What is the slope of this roof?



71. **Road grade.** A local road has a grade of 6%. The grade of a road is its slope expressed as a percent. Find the slope of the road as a fraction and then simplify. What rise and run would reflect this slope or grade?

72. **Highway grade.** A local road rises 2 feet for every 50 feet of highway.

a) What is the slope of the highway?
b) The grade of a highway is its slope expressed as a percent. What is the grade of this highway?

73. **Wheelchair ramp.** The rules for wheelchair ramps require a maximum 1-inch rise for a 12-inch run.

a) How long must the ramp be to accommodate a 24-inch rise to the door?
b) Create a model of this ramp.

74. **Wheelchair ramp.** A 1-inch rise for a 16-inch run makes it easier for the wheelchair rider to ascend a ramp.

a) How long must a ramp be to easily accommodate a 24-inch rise to the door?
b) Create a model of this ramp.

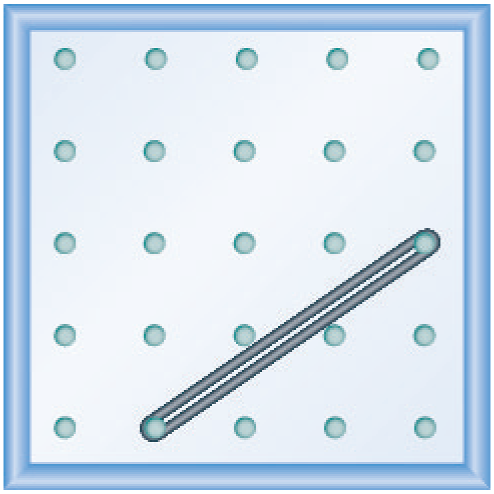
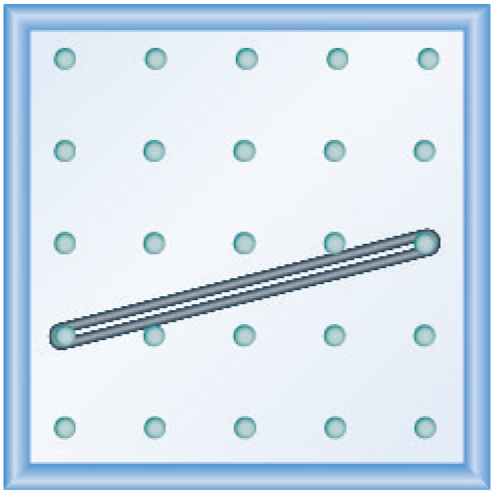
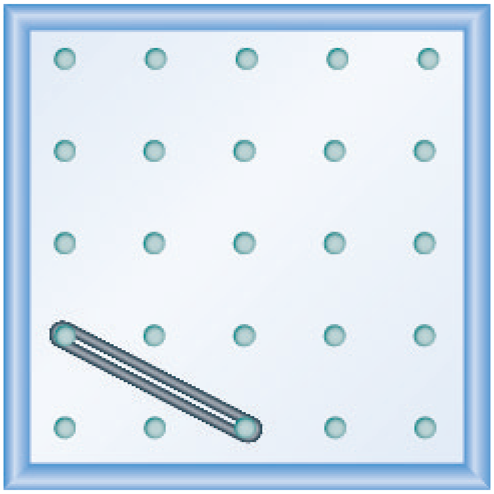
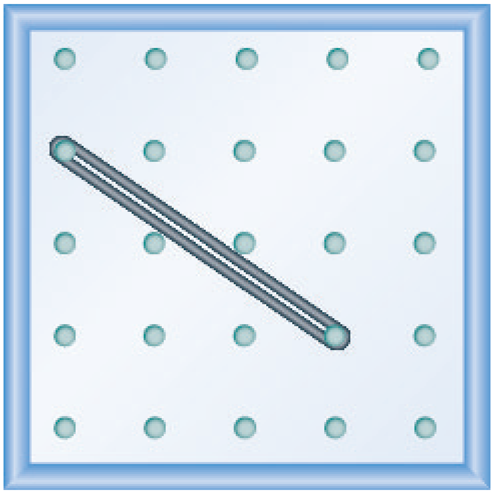
Writing Exercises

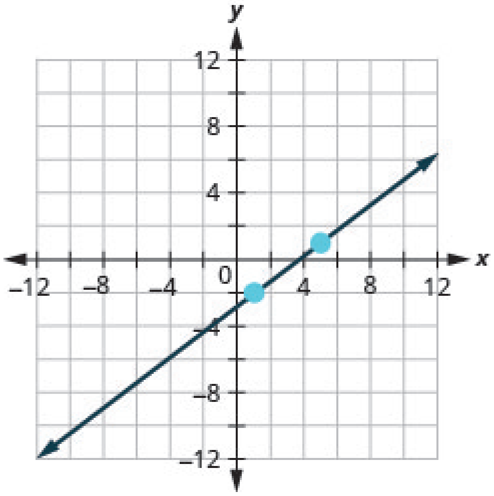
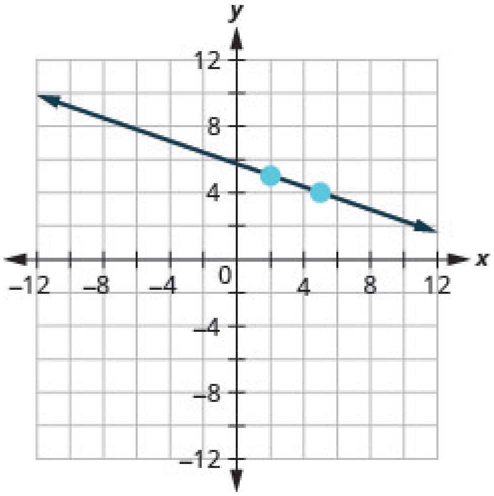
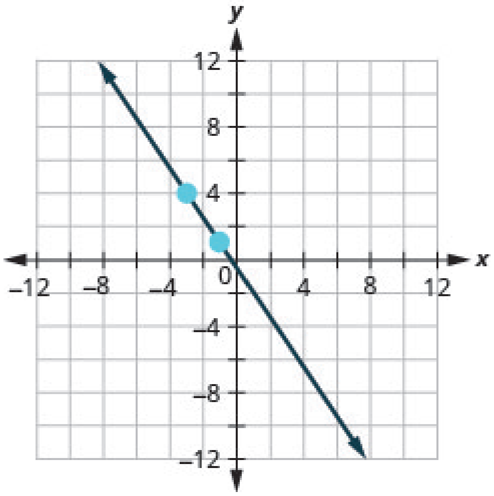
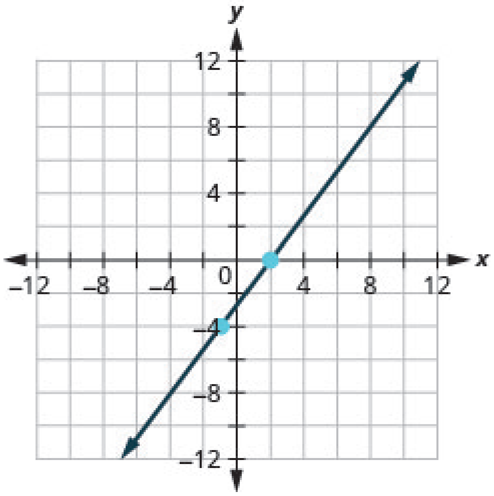
75. What does the sign of the slope tell you about a line?

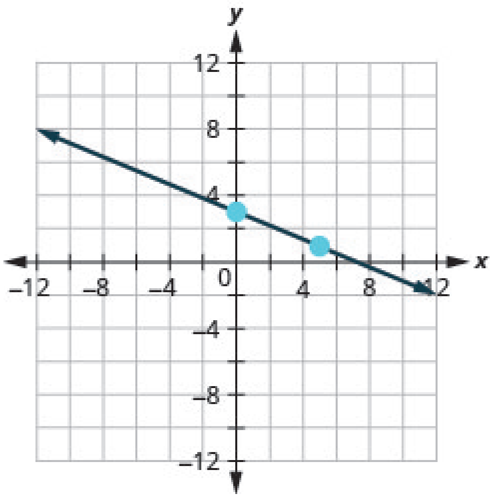
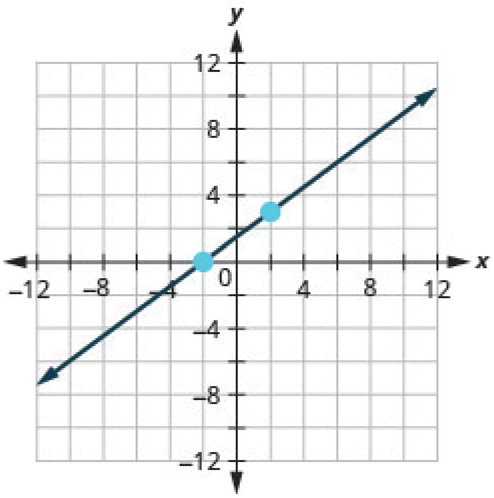
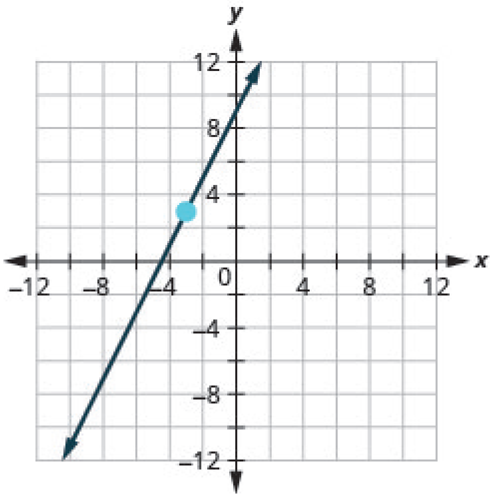
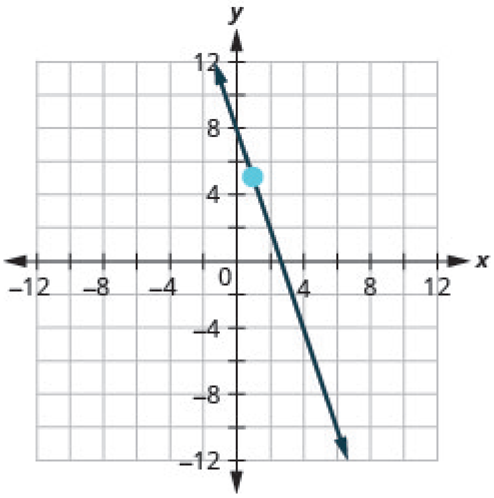
76. How does the graph of a line with slope $m = \frac{1}{2}$ differ from the graph of a line with slope $m = 2$?

77. Why is the slope of a vertical line “undefined”?

Answers

1. $\frac{1}{4}$	3. $\frac{2}{3}$
5. $\frac{-3}{2} = -\frac{3}{2}$	7. $-\frac{3}{4}$
9. 	11. 
13. 	15. 
17. $\frac{2}{5}$	19. $\frac{5}{4}$
21. $-\frac{1}{3}$	23. $-\frac{3}{4}$
25. $\frac{3}{4}$	27. $-\frac{5}{2}$
29. $-\frac{2}{3}$	31. $\frac{1}{4}$

33. 0	35. undefined
37. 0	39. undefined
41. $\frac{5}{2}$	43. $\frac{3}{4}$
45. $-\frac{5}{2}$	47. $-\frac{8}{7}$
49. $\frac{7}{3}$	51. -1
53.  	55. 
57.  	59. 

61. 	63. 
65. 	67. 
69. a) $\frac{1}{3}$ b) 4 12 pitch or 4-in-12 pitch	71. $\frac{3}{50}$; rise = 3, run = 50
73. a) 288 inches (24 feet) b) Models will vary.	75. When the slope is a positive number the line goes up from left to right. When the slope is a negative number the line goes down from left to right.
77. A vertical line has 0 run and since division by 0 is undefined the slope is undefined.	

Attributions

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6.5 Use the Slope–Intercept Form of an Equation of a Line

Learning Objectives

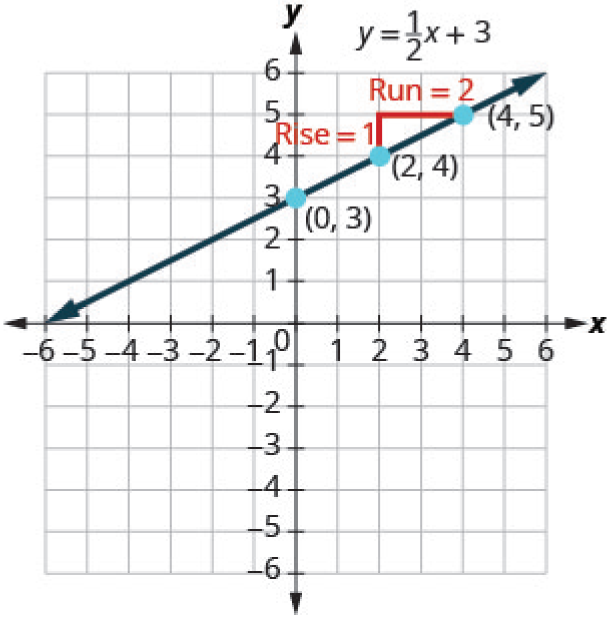
By the end of this section, you will be able to:

- Recognize the relation between the graph and the slope–intercept form of an equation of a line
- Identify the slope and y-intercept form of an equation of a line
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel lines
- Use slopes to identify perpendicular lines

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using the point–slope method. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

In Graph Linear Equations in Two Variables, we graphed the line of the equation $y = \frac{1}{2}x + 3$ by plotting points. See (Figure). Let's find the slope of this line.



The red lines show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$
\begin{aligned}
m &= \frac{\text{rise}}{\text{run}} \\
m &= \frac{1}{2}
\end{aligned}
$$

What is the *y*-intercept of the line? The *y*-intercept is where the line crosses the *y*-axis, so *y*-intercept is $(0, 3)$. The equation of this line is:

$$y = \frac{1}{2}x + 3$$

Notice, the line has:

slope $m = \frac{1}{2}$ and *y*-intercept (0, 3)

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form.

$m = \frac{1}{2}$; *y*-intercept is (0, 3)

$$y = \frac{1}{2}x + 3$$

$$y = mx + b$$

Slope-intercept form of an equation of a line

The slope–intercept form of an equation of a line with slope m and *y*-intercept, $(0, b)$ is,

$y = mx + b$

Sometimes the slope–intercept form is called the “*y*-form.”

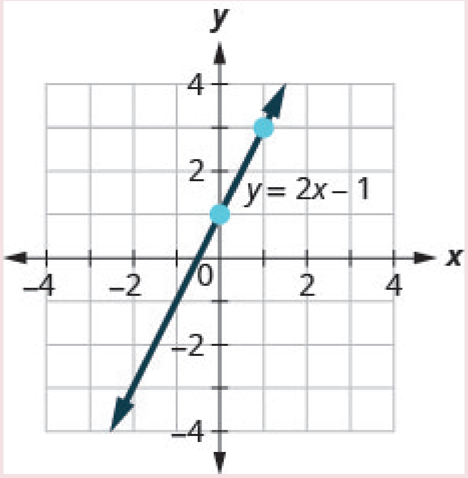
EXAMPLE 1

Use the graph to find the slope and *y*-intercept of the line, $y = 2x + 1$.

Compare these values to the equation$y = mx + b$.

Solution

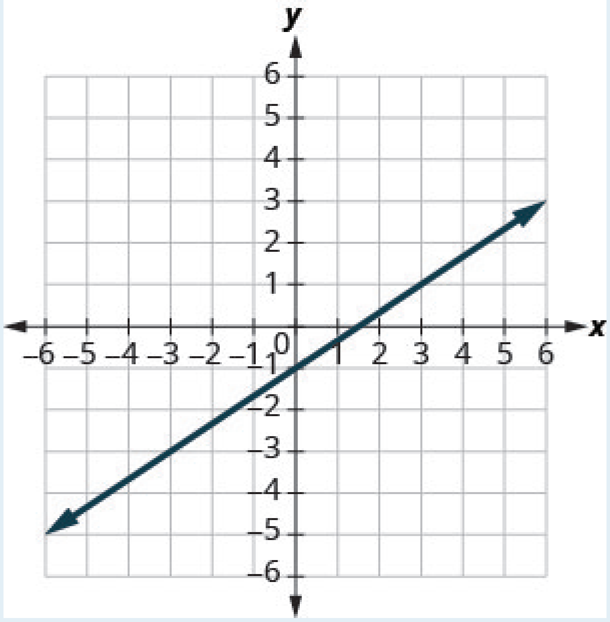
To find the slope of the line, we need to choose two points on the line. We’ll use the points $(0, 1)$ and $(1, 3)$.

	
Find the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{1}$
	$m = 2$
Find the *y*-intercept of the line.	The *y*-intercept is the point (0, 1).
We found slope $m = 2$ and *y*-intercept (0, 1).	$y = 2x + 1$ $y = mx + b$

The slope is the same as the coefficient of x and the *y*-coordinate of the *y*-intercept is the same as the constant term.

TRY IT 1.1

Use the graph to find the slope and *y*-intercept of the line $y = \frac{2}{3}x - 1$. Compare these values to the equation $y = mx + b$.

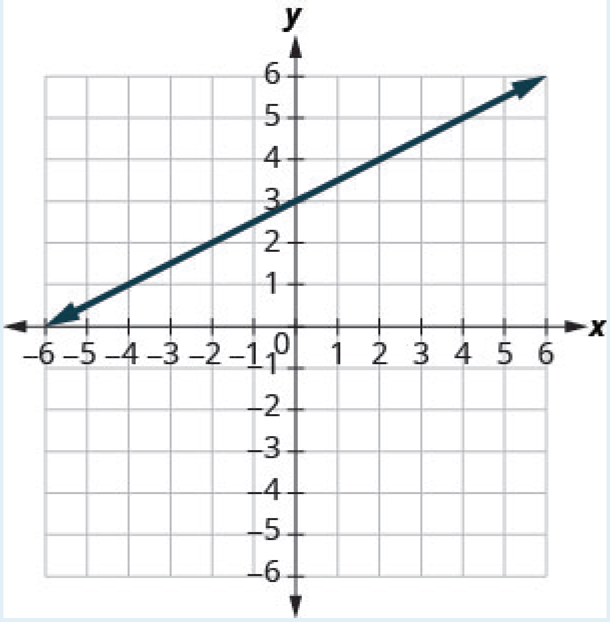


Show answer

slope $m = \frac{2}{3}$ and *y*-intercept $(0, -1)$

TRY IT 1.2

Use the graph to find the slope and *y*-intercept of the line $y = \frac{1}{2}x + 3$. Compare these values to the equation $y = mx + b$.



Show answer

slope $m = \frac{1}{2}$ and *y*-intercept $(0, 3)$

Identify the Slope and *y*-Intercept From an Equation of a Line

In Understand Slope of a Line, we graphed a line using the slope and a point. When we are given an equation in slope–intercept form, we can use the *y*-intercept as the point, and then count out the slope from there. Let’s practice finding the values of the slope and *y*-intercept from the equation of a line.

EXAMPLE 2

Identify the slope and *y*-intercept of the line with equation $y = -3x + 5$.

Solution

We compare our equation to the slope–intercept form of the equation.

	$y = mx + b$
Write the equation of the line.	$y = -3x + 5$
Identify the slope.	$m = -3$
Identify the *y*-intercept.	*y*-intercept is $(0, 5)$

TRY IT 2.1

Identify the slope and *y*-intercept of the line $y = \frac{2}{5}x - 1$.

Show answer

$\frac{2}{5}; (0, -1)$

TRY IT 2.2

Identify the slope and *y*-intercept of the line $y = -\frac{4}{3}x + 1$.

Show answer

$-\frac{4}{3}; (0, 1)$

When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

EXAMPLE 3

Identify the slope and *y*-intercept of the line with equation $x + 2y = 6$.

Solution

This equation is not in slope–intercept form. In order to compare it to the slope–intercept form we must first solve the equation for$y$.

Solve for *y*.	$x + 2y = 6$
Subtract *x* from each side.	$2y = -x + 6$
Divide both sides by 2.	$\frac{2y}{2} = \frac{-x + 6}{2}$
Simplify.	$\frac{2y}{2} = \frac{-x}{2} + \frac{6}{2}$
(Remember: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$)	
Simplify.	$y = -\frac{1}{2}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{2}x + 3$
Identify the slope.	$m = -\frac{1}{2}$
Identify the *y*-intercept.	*y*-intercept is (0, 3)

TRY IT 3.1

Identify the slope and *y*-intercept of the line $x + 4y = 8$.

Show answer

$-\frac{1}{4}; (0, 2)$

TRY IT 3.2

Identify the slope and *y*-intercept of the line $3x + 2y = 12$.

Show answer

$-\frac{3}{2}; (0, 6)$

Graph a Line Using its Slope and Intercept

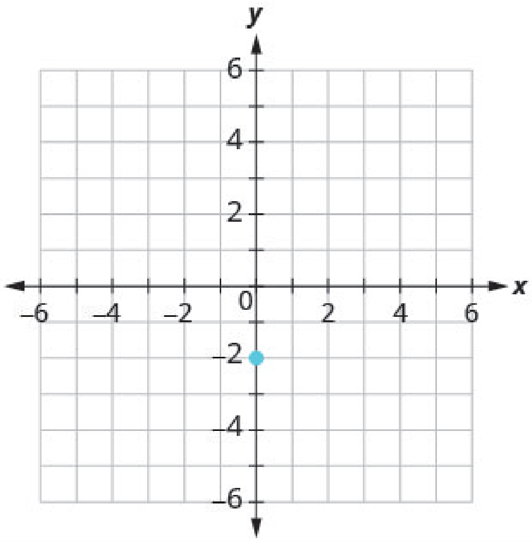
Now that we know how to find the slope and *y*-intercept of a line from its equation, we can graph the line by plotting the *y*-intercept and then using the slope to find another point.

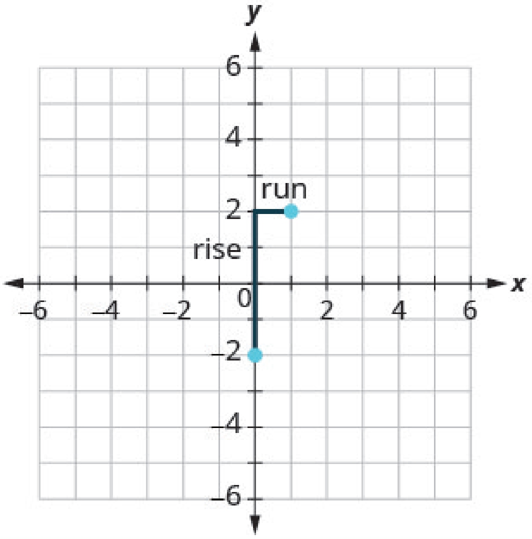
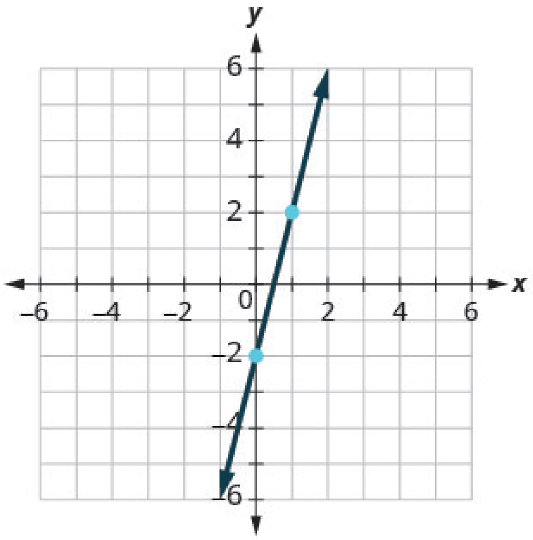
EXAMPLE 4

How to Graph a Line Using its Slope and Intercept

Graph the line of the equation $y = 4x - 2$ using its slope and *y*-intercept.

Solution

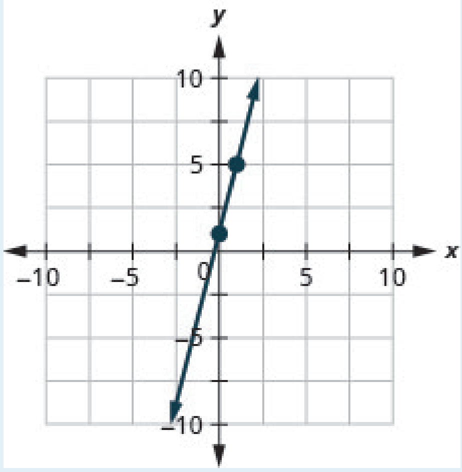
Step 1. Find the slope–intercept form of the equation.	This equation is in slope–intercept form.	$y = 4x - 2$
Step 2. Identify the slope and *y*-intercept.	Use $y = mx + b$ Find the slope. Find the *y*-intercept.	$y = mx + b$ $y = 4x + (-2)$ $m = 4$ $b = -2$, $(0, -2)$
Step 3. Plot the *y*-intercept.	Plot (0, –2).	

Step 4. Use the slope formula $m = \dfrac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = 4$ $\dfrac{\text{rise}}{\text{run}} = \dfrac{4}{1}$ $\text{rise} = 4$ $\text{run} = 1$
Step 5. Starting at the *y*-intercept, count out the rise and run to mark the second point.	Start at (0, –2) and count the rise and the run. Up 4, right 1.	
Step 6. Connect the points with a line.	Connect the two points with a line.	

TRY IT 4.1

Graph the line of the equation $y = 4x + 1$ using its slope and *y*-intercept.

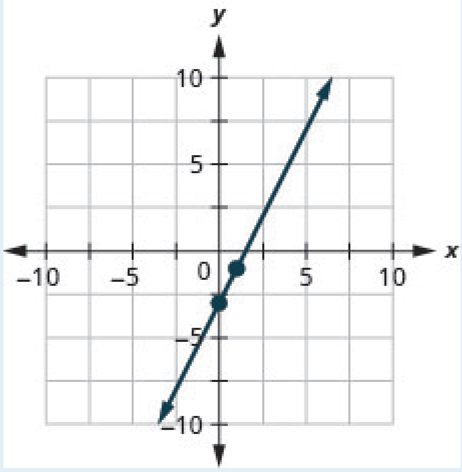
Show answer



TRY IT 4.2

Graph the line of the equation $y = 2x - 3$ using its slope and *y*-intercept.

Show answer



HOW TO: Graph a line using its slope and *y*-intercept

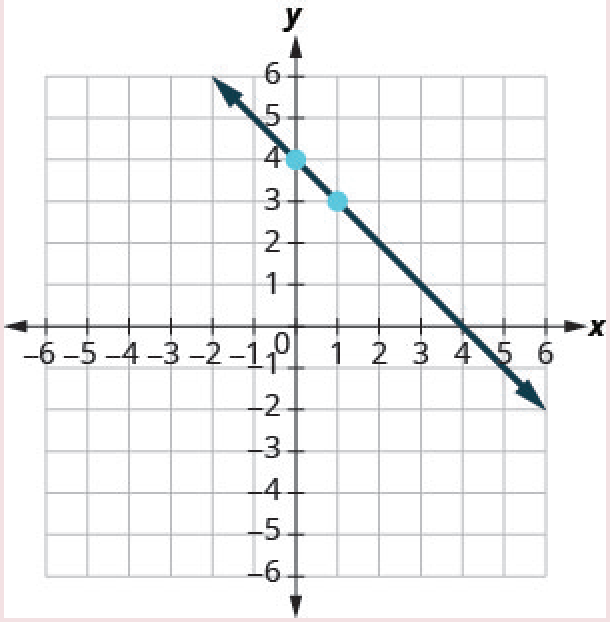
1. Find the slope-intercept form of the equation of the line.

2. Identify the slope and *y*-intercept.
3. Plot the *y*-intercept.
4. Use the slope formula $m = \dfrac{\text{rise}}{\text{run}}$ to identify the rise and the run.
5. Starting at the *y*-intercept, count out the rise and run to mark the second point.
6. Connect the points with a line.

EXAMPLE 5

Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

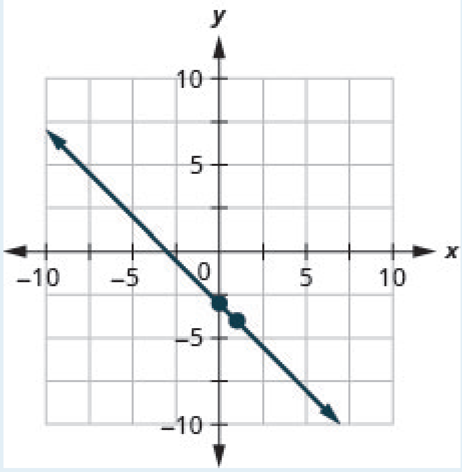
Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$
	y-intercept is (0, 4)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise −1, run 1
Draw the line.	
To check your work, you can find another point on the line and make sure it is a solution of the equation. In the graph we see the line goes through (4, 0).	
Check. $y = -x + 4$ $0 \overset{?}{=} -4 + 4$ $0 = 0 \checkmark$	

TRY IT 5.1

Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

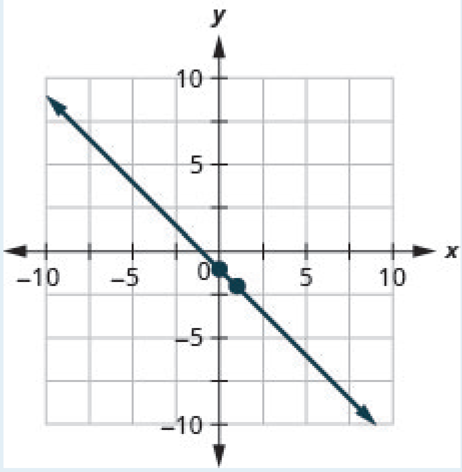
Show answer



TRY IT 5.2

Graph the line of the equation $y = -x - 1$ using its slope and *y*-intercept.

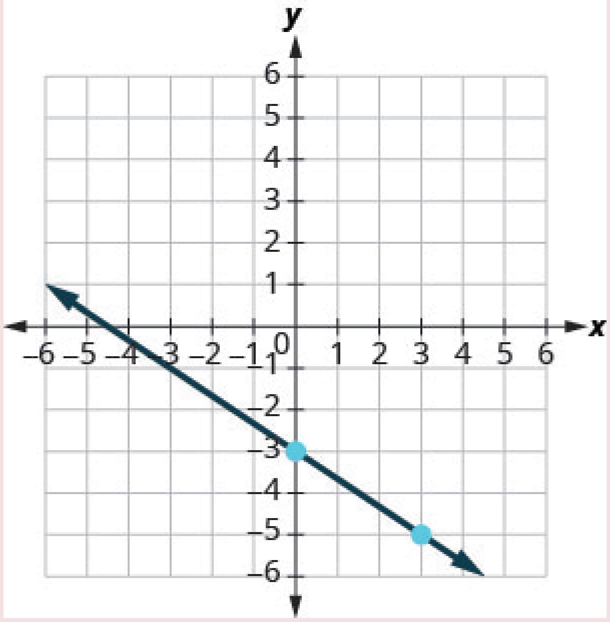
Show answer



EXAMPLE 6

Graph the line of the equation $y = -\frac{2}{3}x - 3$ using its slope and *y*-intercept.

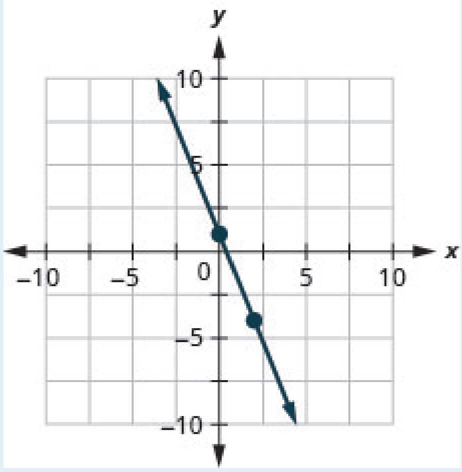
Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -\frac{2}{3}x - 3$
Identify the slope and *y*-intercept.	$m = -\frac{2}{3}$; *y*-intercept is (0, −3)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run.	
Count out the rise and run to mark the second point.	
Draw the line.	

TRY IT 6.1

Graph the line of the equation $y = -\frac{5}{2}x + 1$ using its slope and *y*-intercept.

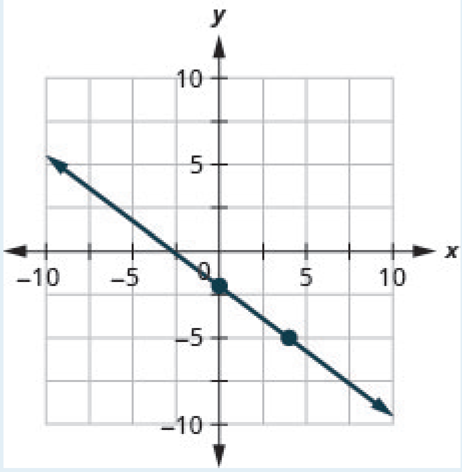
Show answer



TRY IT 6.2

Graph the line of the equation $y = -\frac{3}{4}x - 2$ using its slope and *y*-intercept.

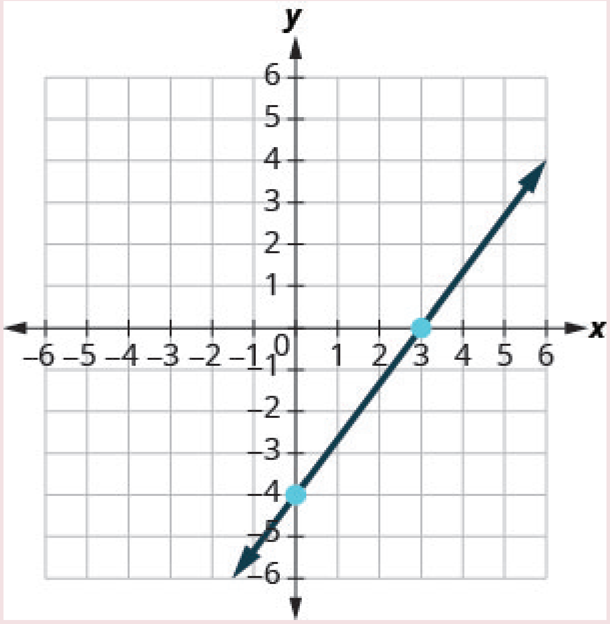
Show answer



EXAMPLE 7

Graph the line of the equation $4x - 3y = 12$ using its slope and *y*-intercept.

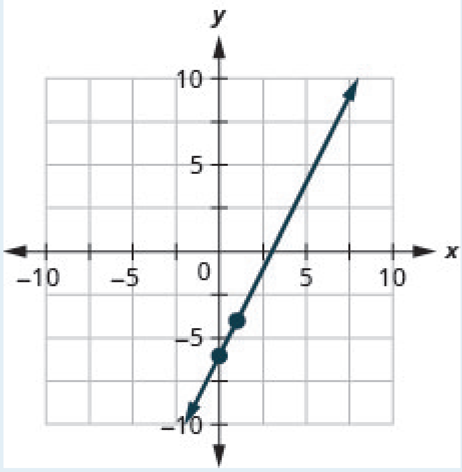
Solution

	$4x - 3y = 12$
Find the slope–intercept form of the equation.	$-3y = -4x + 12$
	$-\frac{3y}{3} = \frac{-4x + 12}{-3}$
The equation is now in slope–intercept form.	$y = \frac{4}{3}x - 4$
Identify the slope and *y*-intercept.	$m = \frac{4}{3}$
	y-intercept is (0, −4)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run; count out the rise and run to mark the second point.	
Draw the line.	

TRY IT 7.1

Graph the line of the equation $2x - y = 6$ using its slope and *y*-intercept.

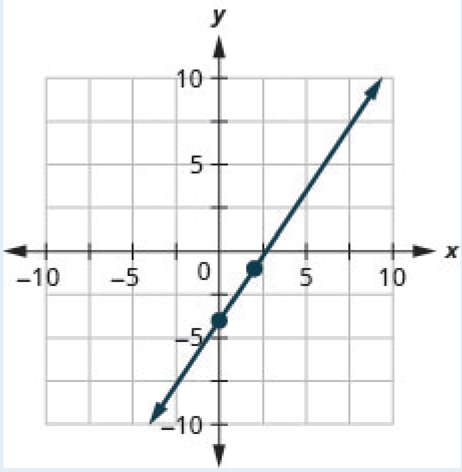
Show answer



TRY IT 7.2

Graph the line of the equation $3x - 2y = 8$ using its slope and *y*-intercept.

Show answer



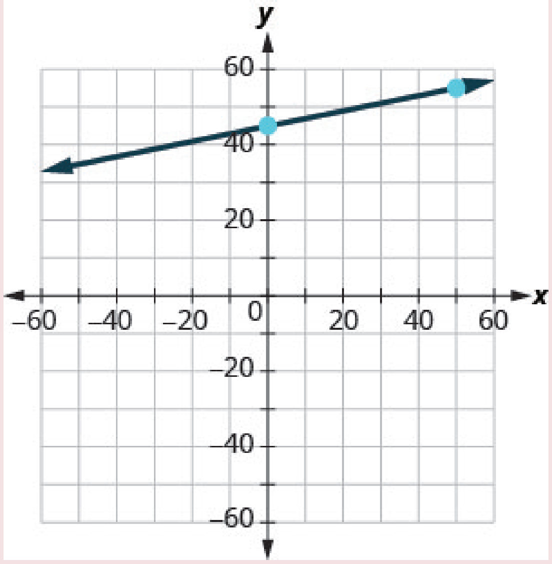
We have used a grid with x and y both going from about -10 to 10 for all the equations we've graphed so far. Not all linear equations can be graphed on this small grid. Often, especially in applications with real-world data, we'll need to extend the axes to bigger positive or smaller negative numbers.

EXAMPLE 8

Graph the line of the equation $y = 0.2x + 45$ using its slope and *y*-intercept.

Solution

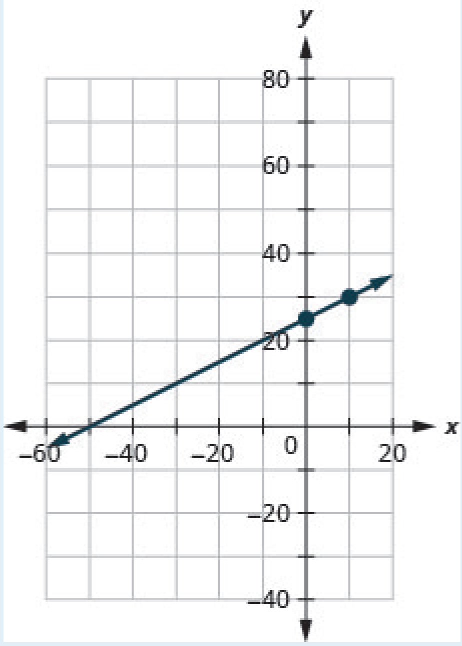
We'll use a grid with the axes going from about -80 to 80.

	$y = mx + b$
The equation is in slope–intercept form.	$y = 0.2x + 45$
Identify the slope and *y*-intercept.	$m = 0.2$
	The *y*-intercept is (0, 45)
Plot the *y*-intercept.	See graph below.
Count out the rise and run to mark the second point. The slope is $m = 0.2$; in fraction form this means $m = \frac{2}{10}$. Given the scale of our graph, it would be easier to use the equivalent fraction $m = \frac{10}{50}$.	
Draw the line.	

TRY IT 8.1

Graph the line of the equation $y = 0.5x + 25$ using its slope and *y*-intercept.

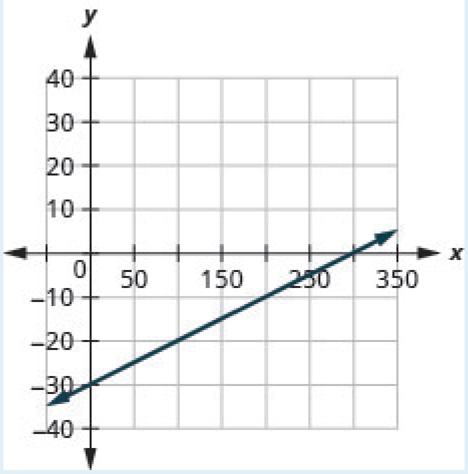
Show answer



TRY IT 8.2

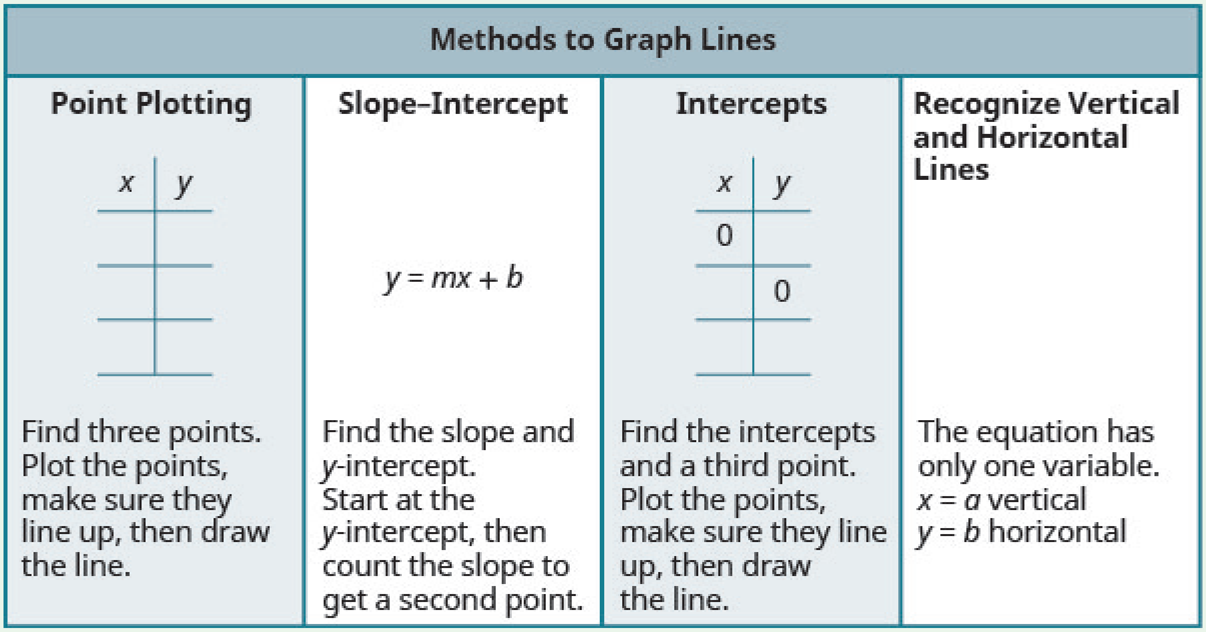
Graph the line of the equation $y = 0.1x - 30$ using its slope and *y*-intercept.

Show answer



Now that we have graphed lines by using the slope and *y*-intercept, let’s summarize all the methods we have used to graph lines. See (Figure).

Methods to graph lines



Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
$\begin{array}{c\|c} x & y \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$	$y = mx + b$	$\begin{array}{c\|c} x & y \\ \hline 0 & \\ \hline & 0 \\ \hline & \\ \hline \end{array}$	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier. Generally, plotting points is not the most efficient way to graph a line. We saw better methods in sections 4.3, 4.4, and earlier in this section. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are six equations we graphed in this chapter, and the method we used to graph each of them.

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = 4$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = 4x - 2$	Slope-intercept
#6	$y = -x + 4$	Slope-intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the *x*-intercept and $x = 0$ to find the *y*-intercept, and then found a third point by choosing another value for x or y.

Equations #5 and #6 are written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Strategy for choosing the most convenient method to graph a line

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
    - $x = a$ is a vertical line passing through the *x*-axis at a.
    - $y = b$ is a horizontal line passing through the *y*-axis at b.
- If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.
    - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
    - Find the *x*– and *y*-intercepts, a third point, and then graph.

EXAMPLE 9

Determine the most convenient method to graph each line.

a) $y = -6$ b)$5x - 3y = 15$ c) $x = 7$ d) $y = \frac{2}{5}x - 1$.

Solution

a. $y = -6$
   This equation has only one variable,y. Its graph is a horizontal line crossing the *y*-axis at -6.

b. $5x - 3y = 15$
   This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.

c. $x = 7$
   There is only one variable, x. The graph is a vertical line crossing the *x*-axis at 7.

d. $y = \frac{2}{5}x - 1$

Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and *y*-intercept.

TRY IT 9.1

Determine the most convenient method to graph each line: a) $3x + 2y = 12$ b) $y = 4$ c) $y = \frac{1}{5}x - 4$ d) $x = -7$.

Show answer

a) intercepts b) horizontal line c) slope–intercept d) vertical line

TRY IT 9.2

Determine the most convenient method to graph each line: a) $x = 6$ b) $y = -\frac{3}{4}x + 1$ c) $y = -8$ d) $4x - 3y = -1$.

Show answer

a) vertical line b) slope–intercept c) horizontal line d) intercepts

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real-world situations.

Usually when a linear equation models a real-world situation, different letters are used for the variables, instead of *x* and *y*. The variable names remind us of what quantities are being measured.

EXAMPLE 10

The equation $F = \frac{9}{5}C + 32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

a) Find the Fahrenheit temperature for a Celsius temperature of 0.
b) Find the Fahrenheit temperature for a Celsius temperature of 20.

c) Interpret the slope and *F*-intercept of the equation.
d) Graph the equation.

Solution

a) Find the Fahrenheit temperature for a Celsius temperature of 0. Find F when $C = 0$. Simplify.	$F = \frac{9}{5}C + 32$ $F = \frac{9}{5}(0) + 32$ $F = 32$
b) Find the Fahrenheit temperature for a Celsius temperature of 20. Find F when $C = 20$. Simplify. Simplify.	$F = \frac{9}{5}C + 32$ $F = \frac{9}{5}(20) + 32$ $F = 36 + 32$ $F = 68$

c) Interpret the slope and *F*-intercept of the equation.

Even though this equation uses $F$and C, it is still in slope–intercept form.

$$y = mx + b$$

$$F = mC + b$$

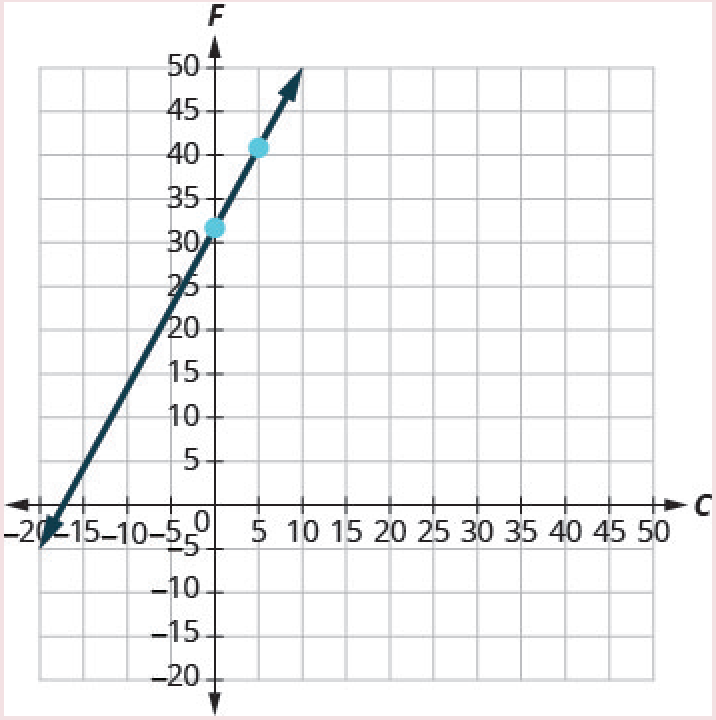
$$F = \frac{9}{5}C + 32$$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (*F*) increases 9 degrees when the temperature Celsius (*C*) increases 5 degrees.

The *F*-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

d) Graph the equation.

We'll need to use a larger scale than our usual. Start at the *F*-intercept $(0, 32)$ then count out the rise of 9 and the run of 5 to get a second point. See (Figure).



TRY IT 10.1

The equation $h = 2s + 50$ is used to estimate a woman’s height in inches, *h*, based on her shoe size, *s*.

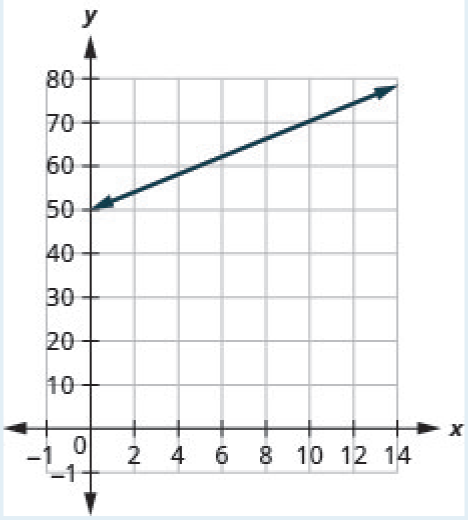
a) Estimate the height of a child who wears women’s shoe size 0.
b) Estimate the height of a woman with shoe size 8.
c) Interpret the slope and *h*-intercept of the equation.
d) Graph the equation.

Show answer

a. 50 inches

b. 66 inches

c. The slope, 2, means that the height, *h*, increases by 2 inches when the shoe size, *s*, increases by 1. The *h*-intercept means that when the shoe size is 0, the height is 50 inches.

d. 

TRY IT 10.2

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, *T*, based on the number of cricket chirps, *n*, in one minute.

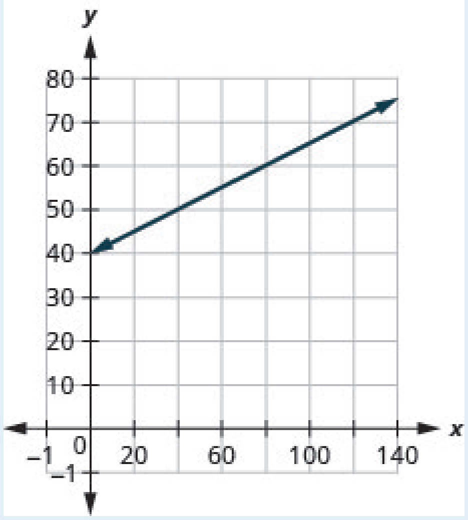
a) Estimate the temperature when there are no chirps.
b) Estimate the temperature when the number of chirps in one minute is 100.
c) Interpret the slope and *T*-intercept of the equation.
d) Graph the equation.

Show answer

a. 40 degrees

b. 65 degrees

c. The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (*F*) increases 1 degree when the number of chirps, *n*, increases by 4. The *T*-intercept means that when the number of chirps is 0, the temperature is 40°.

d. 

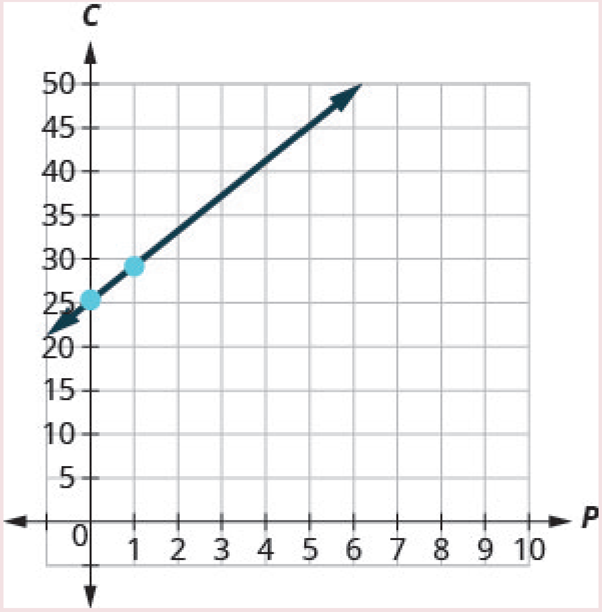
The cost of running some types business has two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labour needed to produce each item.

EXAMPLE 11

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, *C*, in dollars and the number of pizzas, *p*, that she sells.

a) Find Stella’s cost for a week when she sells no pizzas.
b) Find the cost for a week when she sells 15 pizzas.
c) Interpret the slope and *C*-intercept of the equation.
d) Graph the equation.

Solution

a) Find Stella’s cost for a week when she sells no pizzas.	$C = 4p + 25$
Find C when $p = 0$.	$C = 4(0) + 25$
Simplify.	$C = 25$
	Stella’s fixed cost is $25 when she sells no pizzas.
b) Find the cost for a week when she sells 15 pizzas.	$C = 4p + 25$
Find C when $p = 15$.	$C = 4(15) + 25$
Simplify.	$C = 60 + 25$
	$C = 85$
	Stella’s costs are $85 when she sells 15 pizzas.
c) Interpret the slope and C-intercept of the equation.	$y = mx + b$ $C = 4p + 25$
	The slope, 4, means that the cost increases by $4 for each pizza Stella sells. The C-intercept means that even when Stella sells no pizzas, her costs for the week are $25.
d) Graph the equation. We’ll need to use a larger scale than our usual. Start at the C-intercept (0, 25) then count out the rise of 4 and the run of 1 to get a second point.	

TRY IT 11.1

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, *C*, in dollars and the number of miles, *m*, that he drives.

a) Find Sam’s cost for a week when he drives 0 miles.
b) Find the cost for a week when he drives 250 miles.
c) Interpret the slope and *C*-intercept of the equation.
d) Graph the equation.

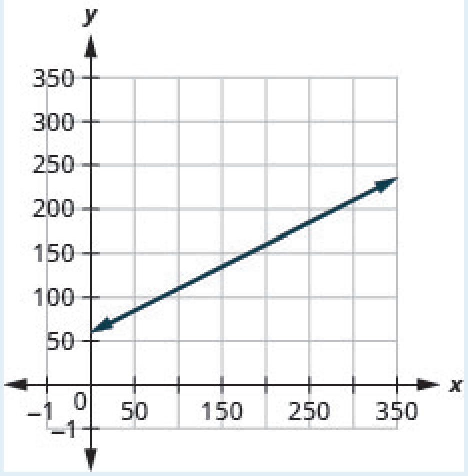
Show answer

a. $60

b. $185

c. The slope, 0.5, means that the weekly cost, *C*, increases by $0.50 when the number of miles driven, *n*, increases by 1. The *C*-intercept means that when the number of miles driven is 0, the weekly cost is $60

d.



TRY IT 11.2

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, *C*, in dollars and the number of wedding invitations, *n*, that she writes.

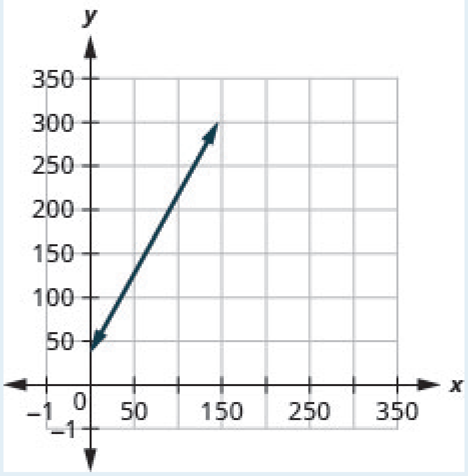
a) Find Loreen’s cost for a week when she writes no invitations.
b) Find the cost for a week when she writes 75 invitations.
c) Interpret the slope and *C*-intercept of the equation.
d) Graph the equation.

Show answer

a. $35

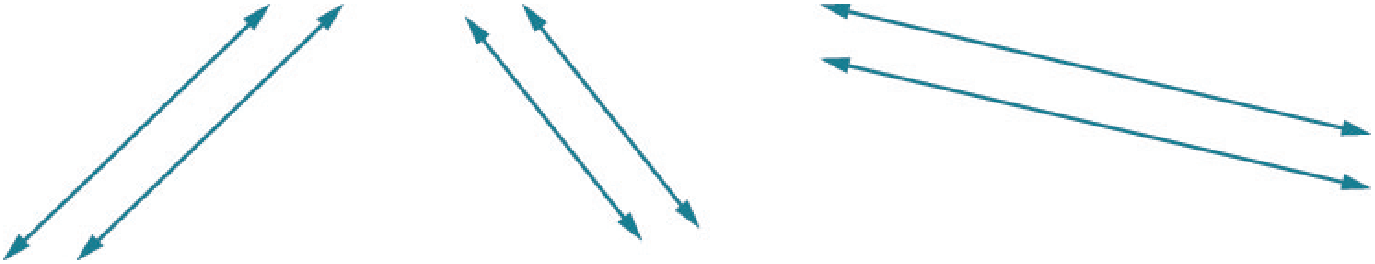
b. \$170

c. The slope, 1.8, means that the weekly cost, C, increases by \$1.80 when the number of invitations, *n*, increases by 1.80.
The *C*-intercept means that when the number of invitations is 0, the weekly cost is \$35.;

d. 

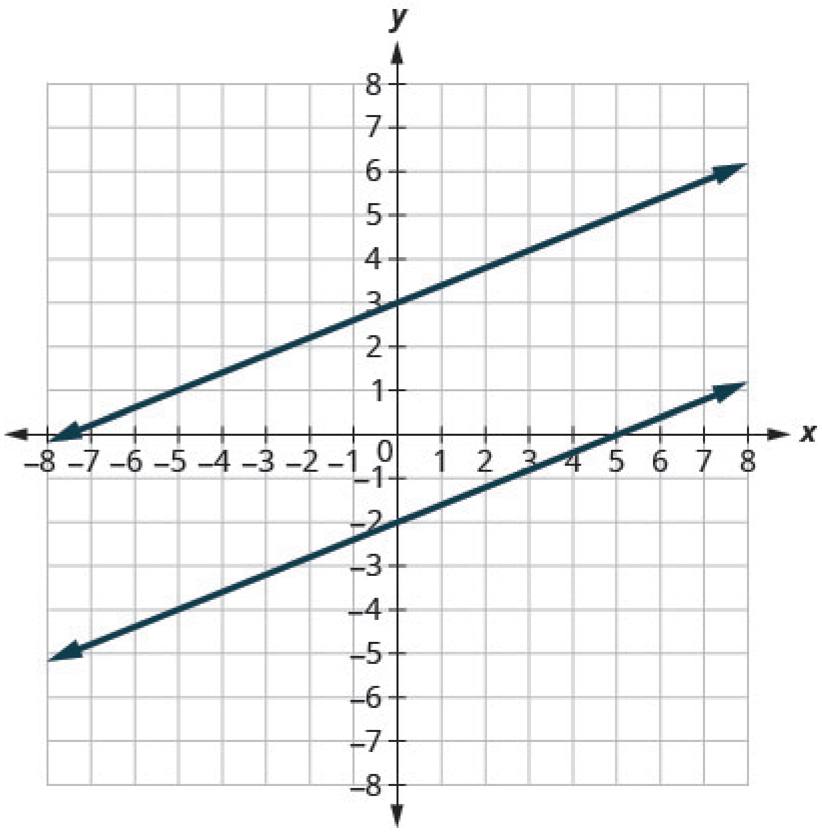
Use Slopes to Identify Parallel Lines

The slope of a line indicates how steep the line is and whether it rises or falls as we read it from left to right. Two lines that have the same slope are called parallel lines. Parallel lines never intersect.



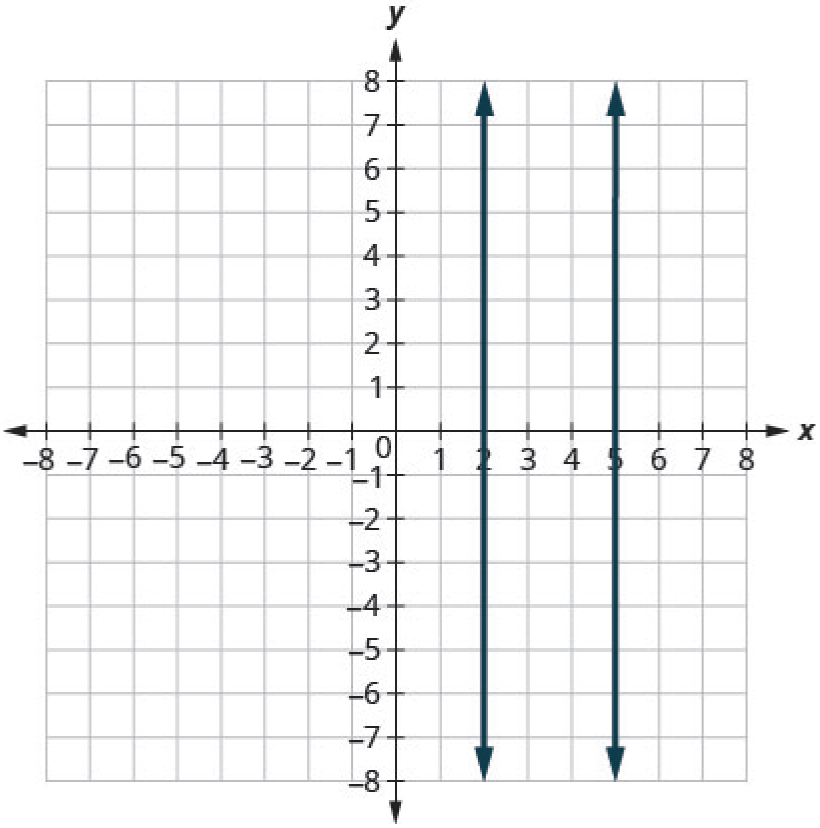
We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called parallel lines. See (Figure).

Verify that both lines have the same slope, $m = \frac{2}{5}$, and different *y*-intercepts.



What about vertical lines? The slope of a vertical line is undefined, so vertical lines don’t fit in the definition above. We say that vertical lines that have different *x*-intercepts are parallel. See (Figure).

Vertical lines with different *x*-intercepts are parallel.



Parallel lines

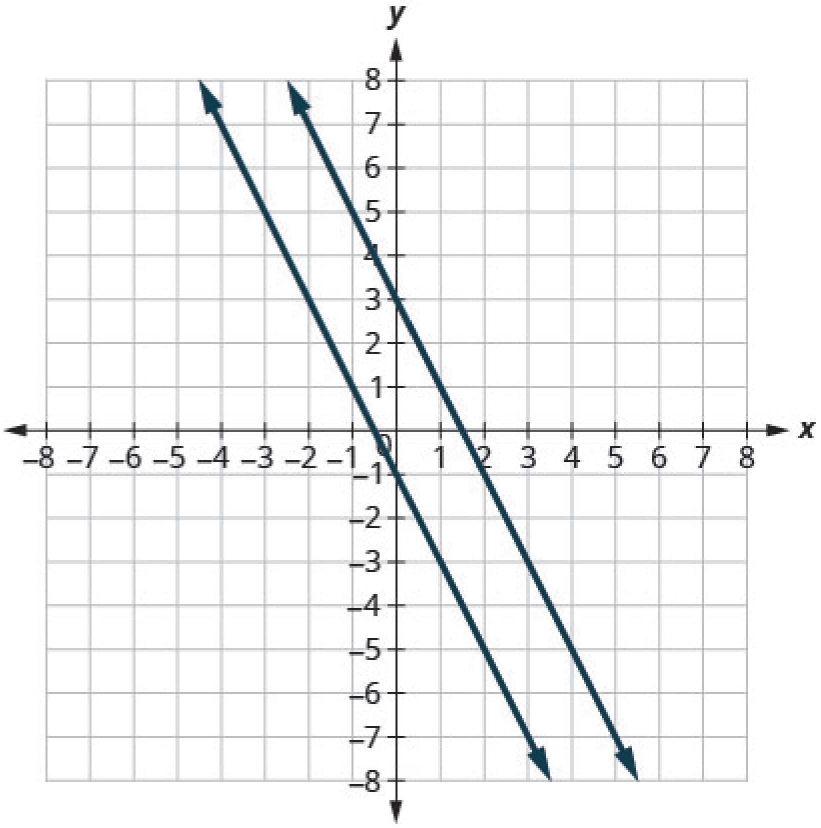
Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different *y*-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then$m_1 = m_2$.
- Parallel vertical lines have different *x*-intercepts.

Let’s graph the equations $y = -2x + 3$ and $2x + y = -1$ on the same grid. The first equation is already in slope–intercept form: $y = -2x + 3$. We solve the second equation for y:

$$\begin{aligned} 2x + y &= -1 \\ y &= -2x - 1 \end{aligned}$$

Graph the lines.



Notice the lines look parallel. What is the slope of each line? What is the *y*-intercept of each line?

$$\begin{aligned} y &= mx+b \\ y &= -2x+3 \\ m &= -2 \\ b &= 3, (0,\ 3) \end{aligned} \qquad \begin{aligned} y &= mx+b \\ y &= -2x-1 \\ m &= -2 \\ b &= -1, (0,\ \text{-1}) \end{aligned}$$

The slopes of the lines are the same and the *y*-intercept of each line is different. So we know these lines are parallel.

Since parallel lines have the same slope and different *y*-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

EXAMPLE 12

Use slopes and *y*-intercepts to determine if the lines $3x-2y=6$ and $y=\frac{3}{2}x+1$ are parallel.

Solution

Solve the first equation for y.	$\begin{aligned} 3x - 2y &= 6 \\ -2y &= -3x + 6 \\ \frac{-2y}{-2} &= \frac{-3x+6}{-2} \end{aligned}$	and	$y = \frac{3}{2}x + 1$
The equation is now in slope-intercept form.	$y = \frac{3}{2}x - 3$		
The equation of the second line is already in slope-intercept form.			$y = \frac{3}{2}x + 1$
Identify the slope and y-intercept of both lines.	$y = \frac{3}{2}x - 3$ $y = mx + b$ $m = \frac{3}{2}$		$y = \frac{3}{2}x + 1$ $y = mx + b$ $m = \frac{3}{2}$
	y-intercept is (0, −3)		y-intercept is (0, 1)

The lines have the same slope and different *y*-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.

TRY IT 12.1

Use slopes and *y*-intercepts to determine if the lines $2x + 5y = 5$ and $y = -\frac{2}{5}x - 4$ are parallel.

Show answer
parallel

TRY IT 12.2

Use slopes and *y*-intercepts to determine if the lines $4x - 3y = 6$ and $y = \frac{4}{3}x - 1$ are parallel.

Show answer
parallel

EXAMPLE 13

Use slopes and *y*-intercepts to determine if the lines $y=-4$ and $y=3$ are parallel.

Solution

	$y=-4$ $y=0x-4$	and	$y=3$ $y=0x+3$
Write each equation in slope-intercept form.	$y=0x-4$		$y=0x+3$
Since there is no x term we write $0x$.	$y=mx+b$		$y=mx+b$
Identify the slope and y-intercept of both lines.	$m=0$		$m=0$
	y-intercept is (0, 4)		y-intercept is (0, 3)

The lines have the same slope and different *y*-intercepts and so they are parallel.

There is another way you can look at this example. If you recognize right away from the equations that these are horizontal lines, you know their slopes are both 0. Since the horizontal lines cross the *y*-axis at $y=-4$ and at $y=3$, we know the *y*-intercepts are $(0,-4)$ and $(0,3)$. The lines have the same slope and different *y*-intercepts and so they are parallel.

TRY IT 13.1

Use slopes and *y*-intercepts to determine if the lines $y=8$ and $y=-6$ are parallel.

Show answer
parallel

TRY IT 13.2

Use slopes and *y*-intercepts to determine if the lines $y=1$ and $y=-5$ are parallel.

Show answer
parallel

EXAMPLE 14

Use slopes and *y*-intercepts to determine if the lines $x = -2$ and $x = -5$ are parallel.

Solution

$x = -2$ and $x = -5$

Since there is no y, the equations cannot be put in slope–intercept form. But we recognize them as equations of vertical lines. Their *x*-intercepts are -2 and -5. Since their *x*-intercepts are different, the vertical lines are parallel.

TRY IT 14.1

Use slopes and *y*-intercepts to determine if the lines $x = 1$ and $x = -5$ are parallel.

Show answer

parallel

TRY IT 14.2

Use slopes and *y*-intercepts to determine if the lines $x = 8$ and $x = -6$ are parallel.

Show answer

parallel

EXAMPLE 15

Use slopes and *y*-intercepts to determine if the lines $y = 2x - 3$ and $-6x + 3y = -9$ are parallel. You may want to graph these lines, too, to see what they look like.

Solution

	$y = 2x - 3$	and	$-6x + 3y = -9$
The first equation is already in slope-intercept form.	$y = 2x - 3$		
Solve the second equation for y.			$\begin{aligned} -6x + 3y &= -9 \\ 3y &= 6x - 9 \\ \frac{3y}{3} &= \frac{6x - 9}{3} \\ y &= 2x - 3 \end{aligned}$
The second equation is now in slope-intercept form.	$y = 2x - 3$		
Identify the slope and y -intercept of both lines.	$y = 2x - 3$ $y = mx + b$ $m = 2$		$y = 2x - 3$ $y = mx + b$ $m = 2$
	y-intercept is (0, -3)		y-intercept is (0, -3)

The lines have the same slope, but they also have the same *y*-intercepts. Their equations represent the same line. They are not parallel; they are the same line.

TRY IT 15.1

Use slopes and *y*-intercepts to determine if the lines $y = -\frac{1}{2}x - 1$ and $x + 2y = 2$ are parallel.

Show answer
not parallel; same line

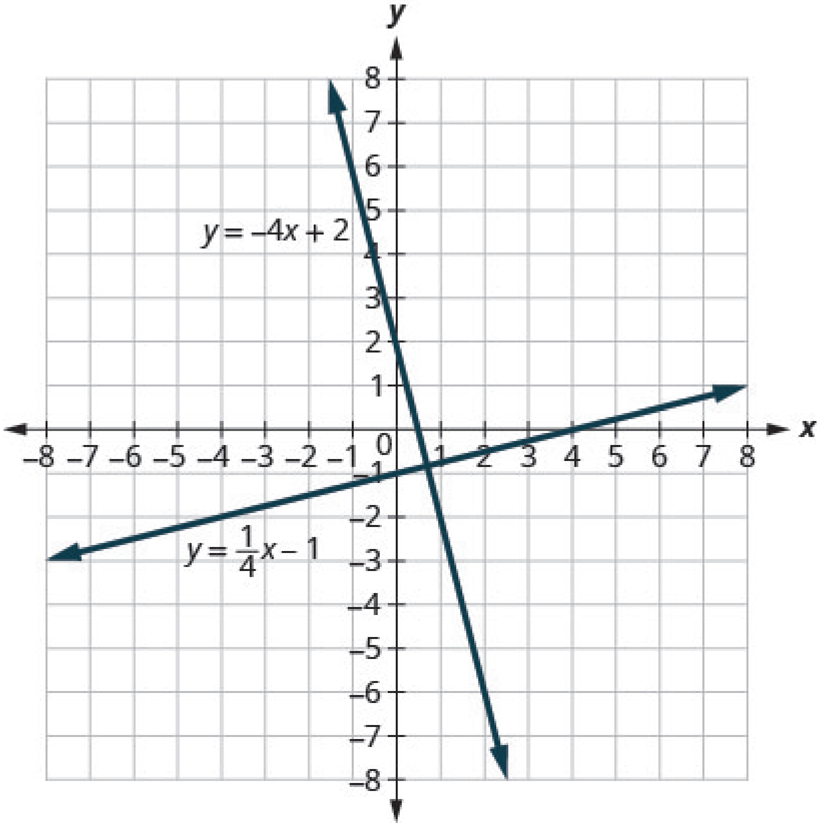
TRY IT 15.2

Use slopes and *y*-intercepts to determine if the lines $y = \frac{3}{4}x - 3$ and $3x - 4y = 12$ are parallel.

Show answer
not parallel; same line

Use Slopes to Identify Perpendicular Lines

Let’s look at the lines whose equations are $y=\frac{1}{4}x-1$ and $y=-4x+2$, shown in (Figure).



These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

What do you notice about the slopes of these two lines? As we read from left to right, the line $y=\frac{1}{4}x-1$ rises, so its slope is positive. The line$y=-4x+2$ drops from left to right, so it has a negative slope. Does it make sense to you that the slopes of two perpendicular lines will have opposite signs?

If we look at the slope of the first line, $m_1=\frac{1}{4}$, and the slope of the second line, $m_2=-4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$m_1\cdot m_2$$
$$\frac{1}{4}(-4)$$
$$-1$$

This is always true for perpendicular lines and leads us to this definition.

Perpendicular lines

Perpendicular lines are lines in the same plane that form a right angle.

If m_1 and m_2 are the slopes of two perpendicular lines, then:

$m_1 \cdot m_2 = -1$ and $m_1 = \dfrac{-1}{m_2}$

Vertical lines and horizontal lines are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are negative reciprocals. If the product of the slopes is -1, the lines are perpendicular. Perpendicular lines may have the same *y*-intercepts.

EXAMPLE 16

Use slopes to determine if the lines, $y = -5x - 4$ and $x - 5y = 5$ are perpendicular.

Solution

The first equation is already in slope-intercept form.	$y = -5x - 4$	
Solve the second equation for y.	$\begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x - 1 \end{aligned}$	
Identify the slope of each line.	$\begin{aligned} y &= -5x - 4 \\ y &= mx + b \\ m_1 &= -5 \end{aligned}$	$\begin{aligned} y &= \frac{1}{5}x - 1 \\ y &= mx + b \\ m_2 &= \frac{1}{5} \end{aligned}$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes,

$$m_1 \cdot m_2$$

$$-5\left(\frac{1}{5}\right)$$

$$-1\checkmark$$

TRY IT 16.1

Use slopes to determine if the lines $y = -3x + 2$ and $x - 3y = 4$ are perpendicular.

Show answer
perpendicular

TRY IT 16.2

Use slopes to determine if the lines $y = 2x - 5$ and $x + 2y = -6$ are perpendicular.

Show answer
perpendicular

EXAMPLE 17

Use slopes to determine if the lines, $7x + 2y = 3$ and $2x + 7y = 5$ are perpendicular.

Solution

Solve the equations for y.	$\begin{aligned} 7x+2y &= 3 \\ 2y &= -7x+3 \\ \frac{2y}{2} &= \frac{-7x+3}{2} \\ y &= -\frac{7}{2}x+\frac{3}{2} \end{aligned}$	$\begin{aligned} 2x+7y &= 5 \\ 7y &= -2x+5 \\ \frac{7y}{7} &= \frac{-2x+5}{7} \\ y &= -\frac{2}{7}x+\frac{5}{7} \end{aligned}$
Identify the slope of each line.	$\begin{aligned} y &= mx+b \\ m_1 &= -\frac{7}{2} \end{aligned}$	$\begin{aligned} y &= mx+b \\ m_2 &= -\frac{2}{7} \end{aligned}$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

TRY IT 17.1

Use slopes to determine if the lines $5x + 4y = 1$ and $4x + 5y = 3$ are perpendicular.

Show answer
not perpendicular

TRY IT 17.2

Use slopes to determine if the lines $2x - 9y = 3$ and $9x - 2y = 1$ are perpendicular.

Show answer
not perpendicular

Access this online resource for additional instruction and practice with graphs.

- Explore the Relation Between a Graph and the Slope–Intercept Form of an Equation of a Line

Key Concepts

- **The slope–intercept form of an equation of a line with slope** m and *y*-intercept, $(0, b)$ is, $y = mx + b$.
- **Graph a Line Using its Slope and *y*-Intercept**
    1. Find the slope-intercept form of the equation of the line.
    2. Identify the slope and *y*-intercept.
    3. Plot the *y*-intercept.
    4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
    5. Starting at the *y*-intercept, count out the rise and run to mark the second point.
    6. Connect the points with a line.
- **Strategy for Choosing the Most Convenient Method to Graph a Line:** Consider the form of the equation.
    - If it only has one variable, it is a vertical or horizontal line.
      $x = a$ is a vertical line passing through the *x*-axis at a.
      $y = b$ is a horizontal line passing through the *y*-axis at b.
    - If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using

    the slope and *y*-intercept.
    Identify the slope and *y*-intercept and then graph.
  - If the equation is of the form $Ax + By = C$, find the intercepts.
    Find the *x*– and *y*-intercepts, a third point, and then graph.
- **Parallel lines are lines in the same plane that do not intersect.**
  - Parallel lines have the same slope and different *y*-intercepts.
  - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
  - Parallel vertical lines have different *x*-intercepts.
- **Perpendicular lines are lines in the same plane that form a right angle.**
  - If $m_1 \text{ and } m_2$ are the slopes of two perpendicular lines, then $m_1 \cdot m_2 = -1$ and $m_1 = \frac{-1}{m_2}$.
  - Vertical lines and horizontal lines are always perpendicular to each other.

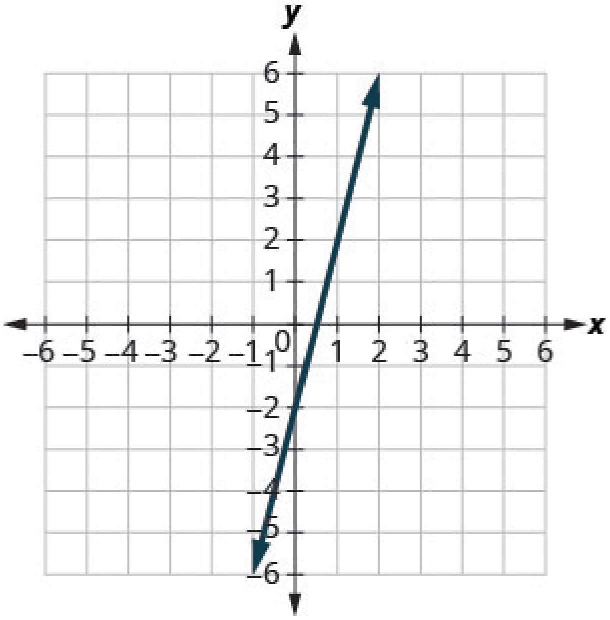
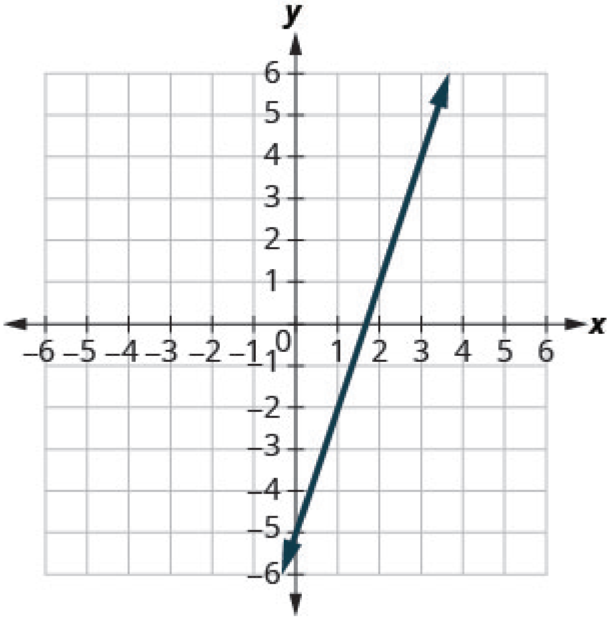
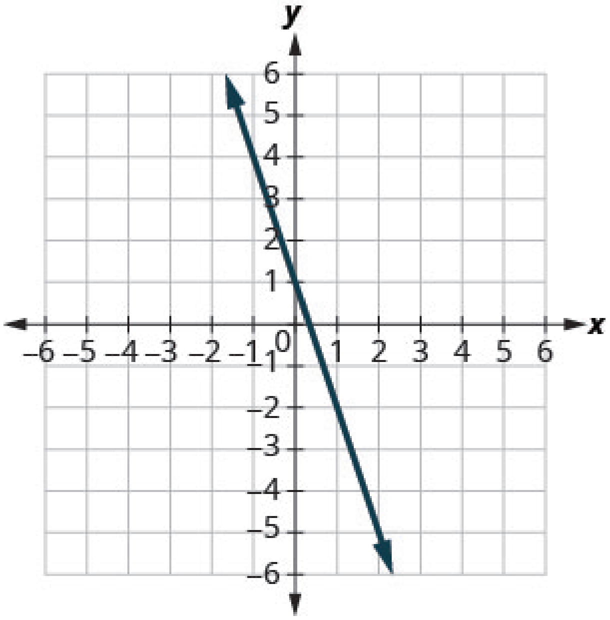
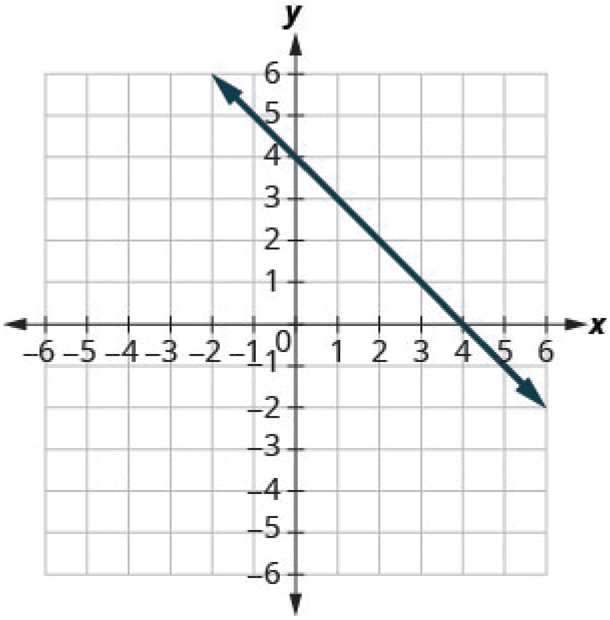
Glossary

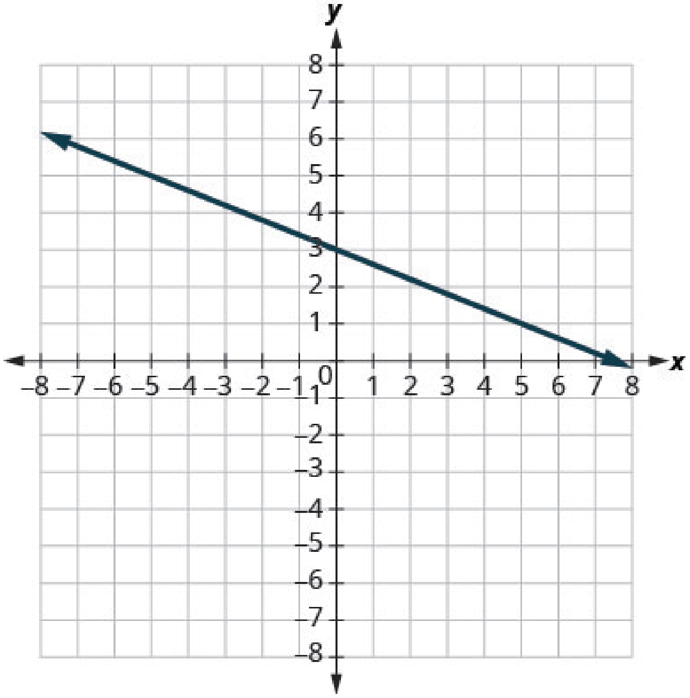
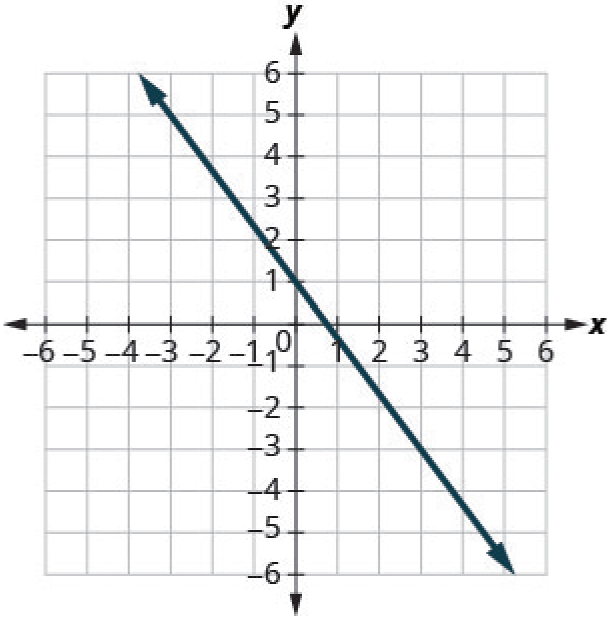
parallel lines
Lines in the same plane that do not intersect.
perpendicular lines
Lines in the same plane that form a right angle.
slope-intercept form of an equation of a line
The slope–intercept form of an equation of a line with slope and *y*-intercept, is, .

Practice Makes Perfect

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

In the following exercises, use the graph to find the slope and y-intercept of each line. Compare the values to the equation $y = mx + b$.

1.  $y = 4x - 2$	2.  $y = 3x - 5$
3.  $y = -3x + 1$	4.  $y = -x + 4$

5.  $y = -\frac{2}{5}x + 3$	6.  $y = -\frac{4}{3}x + 1$

Identify the Slope and y-Intercept From an Equation of a Line

In the following exercises, identify the slope and y-intercept of each line.

7. $y = -9x + 7$	8. $y = -7x + 3$
9. $y = 4x - 10$	10. $y = 6x - 8$
11. $4x + y = 8$	12. $3x + y = 5$
13. $8x + 3y = 12$	14. $6x + 4y = 12$
15. $7x - 3y = 9$	16. $5x - 2y = 6$

Graph a Line Using Its Slope and Intercept

In the following exercises, graph the line of each equation using its slope and y-intercept.

17. $y = x + 4$	18. $y = x + 3$
19. $y = 2x - 3$	20. $y = 3x - 1$
21. $y = -x + 3$	22. $y = -x + 2$
23. $y = -x - 2$	24. $y = -x - 4$
25. $y = -\frac{2}{5} - 3$	26. $y = -\frac{3}{4} - 1$
27. $y = -\frac{2}{3} + 1$	28. $y = -\frac{3}{5} + 2$
29. $4x - 3y = 6$	30. $3x - 4y = 8$
31. $y = 0.1x + 15$	32. $y = 0.1x + 15$

Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

33. $y = 4$	34. $x = 2$
35. $x = -3$	36. $y = 5$
37. $y = -3x + 4$	38. $y = -3x + 4$
39. $x - y = 1$	40. $x - y = 5$
41. $y = \frac{4}{5}x - 3$	42. $y = \frac{2}{3}x - 1$
43. $y = -1$	44. $y = -3$
45. $2x - 5y = -10$	46. $3x - 2y = -12$
47. $y = -\frac{1}{3}x + 5$	48. $y = -\frac{1}{4} + 3$

Graph and Interpret Applications of Slope–Intercept

49. The equation $P = 28 + 2.54w$ models the relation between the amount of Randy’s monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

a. Find the payment for a month when Randy used 0 units of water.
b. Find the payment for a month when Randy used 15 units of water.
c. Interpret the slope and *P*-intercept of the equation.
d. Graph the equation.

50. The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet’s monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

a. Find Tuyet’s payment for a month when 0 units of water are used.
b. Find Tuyet’s payment for a month when 12 units of water are used.
c. Interpret the slope and *P*-intercept of the equation.
d. Graph the equation.

51. Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, *C*, per day and the number of miles, *m*, she drives in one day.

a. Find the cost if Janelle drives the car 0 miles one day.
b. Find the cost on a day when Janelle drives the car 400 miles.
c. Interpret the slope and *C*–intercept of the equation.
d. Graph the equation.

52. Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, *R*, that he is reimbursed and the number of miles, *m*, he drives in one day.

a. Find the amount Bruce is reimbursed on a day when he drives 0 miles.
b. Find the amount Bruce is reimbursed on a day when he drives 220 miles.
c. Interpret the slope and *R*-intercept of the equation.
d. Graph the equation.

53. Patel’s weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, *S*, in dollars and the amount of his sales, *c*, in dollars.

a. Find Patel’s salary for a week when his sales were 0.
b. Find Patel’s salary for a week when his sales were 18,540.
c. Interpret the slope and *S*-intercept of the equation.
d. Graph the equation.

54. Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, *S*, in dollars and the amount of her sales, *c*, in dollars.

a. Find Cherie’s salary for a week when her sales were 0.
b. Find Cherie’s salary for a week when her sales were 3600.
c. Interpret the slope and *S*–intercept of the equation.
d. Graph the equation.

55. Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, *C* of the banquet and the number of guests, *g*.

a. Find the cost if the number of guests is 50.
b. Find the cost if the number of guests is 100.
c. Interpret the slope and *C*–intercept of the equation.
d. Graph the equation.

56. Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, *C*, of the banquet and the number of guests, *g*.

a. Find the cost if the number of guests is 40.
b. Find the cost if the number of guests is 80.
c. Interpret the slope and *C*-intercept of the equation.
d. Graph the equation.

Use Slopes to Identify Parallel Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel.

57. $y = \frac{2}{3}x - 1; \;\; 2x - 3y = -2$
58. $y = \frac{3}{4}x - 3; \;\; 3x - 4y = -2$
59. $3x - 4y = -2; \;\; y = \frac{3}{4}x - 3$
60. $2x - 5y = -3; \;\; y = \frac{2}{5}x + 1$
61. $6x - 3y = 9; \;\; 2x - y = 3$
62. $2x - 4y = 6; \;\; x - 2y = 3$
63. $8x + 6y = 6; \;\; 12x + 9y = 12$
64. $4x + 2y = 6; \;\; 6x + 3y = 3$
65. $x = 7; \;\; x = -8$
66. $x = 5; \;\; x = -6$
67. $x = -3; \;\; x = -2$
68. $x = -4; \;\; x = -1$
69. $y = 5; \;\; y = 1$
70. $y = 2; \;\; y = 6$
71. $y = -1; \;\; y = 2$
72. $y = -4; \;\; y = 3$
73. $4x + 4y = 8; \;\; x + y = 2$
74. $x - y = 2; \;\; 2x - 2y = 4$
75. $5x - 2y = 11; \;\; 5x - y = 7$
76. $x - 3y = 6; \;\; 2x - 6y = 12$
77. $4x - 8y = 16; \;\; x - 2y = 4$
78. $3x - 6y = 12; \;\; 6x - 3y = 3$
79. $x - 5y = 10; \;\; 5x - y = -10$
80. $9x - 3y = 6; \;\; 3x - y = 2$
81. $9x - 5y = 4; \;\; 5x + 9y = -1$
82. $7x - 4y = 8; \;\; 4x + 7y = 14$

Use Slopes to Identify Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are perpendicular.

83. $x - 4y = 8; 4x + y = 2$

84. $3x - 2y = 8; 2x + 3y = 6$

85. $2x + 3y = 5; 3x - 2y = 7$

86. $2x + 5y = 3; 5x - 2y = 6$

87. $3x - 4y = 8; 4x - 3y = 6$

88. $3x - 2y = 1; 2x - 3y = 2$

89. $2x + 4y = 3; 6x + 3y = 2$

90. $5x + 2y = 6; 2x + 5y = 8$

91. $2x - 6y = 4; 12x + 4y = 9$

92. $4x - 2y = 5; 3x + 6y = 8$

93. $8x - 2y = 7; 3x + 12y = 9$

94. $6x - 4y = 5; 8x + 12y = 3$

Everyday Math

95. The equation $n = 4T - 160$ is used to estimate the number of cricket chirps, *n*, in one minute based on the temperature in degrees Fahrenheit, *T*.

a. Explain what the slope of the equation means.

b. Explain what the *n*–intercept of the equation means. Is this a realistic situation?

96. The equation $C = \frac{5}{9}F - 17.8$ can be used to convert temperatures *F*, on the Fahrenheit scale to temperatures, *C*, on the Celsius scale.

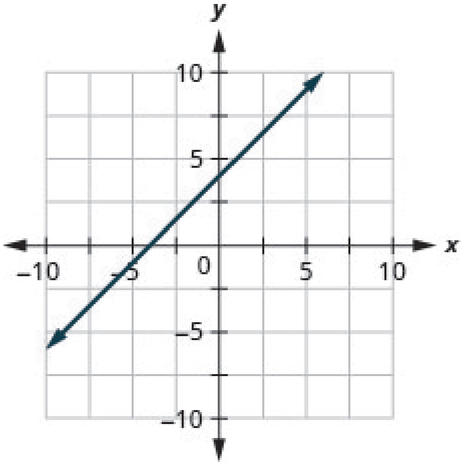
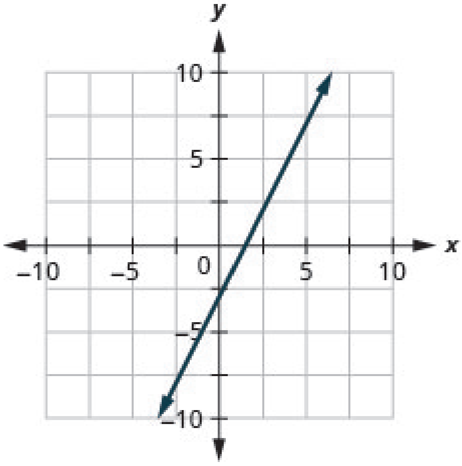
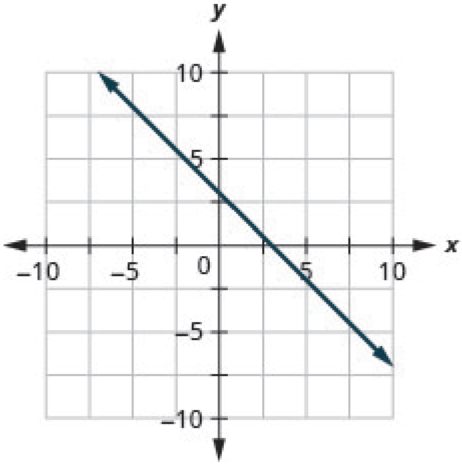
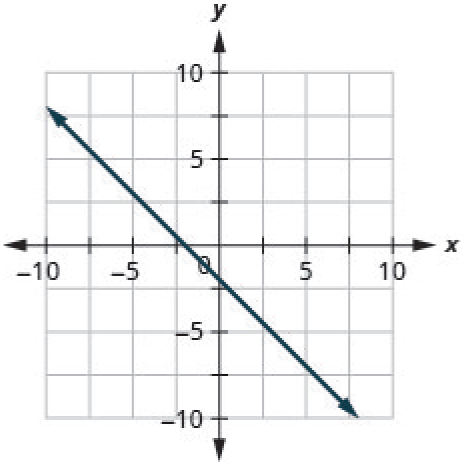
a. Explain what the slope of the equation means.

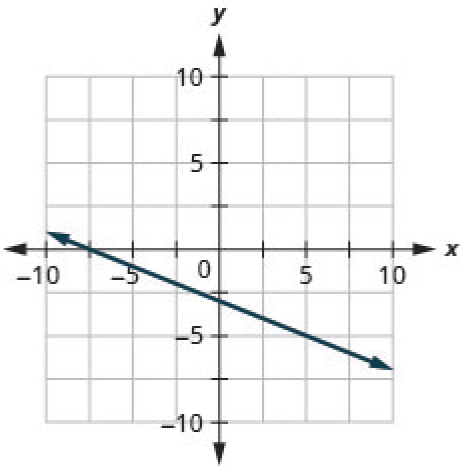
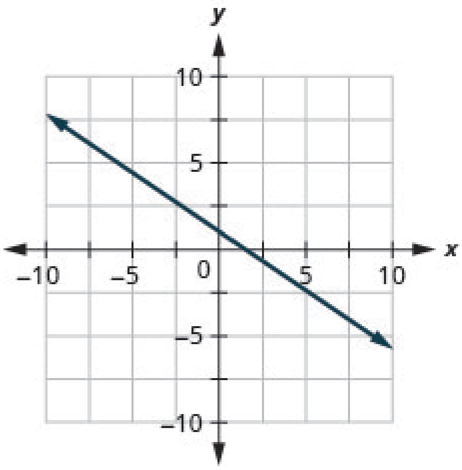
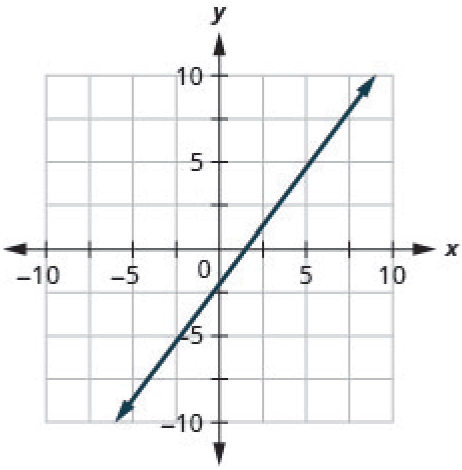
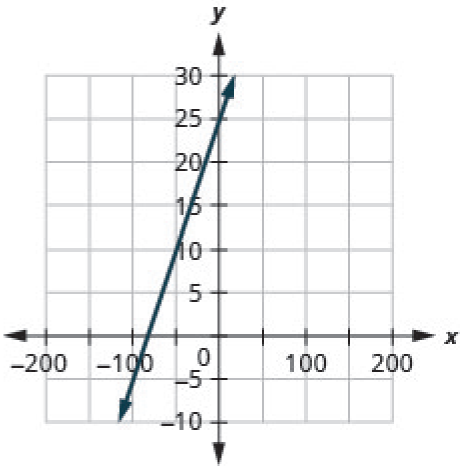
b. Explain what the *C*–intercept of the equation means.

97. Why are all horizontal lines parallel?

98. Explain in your own words how to decide which method to use to graph a line.

Answers

2. slope $m = 4$ and *y*-intercept $(0, -2)$	3. slope $m = -3$ and *y*-intercept $(0, 1)$
6. slope $m = -\frac{2}{5}$ and *y*-intercept $(0, 3)$	7. $-9; (0, 7)$
10. $4; (0, -10)$	11. $-4; (0, 8)$
14. $-\frac{8}{3}; (0, 4)$	15. $\frac{7}{3}; (0, -3)$
18. 	19. 
22. 	23. 

26.  	27. 
30.  	31. 
34. horizontal line	35. vertical line
38. slope–intercept	39. intercepts
42. slope–intercept	43. horizontal line
46. intercepts	47. slope–intercept

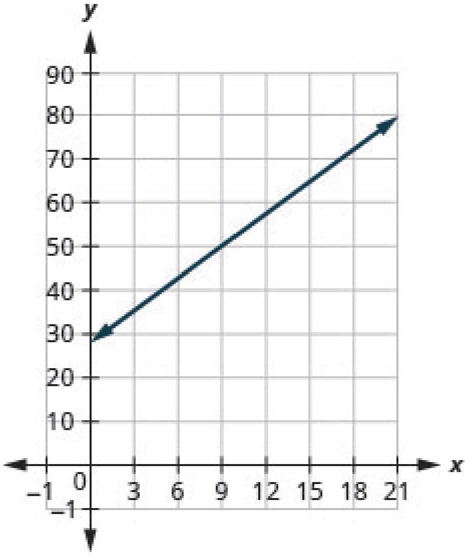
50.

a) \$28

b) \$66.10

c) The slope, 2.54, means that Randy’s payment, *P*, increases by \$2.54 when the number of units of water he used, *w*, increases by 1. The *P*–intercept means that if the number units of water Randy used was 0, the payment would be \$28.

d)



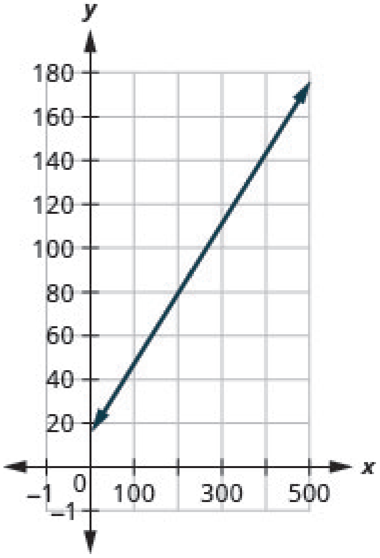
51.

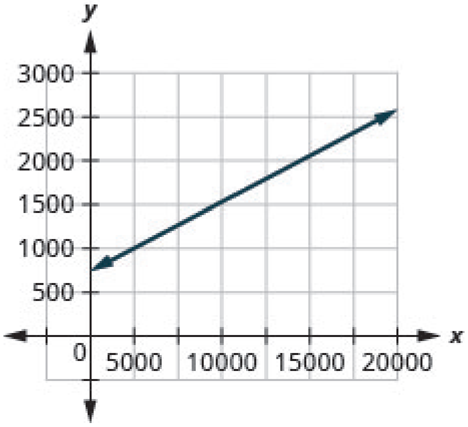
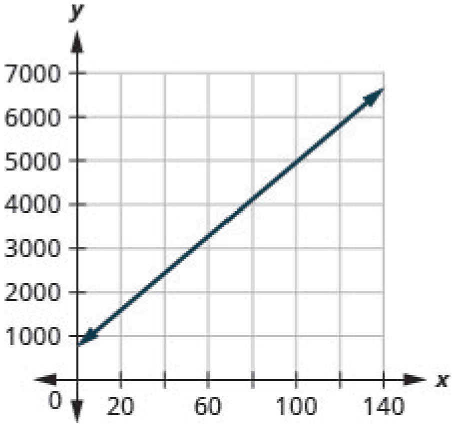
a) \$15

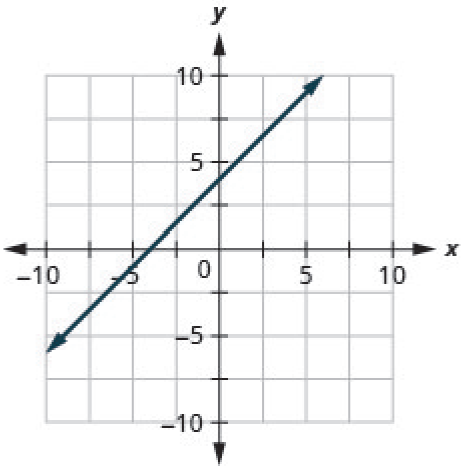
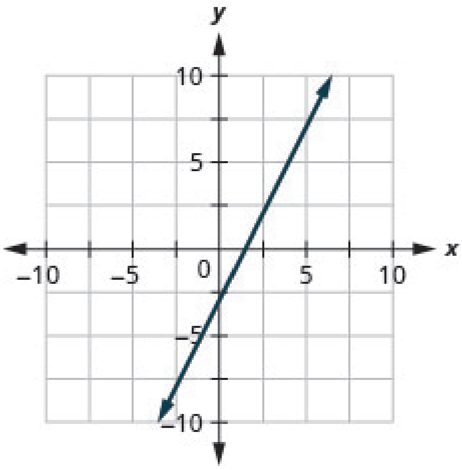
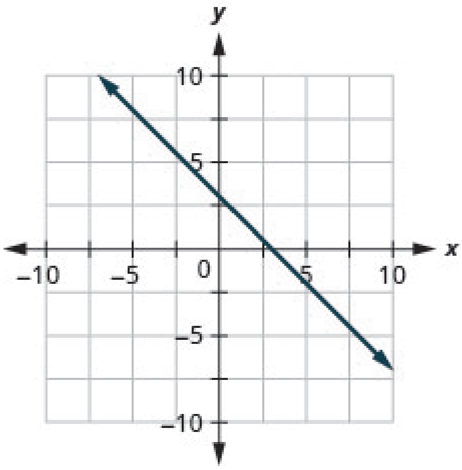
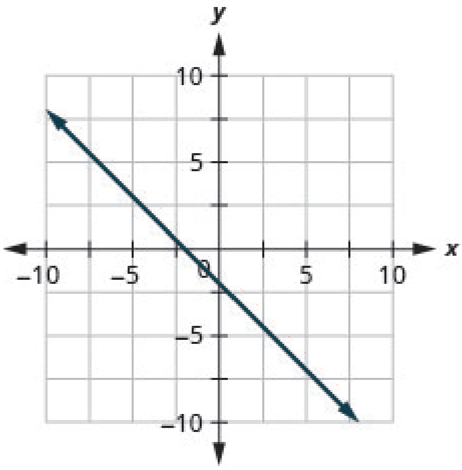
b) \$143

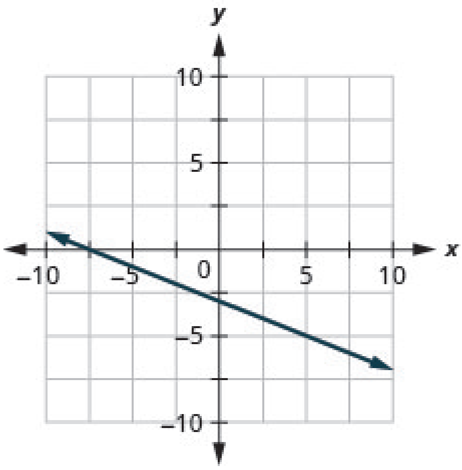
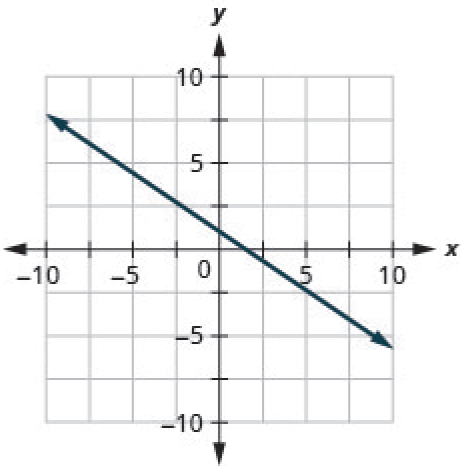
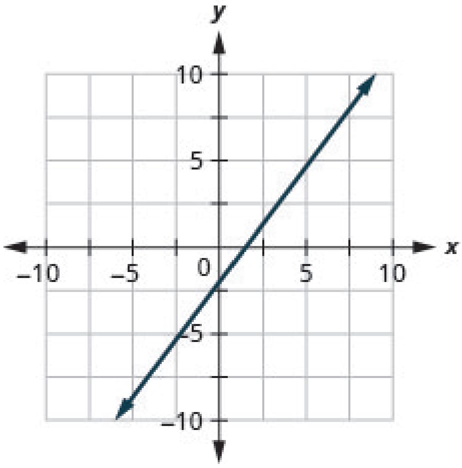
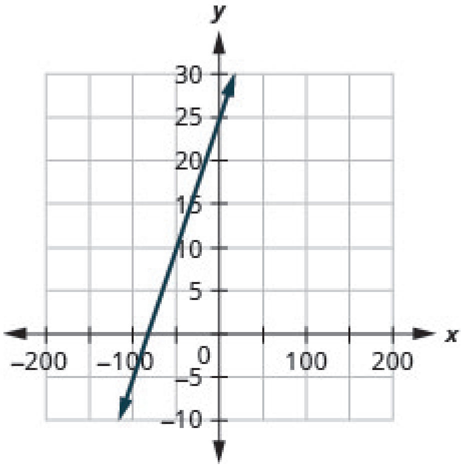
c) The slope, 0.32, means that the cost, *C*, increases by \$0.32 when the number of miles driven, *m*, increases by 1. The *C*-intercept means that if Janelle drives 0 miles one day, the cost would be \$15.

d)



54. a) \$750 b) \$2418.60 c) The slope, 0.09, means that Patel’s salary, *S*, increases by \$0.09 for every \$1 increase in his sales. The *S*-intercept means that when his sales are \$0, his salary is \$750. d) 	55. a) \$2850 b) \$4950 c) The slope, 42, means that the cost, *C*, increases by \$42 for when the number of guests increases by 1. The *C*-intercept means that when the number of guests is 0, the cost would be \$750. d) 
58. parallel	59. parallel
62. parallel	63. parallel
66. parallel	67. parallel
70. parallel	71. parallel
74. not parallel	75. not parallel
78. not parallel	79. not parallel
82. not parallel	83. perpendicular
86. perpendicular	87. not perpendicular
90. not perpendicular	91. perpendicular
94. perpendicular	95. a) For every increase of one degree Fahrenheit, the number of chirps increases by four. b) There would be -160 chirps when the Fahrenheit temperature is 0°. (Notice that this does not make sense; this model cannot be used for all possible temperatures.)
98. Answers will vary.	

1. slope $m = 4$ and *y*-intercept $(0, -2)$	3. slope $m = -3$ and *y*-intercept $(0, 1)$
5. slope $m = -\frac{2}{5}$ and *y*-intercept $(0, 3)$	7. -9; $(0, 7)$
9. 4; $(0, -10)$	11. -4; $(0, 8)$
13. $-\frac{8}{3}$; $(0, 4)$	15. $\frac{7}{3}$; $(0, -3)$
17. 	19. 
21. 	23. 

25. 	27. 
29. 	31. 
33. horizontal line	35. vertical line
37. slope–intercept	39. intercepts
41. slope–intercept	43. horizontal line
45. intercepts	47. slope–intercept

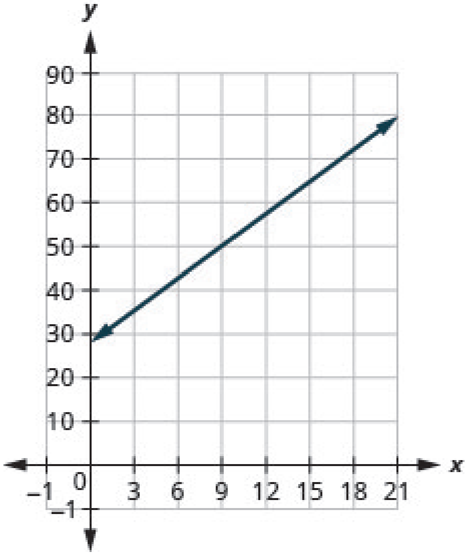
49.

a) $28

b) $66.10

c) The slope, 2.54, means that Randy’s payment, *P*, increases by $2.54 when the number of units of water he used, *w*, increases by 1. The *P*–intercept means that if the number units of water Randy used was 0, the payment would be $28.

d)



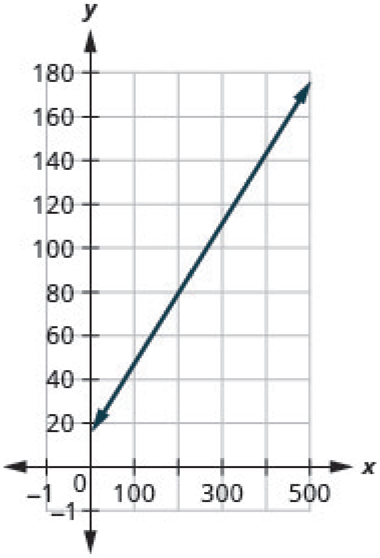
51.

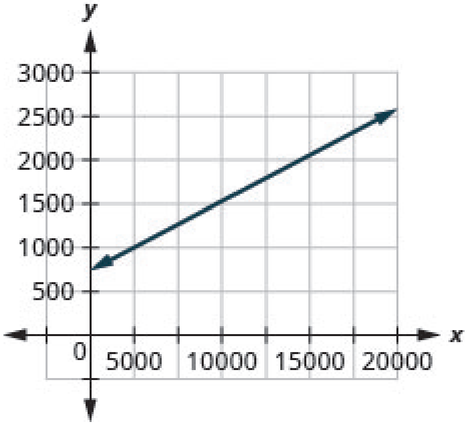
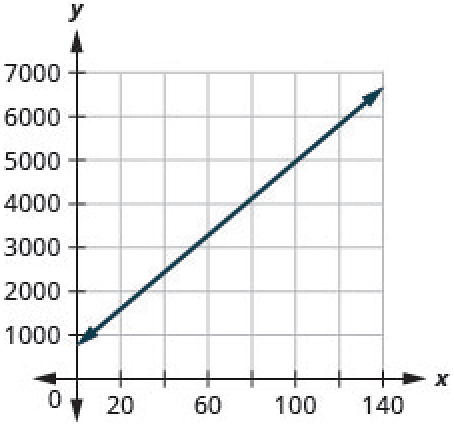
a) $15

b) $143

c) The slope, 0.32, means that the cost, *C*, increases by $0.32 when the number of miles driven, *m*, increases by 1. The *C*-intercept means that if Janelle drives 0 miles one day, the cost would be $15.

d)



53. a) \$750 b) \$2418.60 c) The slope, 0.09, means that Patel's salary, *S*, increases by \$0.09 for every \$1 increase in his sales. The *S*-intercept means that when his sales are \$0, his salary is \$750. d) 	55. a) \$2850 b) \$4950 c) The slope, 42, means that the cost, *C*, increases by \$42 for when the number of guests increases by 1. The *C*-intercept means that when the number of guests is 0, the cost would be \$750. d) 
57. parallel	59. parallel
61. parallel	63. parallel
65. parallel	67. parallel
69. parallel	71. parallel
73. not parallel	75. not parallel
77. not parallel	79. not parallel
81. not parallel	83. perpendicular
85. perpendicular	87. not perpendicular
89. not perpendicular	91. perpendicular
93. perpendicular	95. a) For every increase of one degree Fahrenheit, the number of chirps increases by four. b) There would be -160 chirps when the Fahrenheit temperature is 0°. (Notice that this does not make sense; this model cannot be used for all possible temperatures.)
97. Answers will vary.	

Attributions

This chapter has been adapted from “Use the Slope–Intercept Form of an Equation of a Line” in *Elementary Algebra* (OpenStax) by Lynn Marecek and MaryAnne Anthony-Smith, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Copyright page for more information.

6.6 Find the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

How do online retailers know that 'you may also like' a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It's all mathematics.

You are at an exciting point in your mathematical journey as the mathematics you are studying has interesting applications in the real world.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. Data is collected and graphed. If the data points appear to form a straight line, an equation of that line can be used to predict the value of one variable based on the value of the other variable.

To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and y-Intercept

We can easily determine the slope and intercept of a line if the equation was written in slope–intercept form, $y = mx + b$. Now, we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

EXAMPLE 1

Find an equation of a line with slope -7 and *y*-intercept $(0, -1)$.

Solution

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope–intercept form, $y = mx + b$.

Name the slope.	$m = -7$
Name the *y*-intercept.	*y*-intercept $(0, -1)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = -7x + (-1)$
	$y = -7x + -1$

TRY IT 1.1

Find an equation of a line with slope $\frac{2}{5}$ and *y*-intercept $(0, 4)$.

Show answer

$y = \frac{2}{5}x + 4$

TRY IT 1.2

Find an equation of a line with slope -1 and *y*-intercept $(0, -3)$.

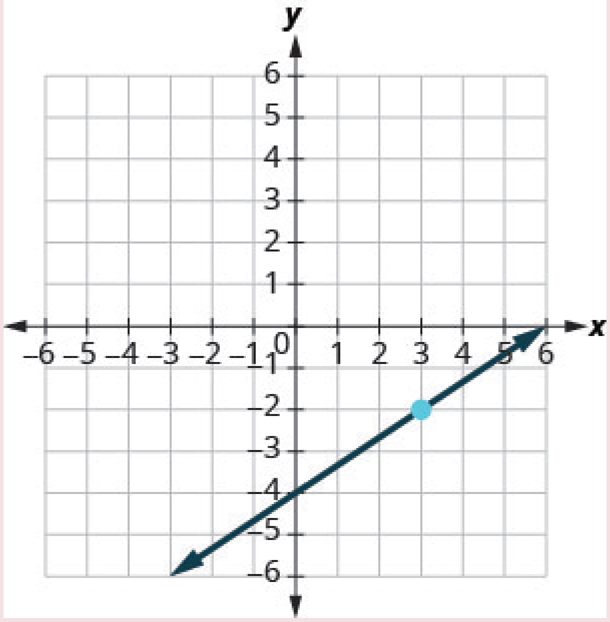
Show answer

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

EXAMPLE 2

Find the equation of the line shown.



Solution

We need to find the slope and *y*-intercept of the line from the graph so we can substitute the needed values into the slope–intercept form, $y = mx + b$.

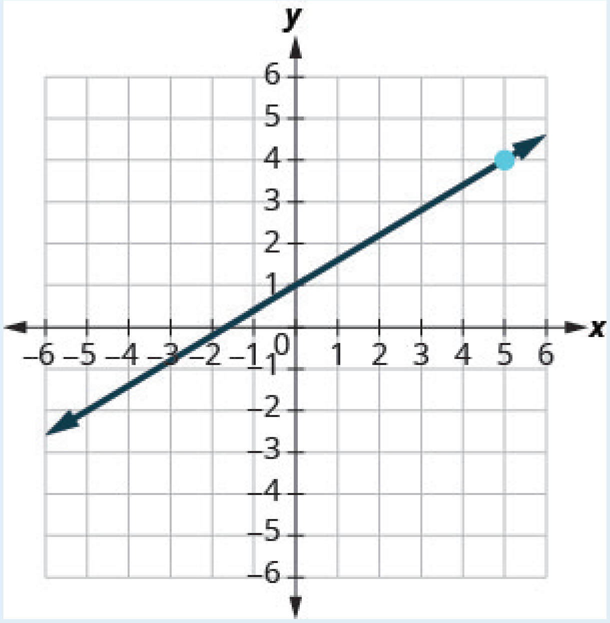
To find the slope, we choose two points on the graph.

The *y*-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$
Find the *y*-intercept.	*y*-intercept $(0, -4)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

TRY IT 2.1

Find the equation of the line shown in the graph.

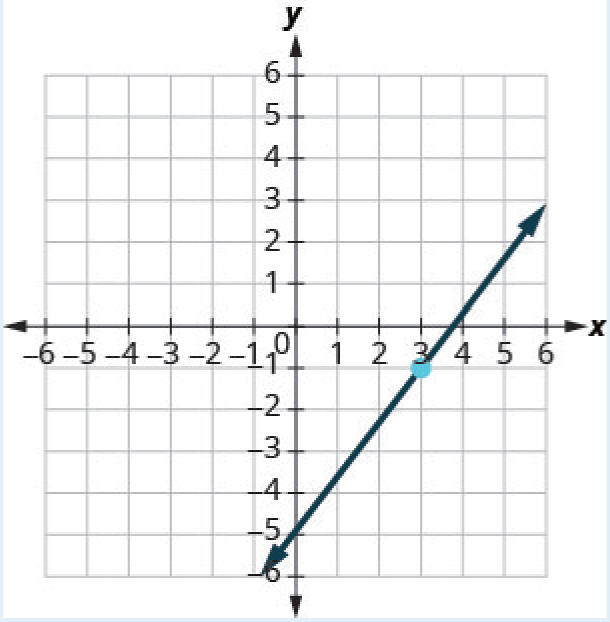


Show answer

$$y = \frac{3}{5}x + 1$$

TRY IT 2.2

Find the equation of the line shown in the graph.



Show answer

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope–intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line. Suppose we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

	$m = \frac{y - y_1}{x - x_1}$
Multiply both sides of the equation by $x - x_1$.	$m\left(x - x_1\right) \quad = \quad \left(\frac{y - y_1}{x - x_1}\right)\left(x - x_1\right)$
Simplify.	$m\left(x - x_1\right) \quad = \quad y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1 \quad = \quad m\left(x - x_1\right)$

This format is called the point–slope form of an equation of a line.

Point–slope form of an equation of a line

The point–slope form of an equation of a line with slope m and containing the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

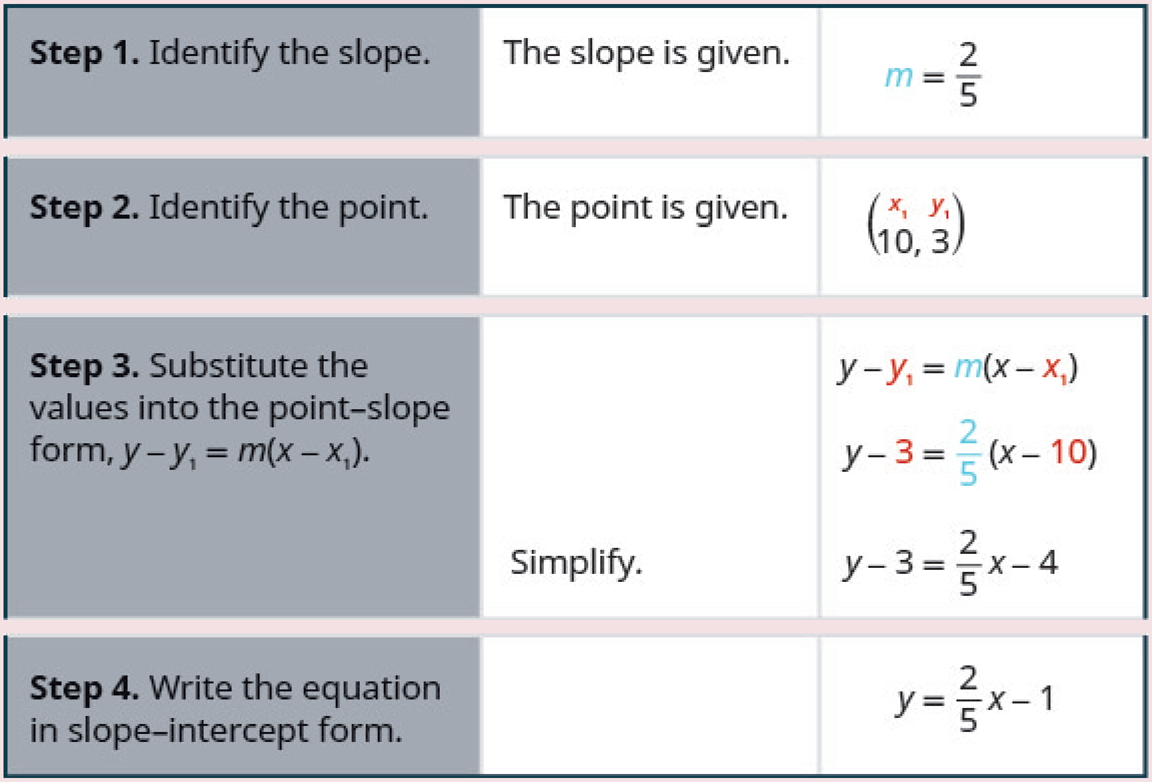
We can use the point–slope form of an equation to find an equation of a line when we are given the slope and one point. Then we will rewrite the equation in slope–intercept form. Most applications of linear equations use the the slope–intercept form.

EXAMPLE 3

Find an Equation of a Line Given the Slope and a Point

Find an equation of a line with slope $m = \frac{2}{5}$ that contains the point $(10, 3)$. Write the equation in slope–intercept form.

Solution



Step 1. Identify the slope.	The slope is given.	$m = \frac{2}{5}$
Step 2. Identify the point.	The point is given.	$\left(\overset{x_1}{10}, \overset{y_1}{3}\right)$
Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.		$y - y_1 = m(x - x_1)$ $y - 3 = \frac{2}{5}(x - 10)$
	Simplify.	$y - 3 = \frac{2}{5}x - 4$
Step 4. Write the equation in slope–intercept form.		$y = \frac{2}{5}x - 1$

TRY IT 3.1

Find an equation of a line with slope $m = \frac{5}{6}$ and containing the point $(6, 3)$.

Show answer

$y = \frac{5}{6}x - 2$

TRY IT 3.2

Find an equation of a line with slope $m = \frac{2}{3}$ and containing thepoint $(9, 2)$.

Show answer

$y = \frac{2}{3}x - 4$

HOW TO: Find an equation of a line given the slope and a point

1. Identify the slope.
2. Identify the point.
3. Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
4. Write the equation in slope–intercept form.

EXAMPLE 4

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope–intercept form.

Solution

Since we are given a point and the slope of the line, we can substitute the needed values into the point–slope form, $y - y_1 = m\left(x - x_1\right)$.

Identify the slope.	$m = -\frac{1}{3}$
Identify the point.	$\left(\overset{x_1}{6}, \overset{y_1}{-4}\right)$
Substitute the values into $y - y_1 = m\left(x - x_1\right)$.	$y - y_1 = m(x - x_1)$
	$y - (-4) = -\frac{1}{3}(x - 6)$
Simplify.	$y + 4 = -\frac{1}{3}x + 2$
Write in slope–intercept form.	$y = -\frac{1}{3}x - 2$

TRY IT 4.1

Find an equation of a line with slope $m = -\frac{2}{5}$ and containing the point $(10, -5)$.

Show answer

$$y = -\frac{2}{5}x - 1$$

TRY IT 4.2

Find an equation of a line with slope $m = -\frac{3}{4}$, and containing the point $(4, -7)$.

Show answer

$$y = -\frac{3}{4}x - 4$$

EXAMPLE 5

Find an equation of a horizontal line that contains the point $(-1, 2)$. Write the equation in slope–intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and points into the point–slope form, $y - y_1 = m\left(x - x_1\right)$.

Identify the slope.	$m = 0$
Identify the point.	$\left(\overset{x_1}{-1}, \overset{y_1}{2}\right)$
Substitute the values into $y - y_1 = m\left(x - x_1\right)$.	$y - y_1 = m(x - x_1)$
	$y - 2 = 0(x - (-1))$
Simplify.	$y - 2 = 0(x + 1)$
	$y - 2 = 0$
	$y = 2$
Write in slope–intercept form.	It is in *y*-form, but could be written $y = 0x + 2$.

Did we end up with the form of a horizontal line, $y = a$?

TRY IT 5.1

Find an equation of a horizontal line containing the point $(-3, 8)$.

Show answer

$y = 8$

TRY IT 5.2

Find an equation of a horizontal line containing the point $(-1, 4)$.

Show answer

$y = 4$

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

We have two options so far for finding an equation of a line: slope–intercept or point–slope. Since we will know two points, it will make more sense to use the point–slope form.

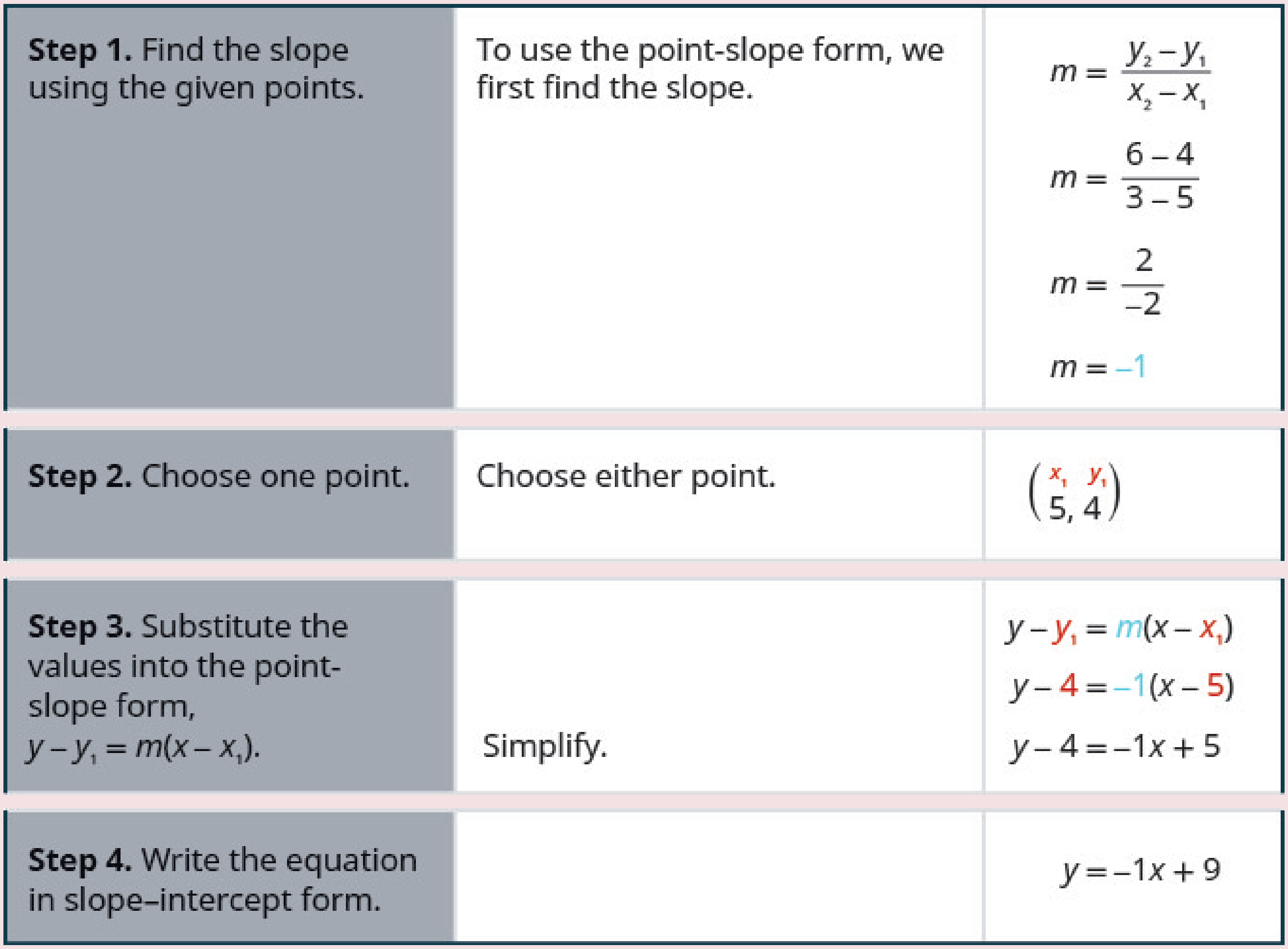
But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

EXAMPLE 6

Find an Equation of a Line Given Two Points

Find an equation of a line that contains the points $(5, 4)$ and $(3, 6)$. Write the equation in slope–intercept form.

Solution



Step 1. Find the slope using the given points.	To use the point-slope form, we first find the slope.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{6 - 4}{3 - 5}$ $m = \frac{2}{-2}$ $m = -1$
Step 2. Choose one point.	Choose either point.	$(\overset{x_1}{5}, \overset{y_1}{4})$
Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 4 = -1(x - 5)$ $y - 4 = -1x + 5$
Step 4. Write the equation in slope–intercept form.		$y = -1x + 9$

Use the point $(3, 6)$ and see that you get the same equation.

TRY IT 6.1

Find an equation of a line containing the points $(3, 1)$ and $(5, 6)$.

Show answer

$$y = \frac{5}{2}x - \frac{13}{2}$$

TRY IT 6.2

Find an equation of a line containing the points $(1, 4)$ and $(6, 2)$.

Show answer

$$y = -\frac{2}{5}x + \frac{22}{5}$$

HOW TO: Find an equation of a line given two points

1. Find the slope using the given points.
2. Choose one point.
3. Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
4. Write the equation in slope–intercept form.

EXAMPLE 7

Find an equation of a line that contains the points $(-3, -1)$ and $(2, -2)$. Write the equation in slope–intercept form.

Solution

Since we have two points, we will find an equation of the line using the point–slope form. The first step will be to find the slope.

Find the slope of the line through (−3, −1) and (2, −2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$
	$m = \frac{-2-(-1)}{2-(-3)}$
	$m = \frac{-1}{5}$
	$m = -\frac{1}{5}$
Choose either point.	$\left(\overset{x_1}{2}, \overset{y_1}{-2}\right)$
Substitute the values into $y - y_1 = m\left(x - x_1\right)$.	$y - y_1 = m(x - x_1)$
	$y - (-2) = -\frac{1}{5}(x - 2)$
	$y + 2 = -\frac{1}{5}x + \frac{2}{5}$
Write in slope–intercept form.	$y = -\frac{1}{5}x - \frac{8}{5}$

TRY IT 7.1

Find an equation of a line containing the points $(-2, -4)$ and $(1, -3)$.

Show answer

$$y = \frac{1}{3}x - \frac{10}{3}$$

TRY IT 7.2

Find an equation of a line containing the points $(-4, -3)$ and $(1, -5)$.

Show answer

$$y = -\frac{2}{5}x - \frac{23}{5}$$

EXAMPLE 8

Find an equation of a line that contains the points $(-2, 4)$ and $(-2, -3)$. Write the equation in slope–intercept form.

Solution
Again, the first step will be to find the slope.

Find the slope of the line through (-2,4) and (-2,-3).	$m = \frac{y_2 - y_1}{x_2 - x_1}$
	$m = \frac{-3-4}{-2-(-2)}$
	$m = \frac{-7}{0}$
	The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -2. So our equation of the line is $x = -2$. Since there is no y, we cannot write it in slope–intercept form.

You may want to sketch a graph using the two given points. Does the graph agree with our conclusion that this is a vertical line?

TRY IT 8.1

Find an equation of a line containing the points $(5, 1)$ and $(5, -4)$.

Show answer
$x = 5$

TRY IT 8.2

Find an equation of a line containing the points $(-4, 4)$ and $(-4, 3)$.

Show answer

$x = -4$

We have seen that we can use either the slope–intercept form or the point–slope form to find an equation of a line. Which form we use will depend on the information we are given. This is summarized in the following table.

To Write an Equation of a Line

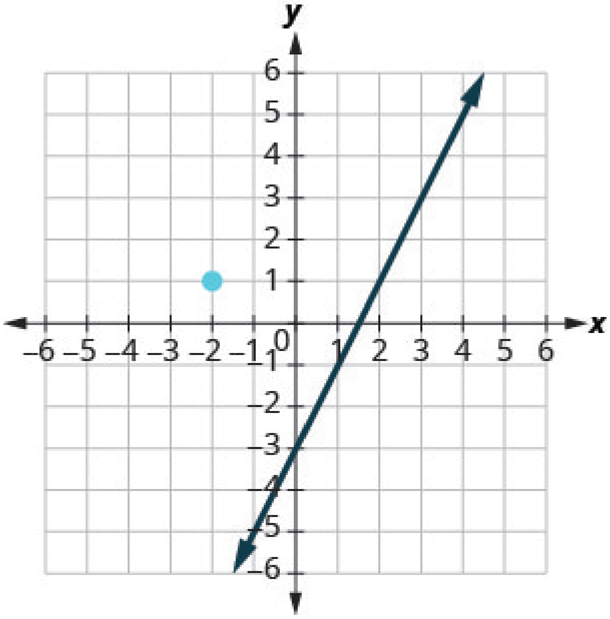
If given:	**Use:**	**Form:**
Slope and *y*-intercept	slope–intercept	$y = mx + b$
Slope and a point	point–slope	$y - y_1 = m(x - x_1)$
Two points	point–slope	$y - y_1 = m(x - x_1)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point–slope equation.

First let’s look at this graphically.

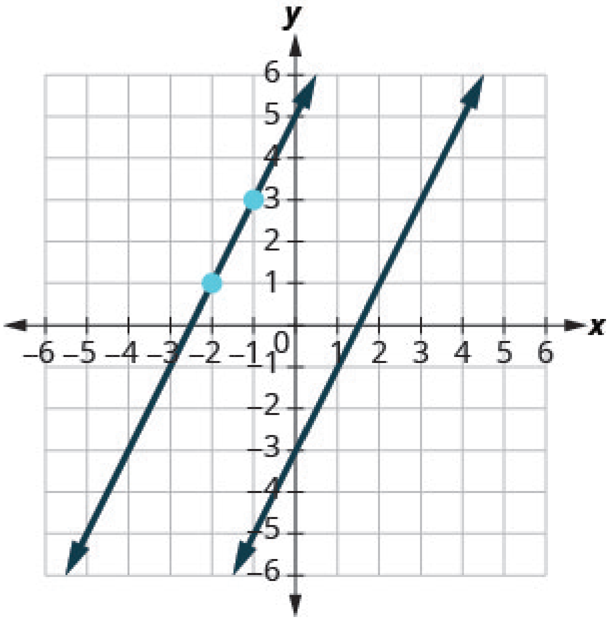
The graph shows the graph of $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2, 1)$.



We know that parallel lines have the same slope. So the second line will have the same slope as

$y = 2x - 3$. That slope is $m_{\|} = 2$. We'll use the notation $m_{\|}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $_{\|}$ looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$. To graph the line, we start at $(-2, 1)$ and count out the rise and run. With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line.



Do the lines appear parallel? Does the second line pass through $(-2, 1)$?

Now, let's see how to do this algebraically.

We can use either the slope–intercept form or the point–slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point–slope form.

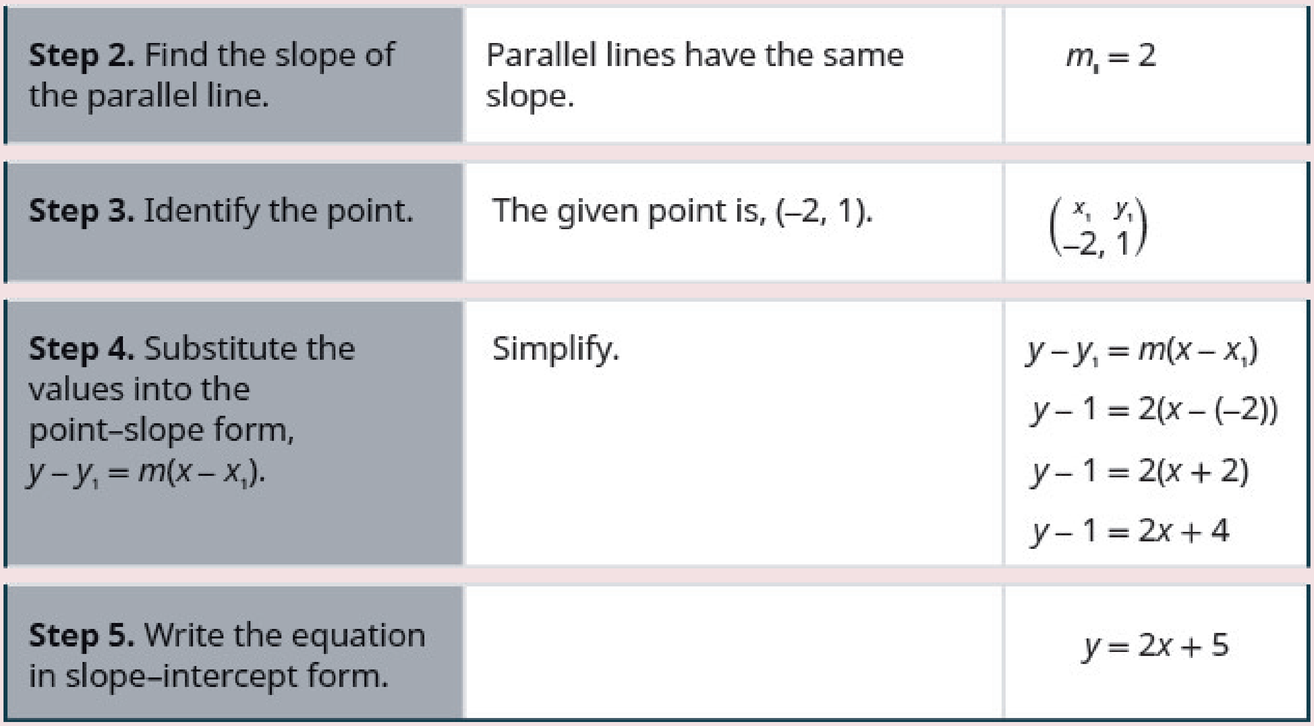
EXAMPLE 9

How to Find an Equation of a Line Parallel to a Given Line

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope–intercept form.

Solution

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$



Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_1 = 2$
Step 3. Identify the point.	The given point is, (–2, 1).	$\begin{pmatrix} x_1, & y_1 \\ -2, & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Step 5. Write the equation in slope–intercept form.		$y = 2x + 5$

Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

TRY IT 9.1

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope–intercept form.

Show answer

$y = 3x - 10$

TRY IT 9.2

Find an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Show answer

$y = \frac{1}{2}x + 1$

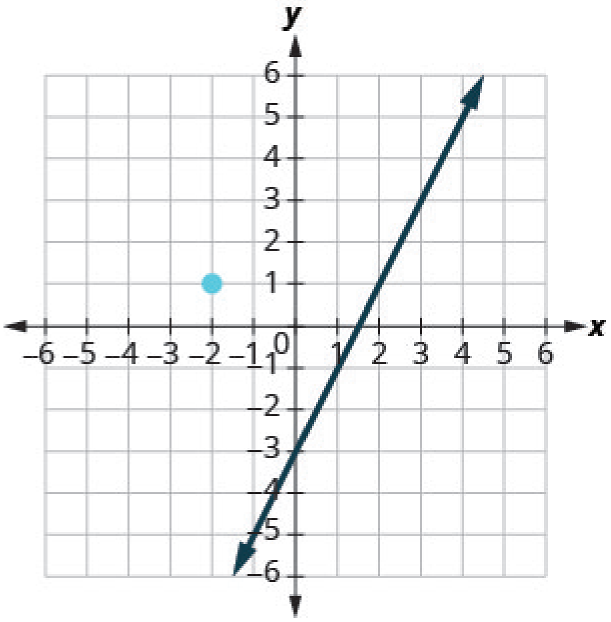
HOW TO: Find an equation of a line parallel to a given line

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point–slope form, $y - y_1 = m\left(x - x_1\right)$.
5. Write the equation in slope–intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let’s consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point–slope equation, like we did with parallel lines.

The graph shows the graph of $y = 2x - 3$. Now, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.

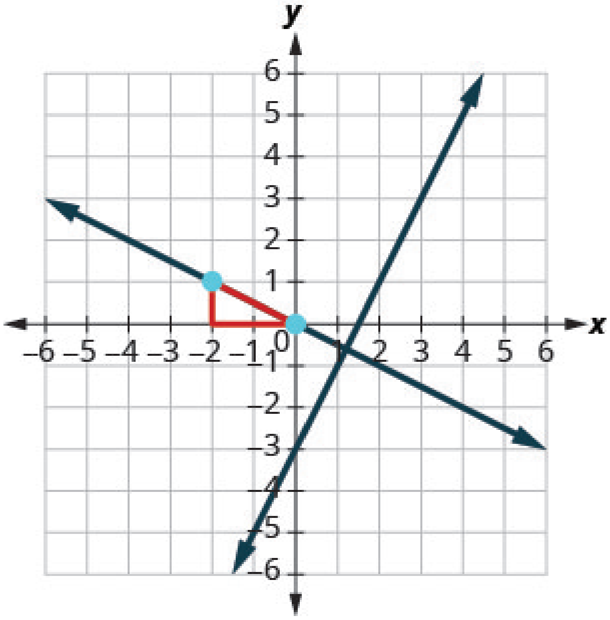


We know that perpendicular lines have slopes that are negative reciprocals. We’ll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$$\begin{array}{ll} y = 2x - 3 & \text{perpendicular line} \\ m = 2 & m_{\perp} = -\dfrac{1}{2} \end{array}$$

We now know the perpendicular line will pass through $(-2, 1)$ with $m_{\perp} = -\frac{1}{2}$.

To graph the line, we will start at $(-2, 1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through $(-2, 1)$?

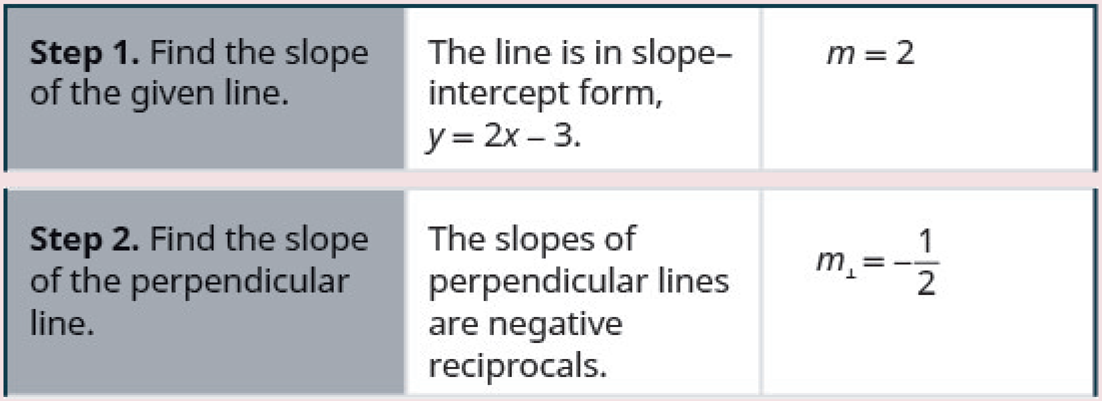
Now, let's see how to do this algebraically. We can use either the slope–intercept form or the point–slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point–slope form.

EXAMPLE 10

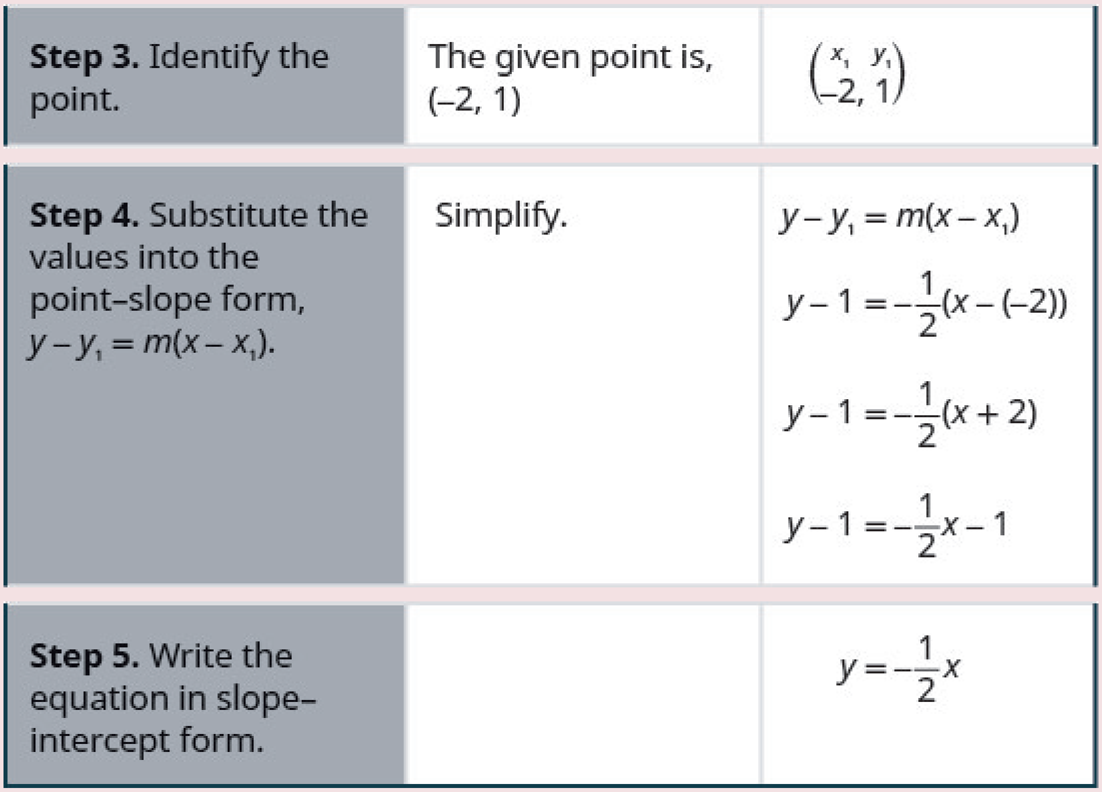
How to Find an Equation of a Line Perpendicular to a Given Line

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope–intercept form.

Solution



Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp} = -\frac{1}{2}$



Step 3. Identify the point.	The given point is, (–2, 1)	$\begin{pmatrix} x_1 & y_1 \\ -2, & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = -\frac{1}{2}(x - (-2))$ $y - 1 = -\frac{1}{2}(x + 2)$ $y - 1 = -\frac{1}{2}x - 1$
Step 5. Write the equation in slope–intercept form.		$y = -\frac{1}{2}x$

TRY IT 10.1

Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope–intercept form.

Show answer

$$y = -\frac{1}{3}x + \frac{10}{3}$$

TRY IT 10.2

Find an equation of a line perpendicular to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Show answer

$$y = -2x + 16$$

HOW TO: Find an equation of a line perpendicular to a given line

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point–slope form, $y - y_1 = m\left(x - x_1\right)$.
5. Write the equation in slope–intercept form.

EXAMPLE 11

Find an equation of a line perpendicular to $x = 5$ that contains the point $\left(3, -2\right)$. Write the equation in slope–intercept form.

Solution
Again, since we know one point, the point–slope option seems more promising than the slope–intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x = 5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_{\perp} = 0$.

Identify the point.	$(3, -2)$
Identify the slope of the perpendicular line.	$m_{\perp} = 0$
Substitute the values into $y - y_1 = m\left(x - x_1\right)$.	$y - y_1 = m\left(x - x_1\right)$ $y - (-2) = 0\left(x - 3\right)$ $y + 2 = 0$
Simplify.	$y = -2$

Sketch the graph of both lines. Do they appear to be perpendicular?

TRY IT 11.1

Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $\left(4, -5\right)$. Write the equation in slope–intercept form.

Show answer

$y = -5$

TRY IT 11.2

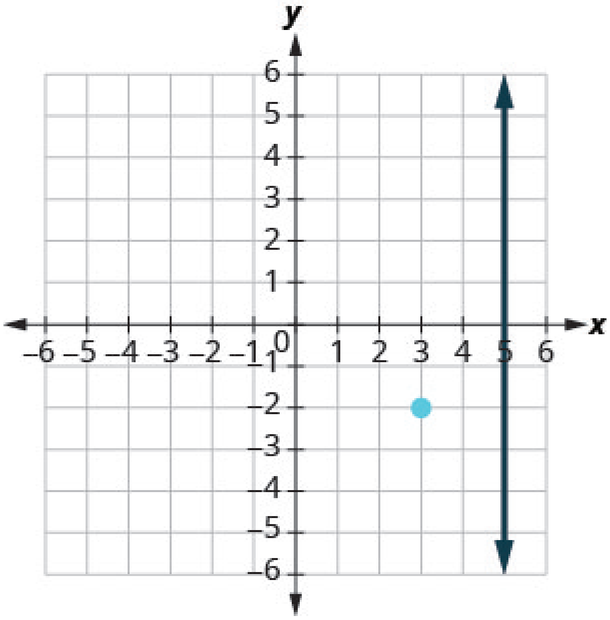
Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope–intercept form.

Show answer

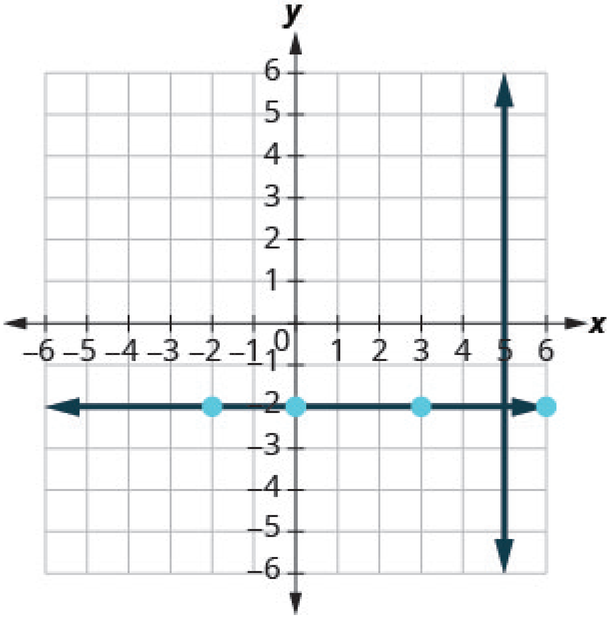
$y = -1$

In (Example 11), we used the point–slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. The graph shows us the line$x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.

EXAMPLE 12

Find an equation of a line that is perpendicular to $y = -4$ that contains the point $(-4, 2)$.

Write the equation in slope–intercept form.

Solution
The line $y = -4$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-4, 2)$, every point on it has an *x*-coordinate of -4. The equation of the perpendicular line is $x = -4$. You may want to sketch the lines. Do they appear perpendicular?

TRY IT 12.1

Find an equation of a line that is perpendicular to the line $y = 1$ that contains the point $(-5, 1)$. Write the equation in slope–intercept form.

Show answer
$x = -5$

TRY IT 12.1

Find an equation of a line that is perpendicular to the line $y = -5$ that contains the point $(-4, -5)$.

Show answer

$x = -4$

Access this online resource for additional instruction and practice with finding the equation of a line.

- Use the Point-Slope Form of an Equation of a Line

Key Concepts

- **To Find an Equation of a Line Given the Slope and a Point**
  1. Identify the slope.
  2. Identify the point.
  3. Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
  4. Write the equation in slope-intercept form.
- **To Find an Equation of a Line Given Two Points**
  1. Find the slope using the given points.
  2. Choose one point.
  3. Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
  4. Write the equation in slope-intercept form.
- **To Write and Equation of a Line**
  - If given slope and *y*-intercept, use slope–intercept form $y = mx + b$.
  - If given slope and a point, use point–slope form $y - y_1 = m\left(x - x_1\right)$.
  - If given two points, use point–slope form $y - y_1 = m\left(x - x_1\right)$.
- **To Find an Equation of a Line Parallel to a Given Line**
  1. Find the slope of the given line.
  2. Find the slope of the parallel line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
  5. Write the equation in slope-intercept form.

- **To Find an Equation of a Line Perpendicular to a Given Line**

    1. Find the slope of the given line.
    2. Find the slope of the perpendicular line.
    3. Identify the point.
    4. Substitute the values into the point-slope form, $y-y_1=m\left(x-x_1\right)$.
    5. Write the equation in slope-intercept form.

Glossary

point–slope form
The point–slope form of an equation of a line with slope m and containing the point $\left(x_1,y_1\right)$ is $y-y_1=m\left(x-x_1\right)$.

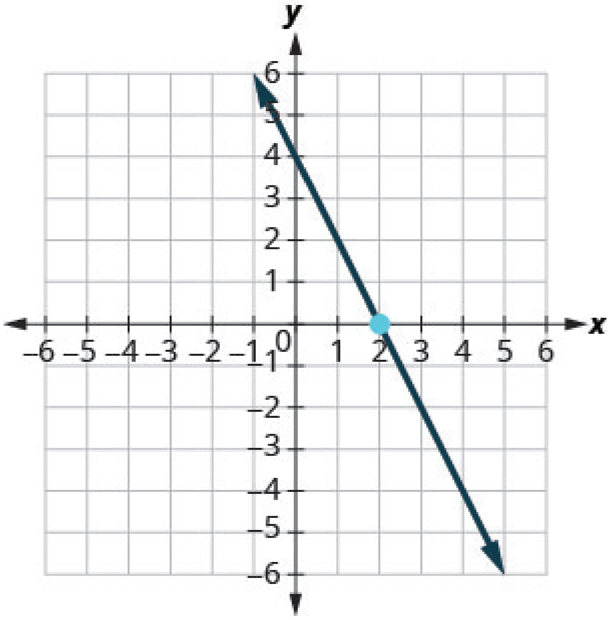
Practice Makes Perfect

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

1. slope 4 and *y*-intercept $(0,1)$	2. slope 3 and *y*-intercept $(0,5)$
3. slope 8 and *y*-intercept $(0,-6)$	4. slope 6 and *y*-intercept $(0,-4)$
5. slope -1 and *y*-intercept $(0,7)$	6. slope -1 and *y*-intercept $(0,3)$
7. slope -3 and *y*-intercept $(0,-1)$	8. slope -2 and *y*-intercept $(0,-3)$
9. slope $\frac{1}{5}$ and *y*-intercept $(0,-5)$	10. slope $\frac{3}{5}$ and *y*-intercept $(0,-1)$
11. slope $-\frac{2}{3}$ and *y*-intercept $(0,-3)$	12. slope $-\frac{3}{4}$ and *y*-intercept $(0,-2)$
13. slope 0 and *y*-intercept $(0,2)$	14. slope 0 and *y*-intercept $(0,-1)$
15. slope -4 and *y*-intercept $(0,0)$	16. slope -3 and *y*-intercept $(0,0)$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.


y
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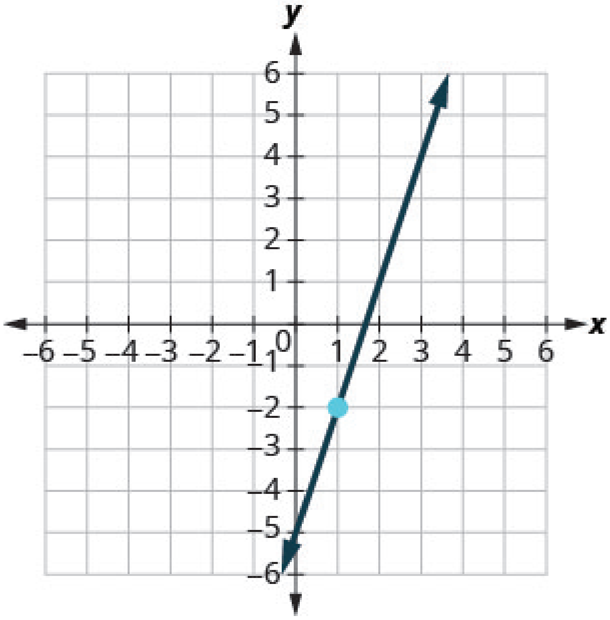
17.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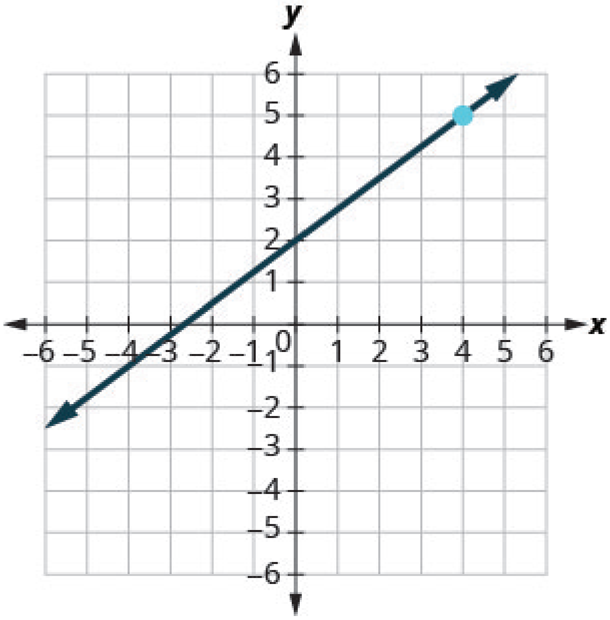
18.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19.

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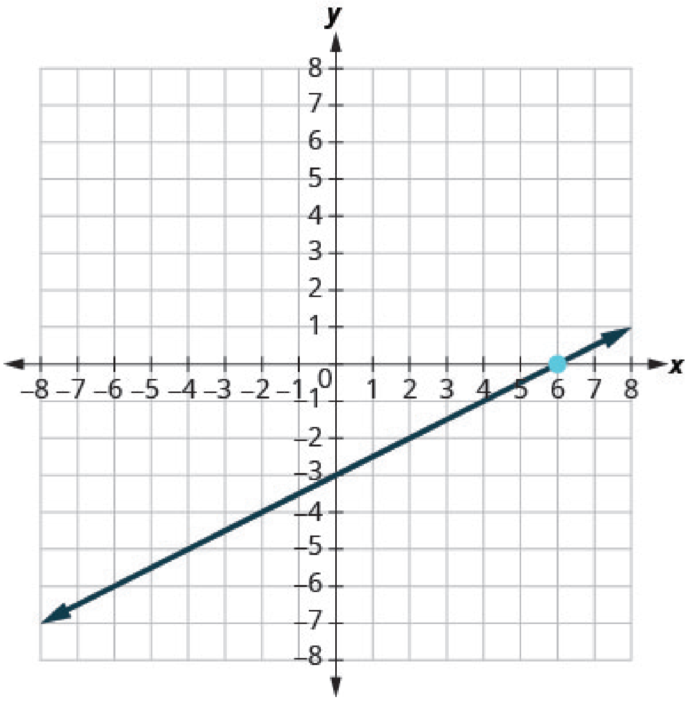
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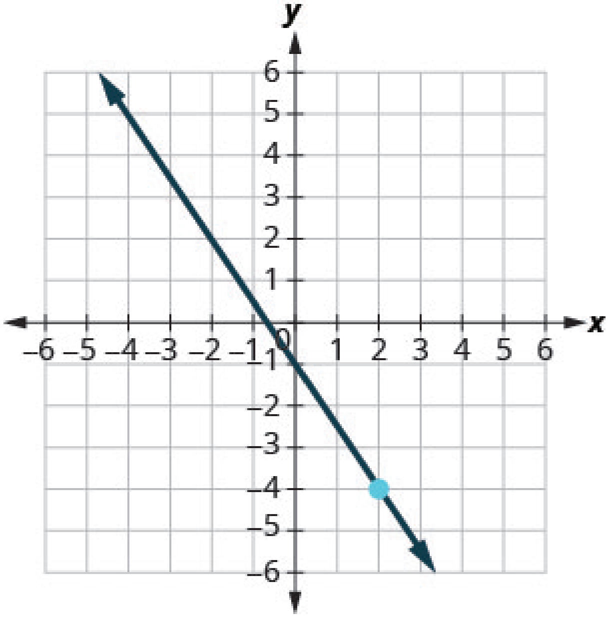

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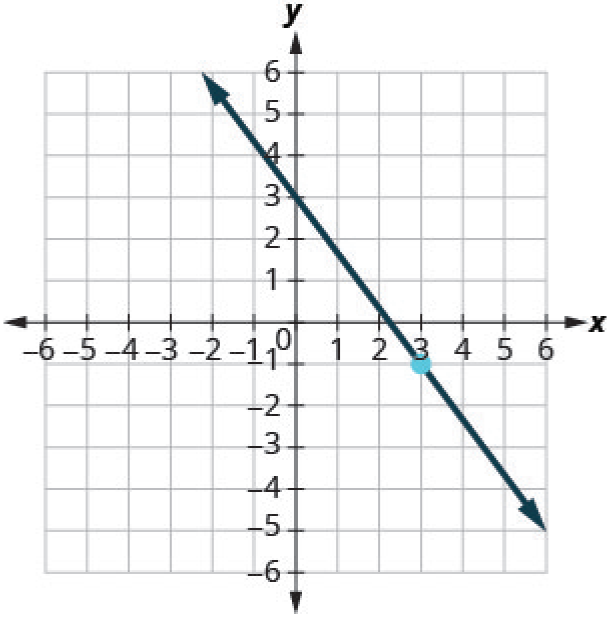

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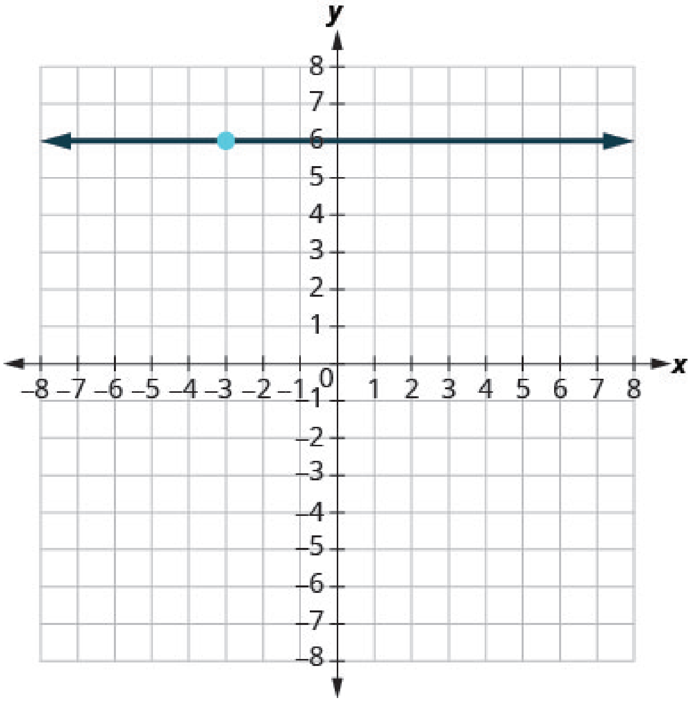

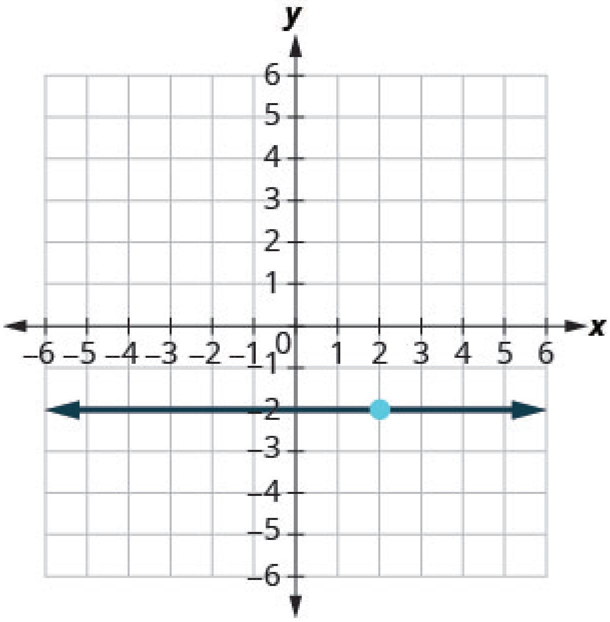

24.

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

25. $m = \frac{3}{8}$, point $(8, 2)$

26. $m = \frac{5}{8}$, point $(8, 3)$

27. $m = \frac{5}{6}$, point $(6, 7)$

28. $m = \frac{1}{6}$, point $(6, 1)$

29. $m = -\frac{3}{5}$, point $(10, -5)$

30. $m = -\frac{3}{4}$, point $(8, -5)$

31. $m = -\frac{1}{3}$, point $(-9, -8)$

32. $m = -\frac{1}{4}$, point $(-12, -6)$

33. Horizontal line containing $(-1, 4)$

34. Horizontal line containing $(-2, 5)$

35. Horizontal line containing $(-1, -7)$

36. Horizontal line containing $(-2, -3)$

37. $m = -\frac{5}{2}$, point $(-8, -2)$

38. $m = -\frac{3}{2}$, point $(-4, -3)$

39. $m = -4$, point $(-2, -3)$

40. $m = -7$, point $(-1, -3)$

41. Horizontal line containing $(4, -8)$

42. Horizontal line containing $(2, -3)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

43. $(3, 1)$ and $(2, 5)$

44. $(2, 6)$ and $(5, 3)$

45. $(2, 7)$ and $(3, 8)$

46. $(4, 3)$ and $(8, 1)$

47. $(-5, -3)$ and $(4, -6)$

48. $(-3, -4)$ and $(5 - 2)$

49. $(-2, 8)$ and $(-4, -6)$

50. $(-1, 3)$ and $(-6, -7)$

51. $(3, -2)$ and $(-4, 4)$

52. $(6, -4)$ and $(-2, 5)$

53. $(0, -2)$ and $(-5, -3)$

54. $(0, 4)$ and $(2, -3)$

55. $(4, 2)$ and $(4, -3)$

56. $(7, 2)$ and $(7, -2)$

57. $(-2, 1)$ and $(-2, -4)$

58. $(-7, -1)$ and $(-7, -4)$

59. $(6, 2)$ and $(-3, 2)$

60. $(6, 1)$ and $(0, 1)$

61. $(-6, -3)$ and $(-1, -3)$

62. $(3, -4)$ and $(5, -4)$

63. $(0, 0)$ and $(1, 4)$

64. $(4, 3)$ and $(8, 0)$

65. $(-3, 0)$ and $(-7, -2)$

66. $(-2, -3)$ and $(-5, -6)$

67. $(3, 5)$ and $(-7, 5)$

68. $(8, -1)$ and $(8, -5)$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

69. line $y = 3x + 4$, point $(2, 5)$	70. line $y = 4x + 2$, point $(1, 2)$
71. line $y = -3x - 1$, point $(2, -3)$	72. line $y = -2x - 3$, point $(-1, 3)$
73. line $2x - y = 6$, point $(3, 0)$	74. line $3x - y = 4$, point $(3, 1)$
75. line $2x + 3y = 6$, point $(0, 5)$	76. line $4x + 3y = 6$, point $(0, -3)$
77. line $x = -4$, point $(-3, -5)$	78. line $x = -3$, point $(-2, -1)$
79. line $x - 6 = 0$, point $(4, -3)$	80. line $x - 2 = 0$, point $(1, -2)$
81. line $y = 1$, point $(3, -4)$	82. line $y = 5$, point $(2, -2)$
83. line $y + 7 = 0$, point $(1, -1)$	84. line $y + 2 = 0$, point $(3, -3)$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

85. line $y = -x + 5$, point $(3, 3)$	86. line $y = -2x + 3$, point $(2, 2)$
87. line $y = \frac{2}{3}x - 4$, point $(2, -4)$	88. line $y = \frac{3}{4}x - 2$, point $(-3, 4)$
89. line $4x - 3y = 5$, point $(-3, 2)$	90. line $2x - 3y = 8$, point $(4, -1)$
91. line $4x + 5y = -3$, point $(0, 0)$	92. line $2x + 5y = 6$, point $(0, 0)$
93. line $y - 6 = 0$, point $(-5, -3)$	94. line $y - 3 = 0$, point $(-2, -4)$
95. line *y*-axis, point $(2, 1)$	96. line *y*-axis, point $(3, 4)$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope–intercept form.

97. Containing the points $(2, 7)$ and $(3, 8)$

98. Containing the points $(4, 3)$ and $(8, 1)$

99. $m = \frac{5}{6}$, containing point $(6, 7)$

100. $m = \frac{1}{6}$, containing point $(6, 1)$

101. Parallel to the line $2x + 3y = 6$, containing point $(0, 5)$

102. Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

103. $m = -\frac{3}{5}$, containing point $(10, -5)$

104. $m = -\frac{3}{4}$, containing point $(8, -5)$

105. Perpendicular to the line *y*-axis, point $(-6, 2)$

106. Perpendicular to the line $y - 1 = 0$, point $(-2, 6)$

107. Containing the points $(-2, 0)$ and $(-3, -2)$

108. Containing the points $(4, 3)$ and $(8, 1)$

109. Parallel to the line $x = -4$, containing point $(-3, -5)$

110. Parallel to the line $x = -3$, containing point $(-2, -1)$

111. Containing the points $(-5, -3)$ and $(4, -6)$

112. Containing the points $(-3, -4)$ and $(2, -5)$

113. Perpendicular to the line $4x + 3y = 1$, containing point $(0, 0)$

114. Perpendicular to the line $x - 2y = 5$, containing point $(-2, 2)$

Everyday Math

115. **Fuel consumption.** The city mpg, x, and highway mpg, y, of two cars are given by the points $(29, 40)$ and $(19, 28)$. Find a linear equation that models the relationship between city mpg and highway mpg.

116. **Cholesterol.** The age, x, and LDL cholesterol level, y, of two men are given by the points $(18, 68)$ and $(27, 122)$. Find a linear equation that models the relationship between age and LDL cholesterol level.

Writing Exercises

117. Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

118. Why are all horizontal lines parallel?

Answers

1. $y = 4x + 1$	3. $y = 8x - 6$
5. $y = -x + 7$	7. $y = -3x - 1$
9. $y = \frac{1}{5}x - 5$	11. $y = -\frac{2}{3}x - 3$
13. $y = 2$	15. $y = -4x$
17. $y = -2x + 4$	19. $y = \frac{3}{4}x + 2$
21. $y = -\dfrac{3}{2}x - 1$	23. $y = 6$
25. $y = \frac{3}{8}x - 1$	27. $y = \frac{5}{6}x + 2$
29. $y = -\frac{3}{5}x + 1$	31. $y = -\frac{1}{3}x - 11$
33. $y = 4$	35. $y = -7$
37. $y = -\frac{5}{2}x - 22$	39. $y = -4x - 11$
41. $y = -8$	43. $y = -4x + 13$
45. $y = x + 5$	47. $y = -\frac{1}{3}x - \frac{14}{3}$
49. $y = 7x + 22$	51. $y = -\frac{6}{7}x + \frac{4}{7}$
53. $y = \frac{1}{5}x - 2$	55. $x = 4$
57. $x = -2$	59. $y = 2$
61. $y = -3$	63. $y = 4x$
65. $y = \frac{1}{2}x + \frac{3}{2}$	67. $y = 5$
69. $y = 3x - 1$	71. $y = -3x + 3$
73. $y = 2x - 6$	75. $y = -\frac{2}{3}x + 5$
77. $x = -3$	79. $x = 4$
81. $y = -4$	83. $y = -1$
85. $y = x$	87. $y = -\frac{3}{2}x - 1$
89. $y = -\frac{3}{4}x - \frac{1}{4}$	91. $y = \frac{5}{4}x$
93. $x = -5$	95. $y = 1$
97. $y = x + 5$	99. $y = \frac{5}{6}x + 2$
101. $y = -\frac{2}{3}x + 5$	103. $y = -\frac{3}{5}x + 1$

105. $y = 2$	107. $y = x + 2$
109. $x = -3$	111. $y = -\frac{1}{3}x - \frac{14}{3}$
113. $y = \frac{3}{4}x$	115. $y = 1.2x + 5.2$
117. Answers will vary.	

Attributions

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6.7 Chapter Review

Review Exercises

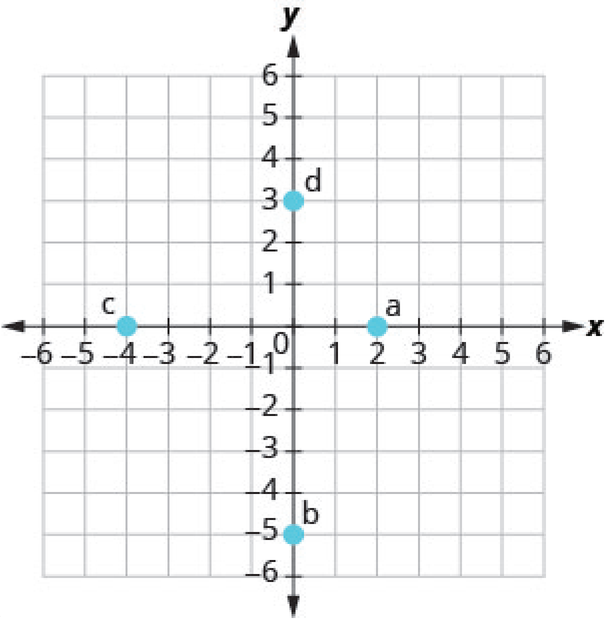
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

1. a) $(4, 3)$ b) $(-4, 3)$ c) $(-4, -3)$ d) $(4, -3)$	2. a) $(-1, -5)$ b) $(-3, 4)$ c) $(2, -3)$ d) $\left(1, \frac{5}{2}\right)$
3. a) $\left(2, \frac{3}{2}\right)$ b) $\left(3, \frac{4}{3}\right)$ c) $\left(\frac{1}{3}, -4\right)$ d) $\left(\frac{1}{2}, -5\right)$	4. a) $(-2, 0)$ b) $(0, -4)$ c) $(0, 5)$ d) $(3, 0)$

Identify Points on a Graph

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.



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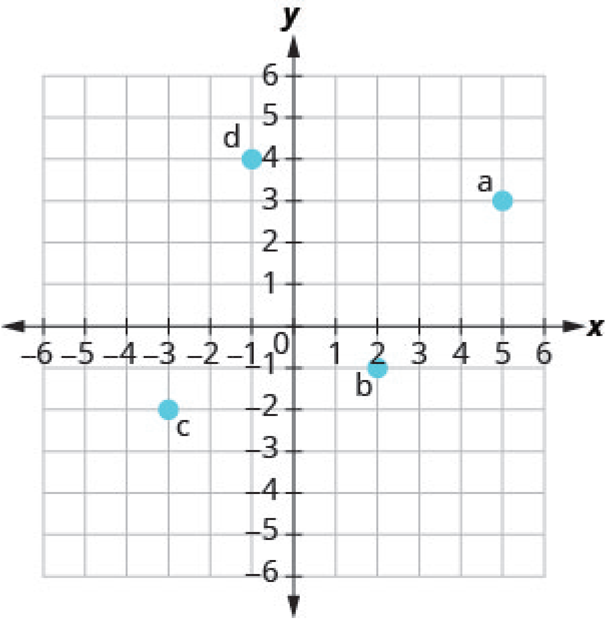


6.

Verify Solutions to an Equation in Two Variables

In the following exercises, which ordered pairs are solutions to the given equations?

7. $y = 6x - 2$ a) $(1, 4)$ b) $\left(\frac{1}{3}, 0\right)$ c) $(6, -2)$	8. $5x + y = 10$ a) $(5, 1)$ b) $(2, 0)$ c) $(4, -10)$

Complete a Table of Solutions to a Linear Equation in Two Variables

In the following exercises, complete the table to find solutions to each linear equation.

9. $y = -\frac{1}{2}x + 3$

x	y	(x, y)
0		
4		
-2		

10. $y = 4x - 1$

x	y	(x, y)
0		
1		
-2		

11. $3x + 2y = 6$

x	y	(x, y)
0		
	0	
-2		

12. $x + 2y = 5$

x	y	(x, y)
	0	
1		
-1		

Find Solutions to a Linear Equation in Two Variables

In the following exercises, find three solutions to each linear equation.

13. $x + y = -4$

14. $x + y = 3$

15. $y =$ -$x - 1$

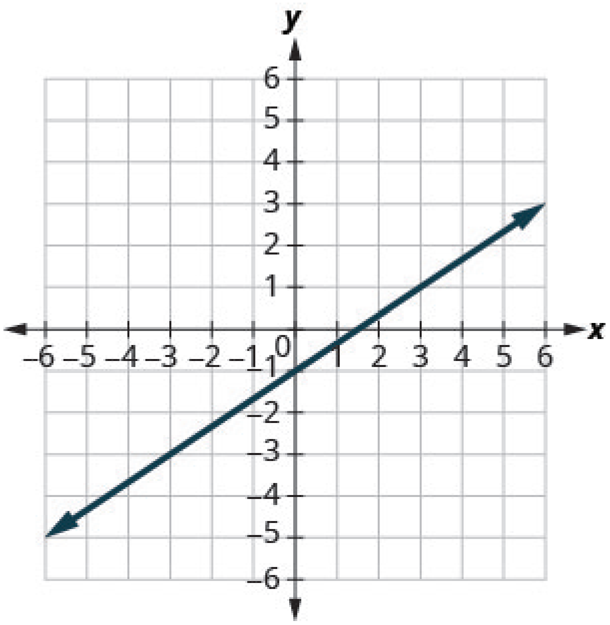
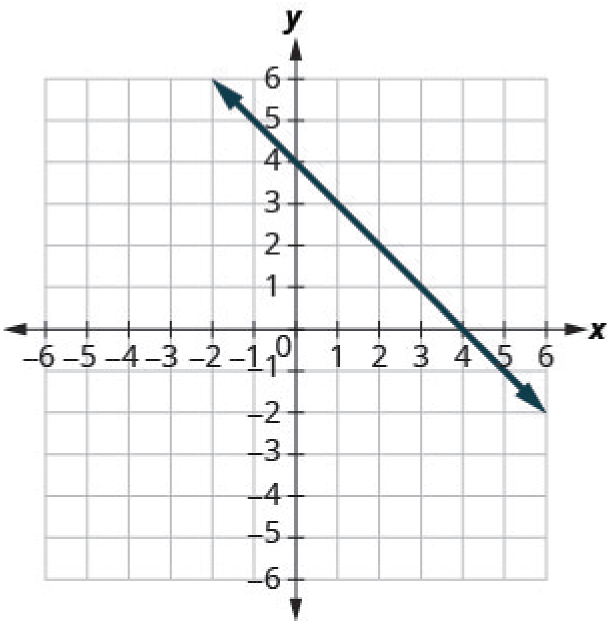
16. $y = 3x + 1$

Recognize the Relation Between the Solutions of an Equation and its Graph

In the following exercises, for each ordered pair, decide:

a) Is the ordered pair a solution to the equation?

b) Is the point on the line?

17. $y = \frac{2}{3}x - 1$ $(0, -1)$ (3, 1) $(-3, -3)$ (6, 4) 	18. $y = -x + 4$ $(0, 4)(-1, 3)$ $(2, 2)(-2, 6)$ 

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

19. $y = -3x$	20. $y = 4x - 3$
21. $x - y = 6$	22. $y = \frac{1}{2}x + 3$
23. $3x - 2y = 6$	24. $2x + y = 7$

Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

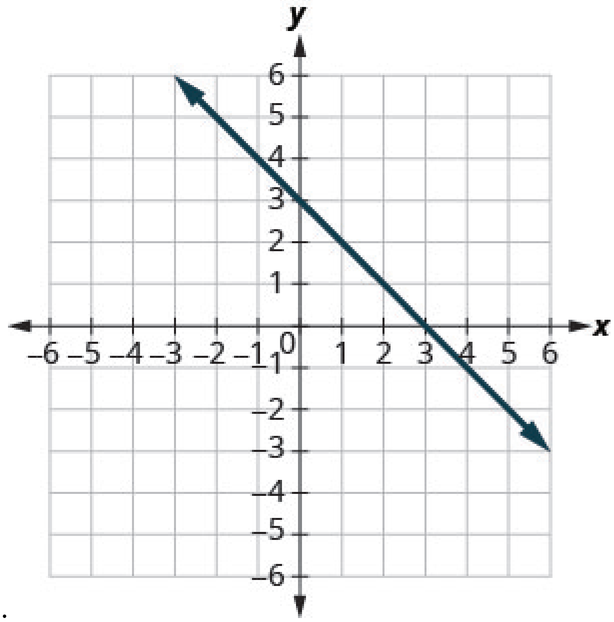
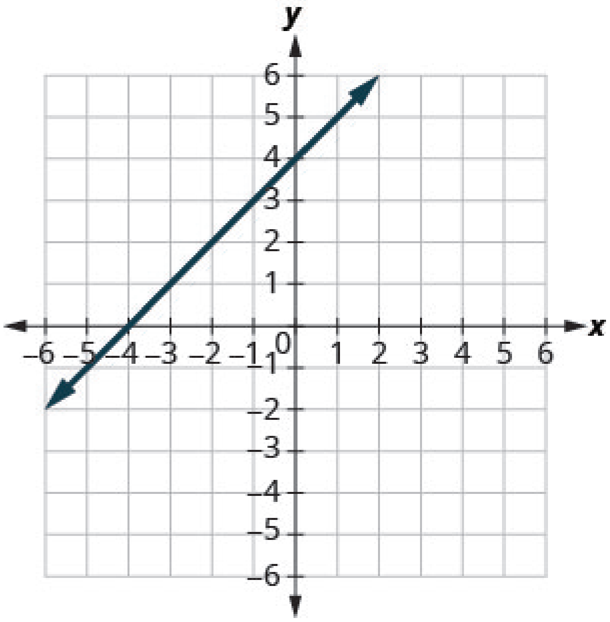
25. $x = 3$	26. $y = -2$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

27. $y = \frac{4}{3}x$ and $y = \frac{4}{3}$	28. $y = -2x$ and $y = -2$

Identify the *x*– and *y*-Intercepts on a Graph

In the following exercises, find the *x*– and *y*-intercepts.

 29.	 30.

Find the *x*– and *y*-Intercepts from an Equation of a Line

In the following exercises, find the intercepts of each equation.

31. $x - y = -1$	32. $x + y = 5$
33. $2x + 3y = 12$	34. $x + 2y = 6$
35. $y = 3x$	36. $y = \frac{3}{4}x - 12$

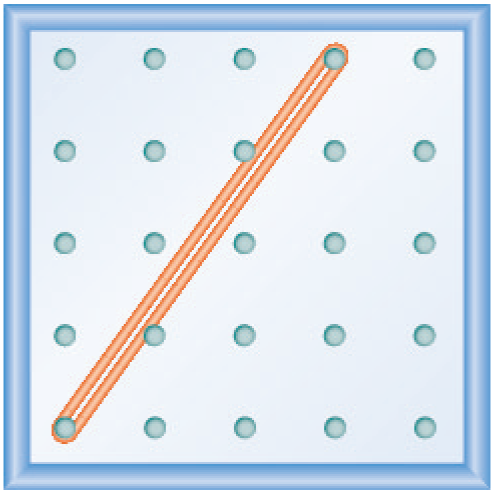
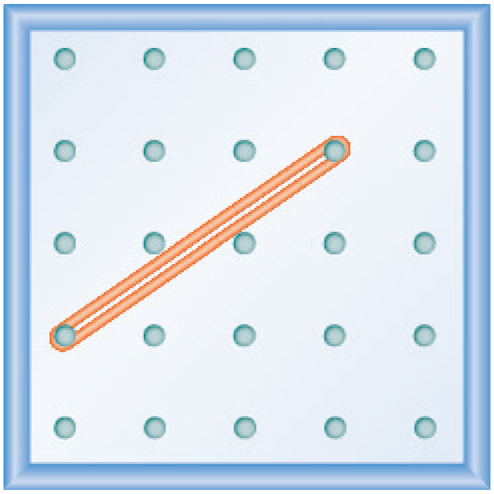
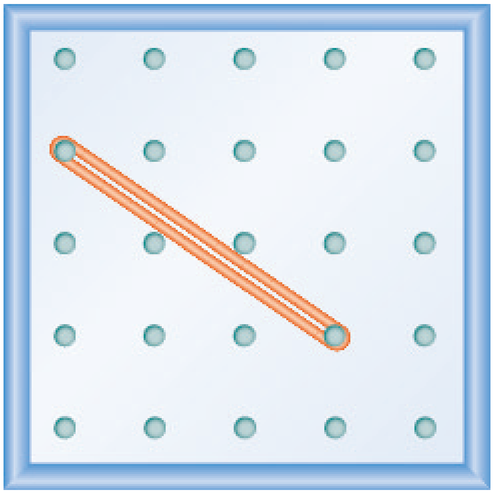
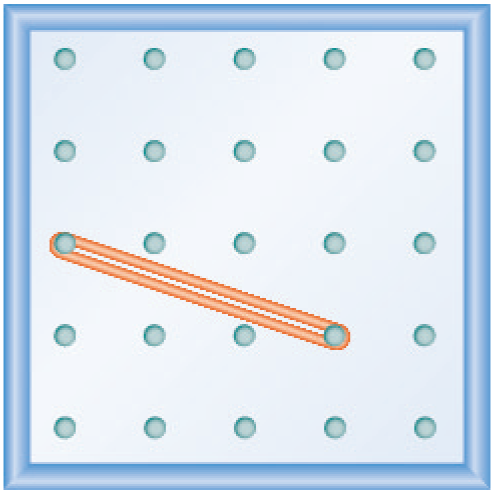
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

37. $-x + 3y = 3$	38. $x + y = -2$
39. $x - y = 4$	40. $2x - y = 5$
41. $2x - 4y = 8$	42. $y = 2x$

Use Geoboards to Model Slope

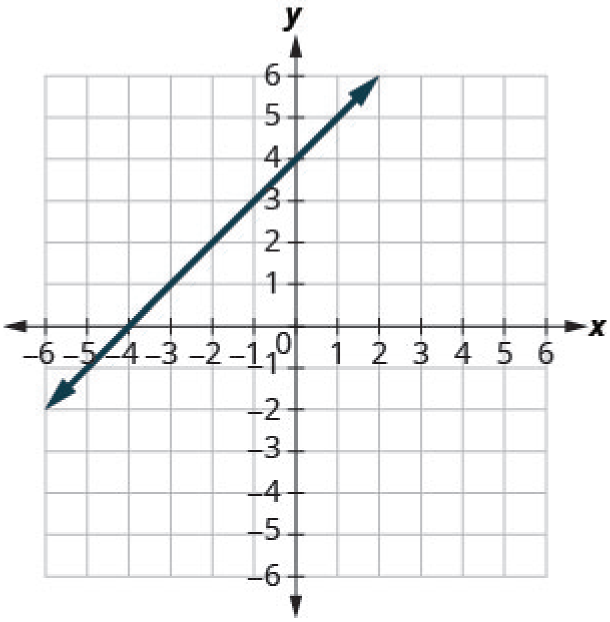
In the following exercises, find the slope modeled on each geoboard.

 43.	 44.
 45.	 46.

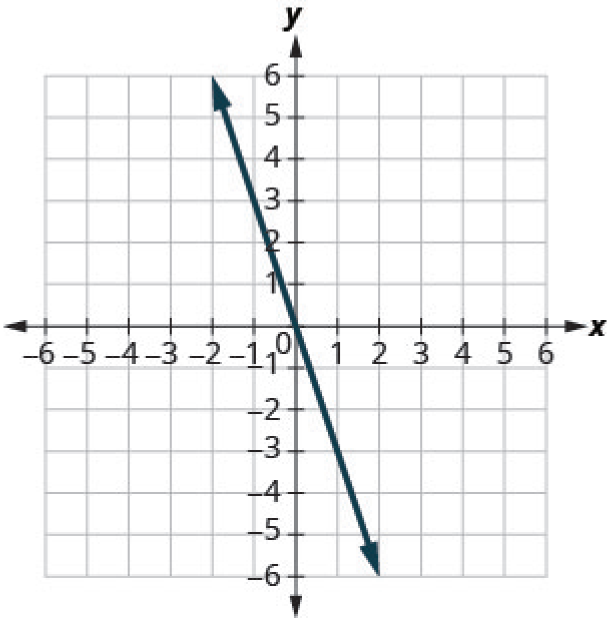
In the following exercises, model each slope. Draw a picture to show your results.

47. $\frac{1}{3}$	48. $\frac{3}{2}$
49. $-\frac{2}{3}$	50. $-\frac{1}{2}$

In the following exercises, find the slope of each line shown. Use $m = \frac{\text{rise}}{\text{run}}$ to find the slope of a line from its graph.

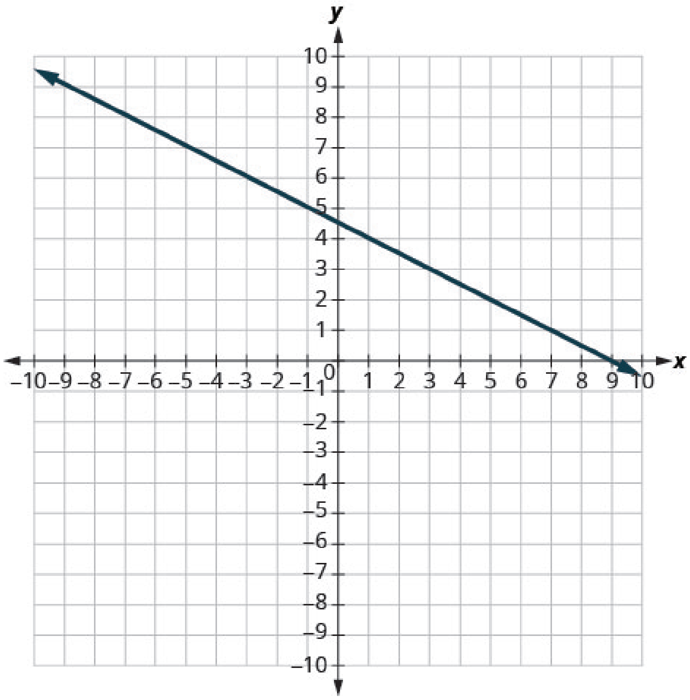


51.

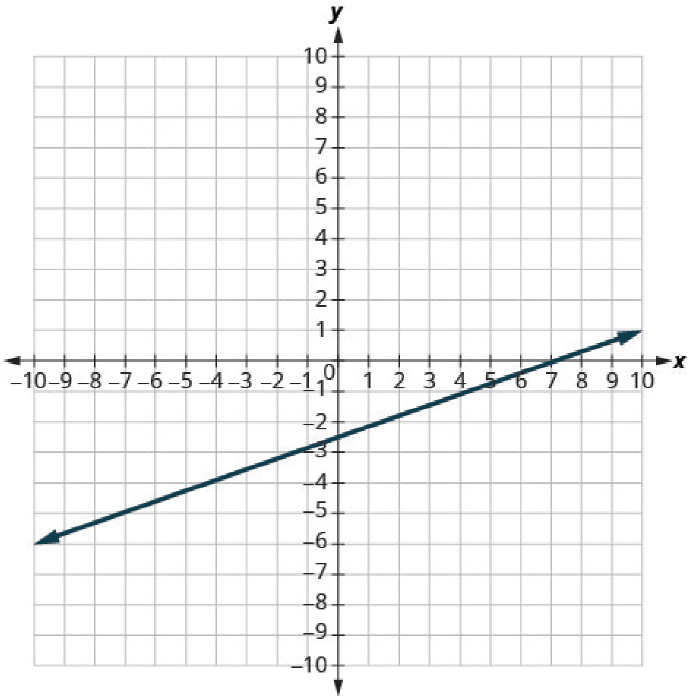


52.

53.



54.



Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

55. $x = 5$

56. $y = 2$

57. $y = -1$

58. $x = -3$

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

59. $(3, 5), (4, -1)$	60. $(-1, -1), (0, 5)$
61. $(2, 1), (4, 6)$	62. $(-5, -2), (3, 2)$

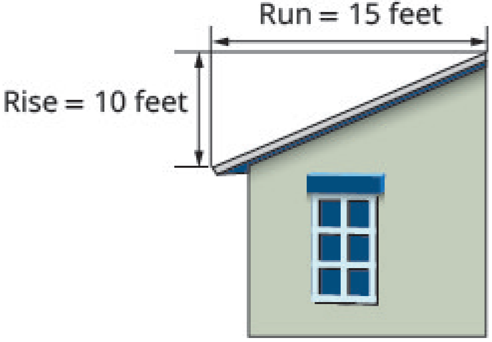
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

63. $(-3, 4); m = -\frac{1}{3}$	64. $(2, -2); m = \frac{5}{2}$
65. *y*-intercept 1; $m = -\frac{3}{4}$	66. *x*-intercept -4; $m = 3$

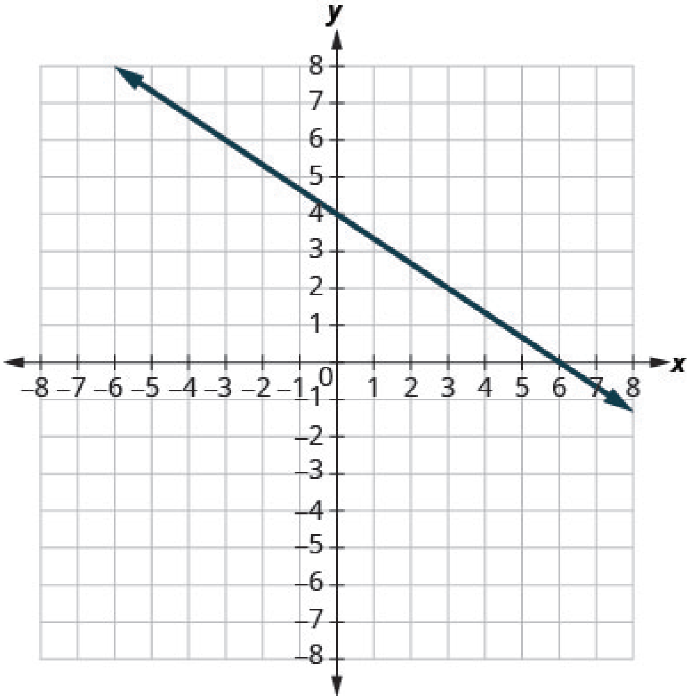
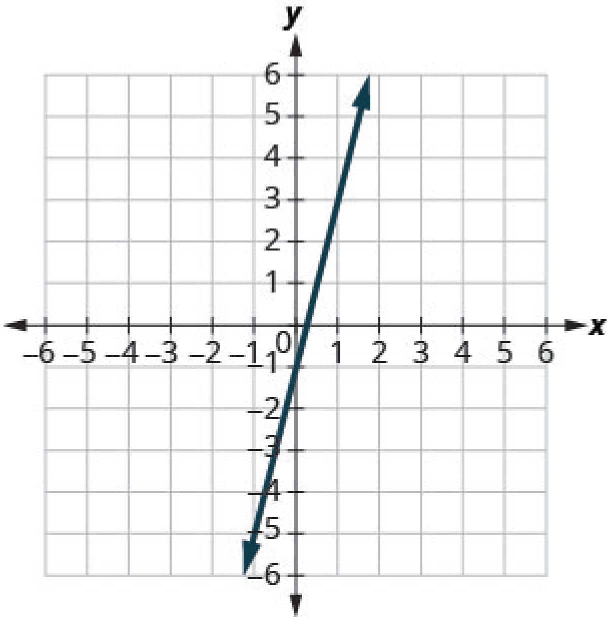
Solve Slope Applications

In the following exercises, solve these slope applications.

67. A mountain road rises 50 feet for a 500-foot run. What is its slope?	68. The roof pictured below has a rise of 10 feet and a run of 15 feet. What is its slope? 

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

In the following exercises, use the graph to find the slope and *y*-intercept of each line. Compare the values to the equation $y = mx + b$.

69.  $y = -\frac{2}{3}x + 4$	70.  $y = 4x - 1$

Identify the Slope and y-Intercept from an Equation of a Line

In the following exercises, identify the slope and *y*-intercept of each line.

71. $y = \frac{5}{3}x - 6$	72. $y = -4x + 9$
73. $4x - 5y = 8$	74. $5x + y = 10$

Graph a Line Using Its Slope and Intercept

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

75. $y = -x - 1$	76. $y = 2x + 3$
77. $4x - 3y = 12$	78. $y = -\frac{2}{5}x + 3$

In the following exercises, determine the most convenient method to graph each line.

79. $y=-3$	80. $x=5$
81. $x-y=2$	82. $2x+y=5$
83. $y=\frac{3}{4}x-1$	84. $y=x+2$

Graph and Interpret Applications of Slope–Intercept

85. Marjorie teaches piano. The equation $P=35h-250$ models the relation between her weekly profit, *P*, in dollars and the number of student lessons, *s*, that she teaches. a. Find Marjorie’s profit for a week when she teaches no student lessons. b. Find the profit for a week when she teaches 20 student lessons. c. Interpret the slope and *P*–intercept of the equation. d. Graph the equation.	86. Katherine is a private chef. The equation $C=6.5m+42$ models the relation between her weekly cost, *C*, in dollars and the number of meals, *m*, that she serves. a. Find Katherine’s cost for a week when she serves no meals. b. Find the cost for a week when she serves 14 meals. c. Interpret the slope and *C*-intercept of the equation. d. Graph the equation.

Use Slopes to Identify Parallel Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel.

87. $2x-y=8;\ x-2y=4$	88. $4x-3y=-1;\ y=\frac{4}{3}x-3$

Use Slopes to Identify Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are perpendicular.

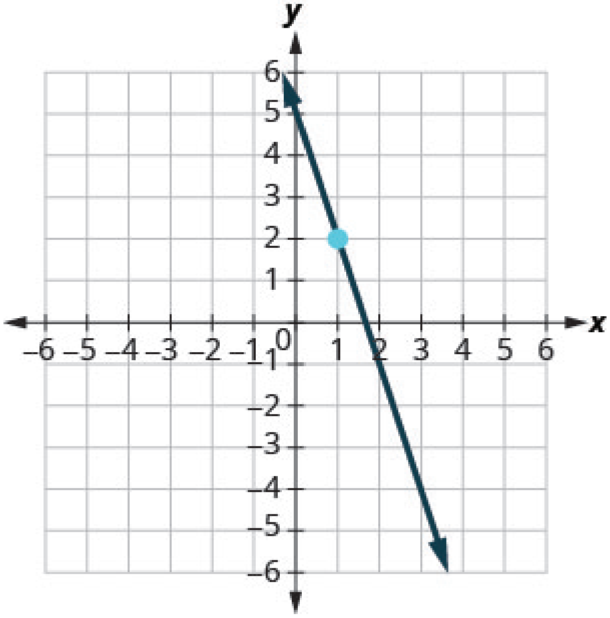
89. $3x-2y=5; 2x+3y=6$	90. $y=5x-1; 10x+2y=0$

Find an Equation of the Line Given the Slope and *y*-Intercept

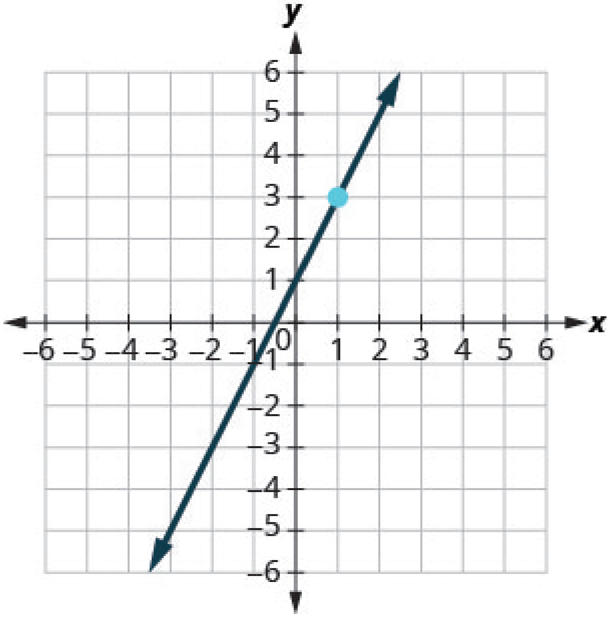
In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

91. slope -5 and $y\text{-intercept }(0,-3)$	92. slope $\frac{1}{3}$ and $y\text{-intercept }(0,-6)$
93. slope -2 and $y\text{-intercept }(0,0)$	94. slope 0 and $y\text{-intercept }(0,4)$

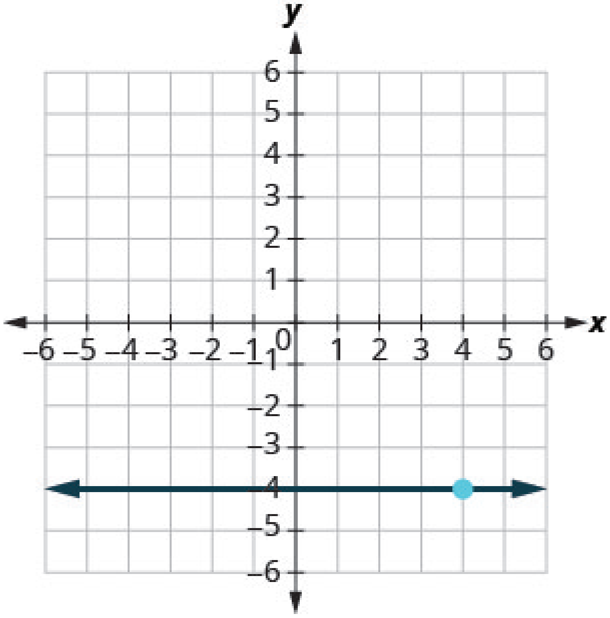
In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.



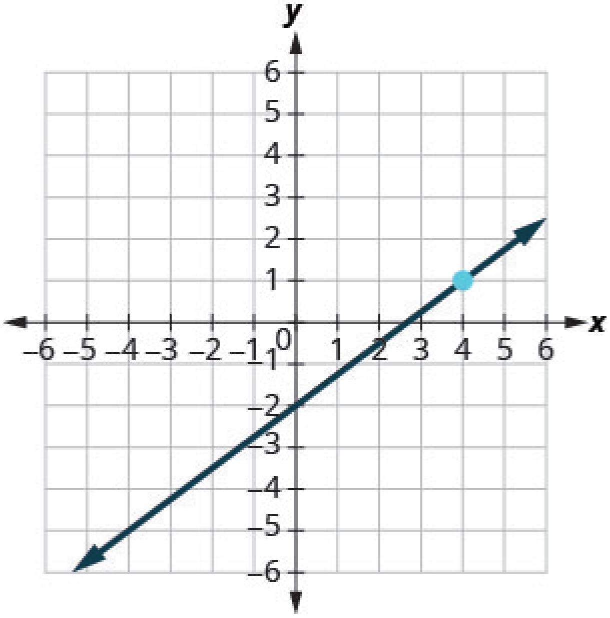
95.



96.



97.



98.

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

99. $m = \frac{3}{5}$, point $(10, 6)$	100. $m = -\frac{1}{4}$, point $(-8, 3)$
101. $m = -2$, point $(-1, -3)$	102. Horizontal line containing $(-2, 7)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

103. $(7, 1)$ and $(5, 0)$	104. $(2, 10)$ and $(-2, -2)$
105. $(5, 2)$ and $(-1, 2)$	106. $(3, 8)$ and $(3, -4)$.

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

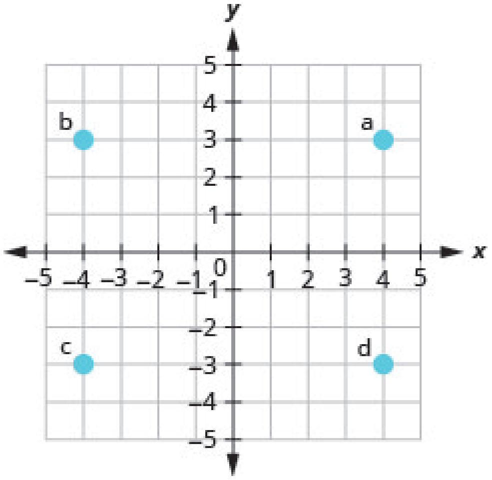
107. line $2x + 5y = -10$, point $(10, 4)$	108. line $y = -3x + 6$, point $(1, -5)$
109. line $y = -5$, point $(-4, 3)$	110. line $x = 4$, point $(-2, -1)$

Find an Equation of a Line Perpendicular to a Given Line

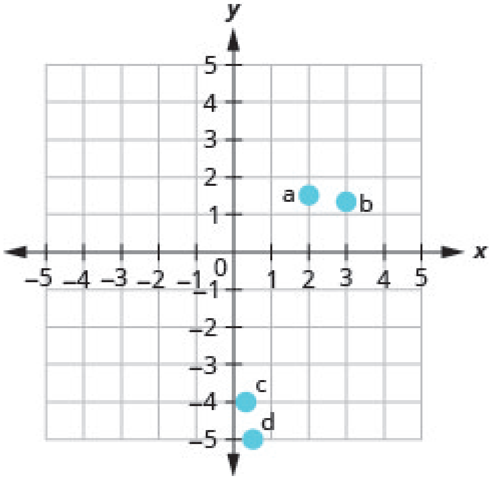
In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

112. line $2x - 3y = 9$, point $(-4, 0)$	111. line $y = -\frac{4}{5}x + 2$, point $(8, 9)$
114. line $x = -5$ point $(2, 1)$	113. line $y = 3$, point $(-1, -3)$

Review Answers



1.



3.

5. a) $(2, 0)$ b) $(0, -5)$ c) (-4.0) d) $(0, 3)$

7. a, b

9.

x	y	(x, y)
0	3	$(0, 3)$
4	1	(4, 1)
-2	4	$(-2, 4)$

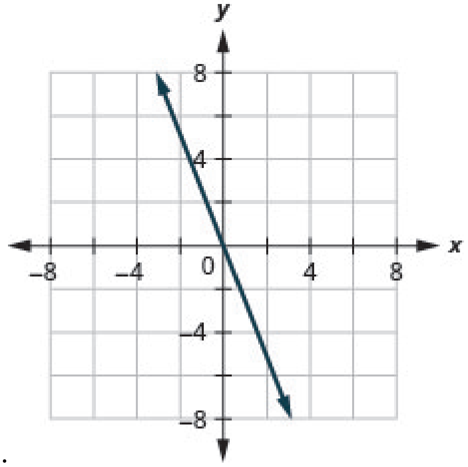
11.

x	y	(x, y)
0	-3	$(0, -3)$
2	0	$(2, 0)$
-2	-6	$(-2, -6)$

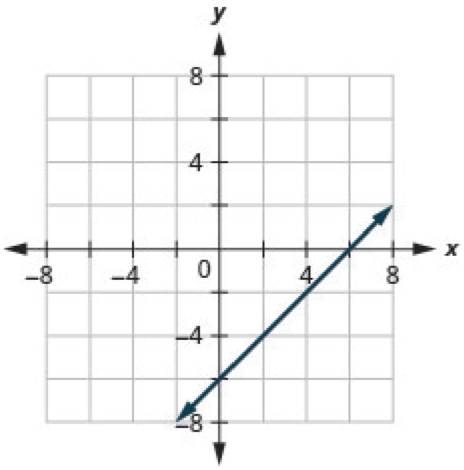
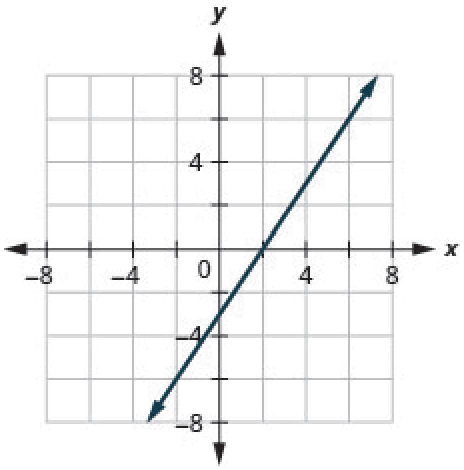
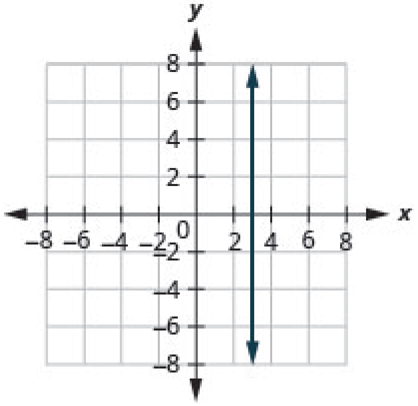
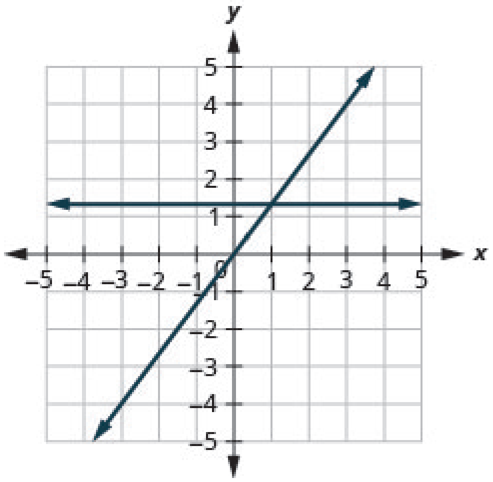
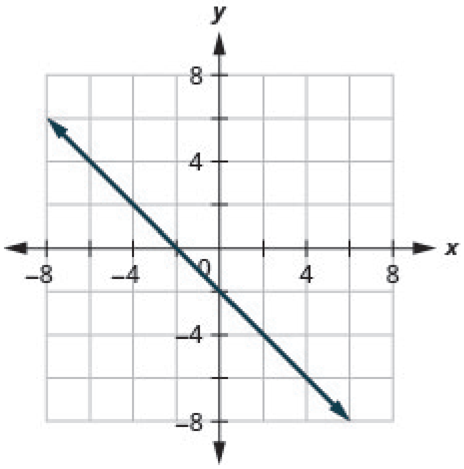
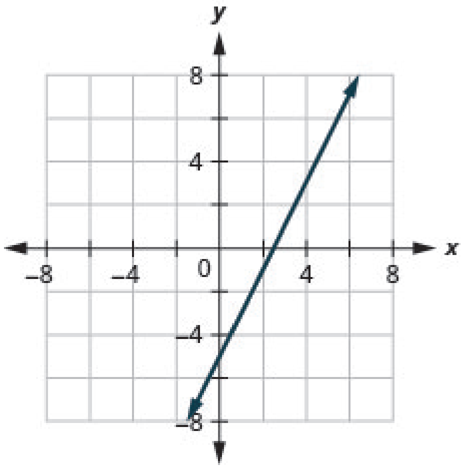
13. Answers will vary.

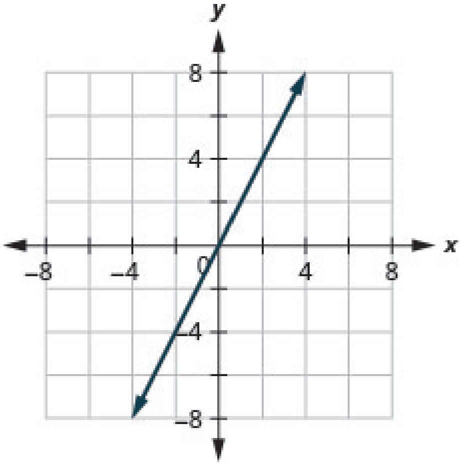
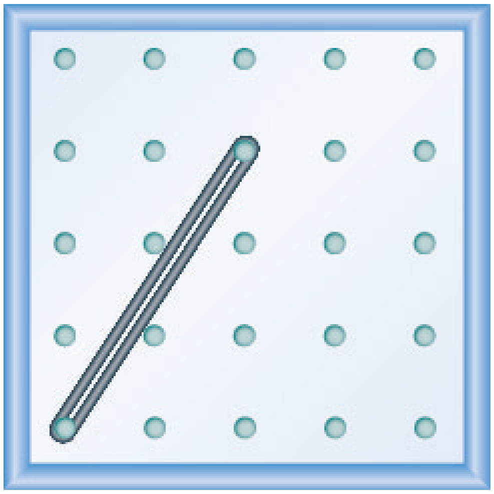
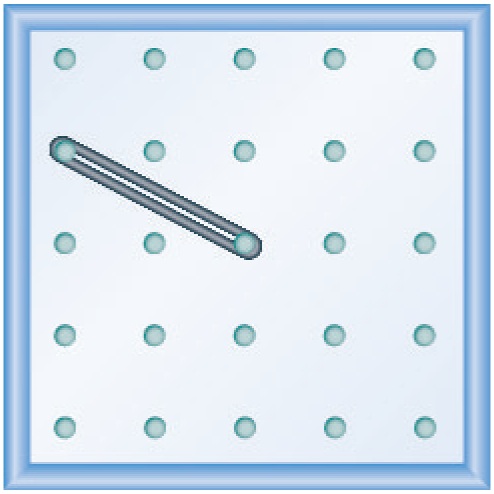
15. Answers will vary.

17. a) yes; yes b) yes; no

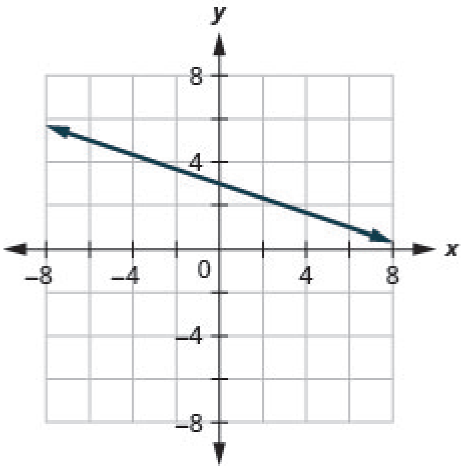


19.

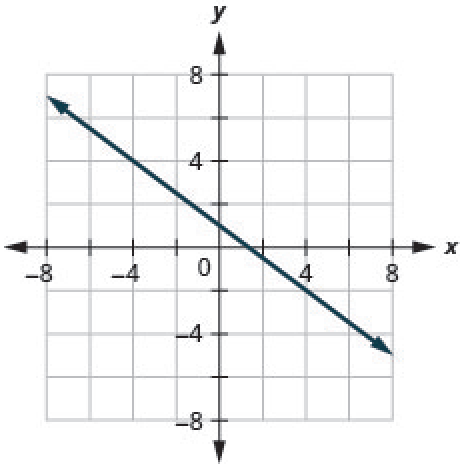
 21.	 23.
 25.	 27.
29. $(3,0)\,,(0,3)$	31. $(-1,0)\,,(0,1)$
33. $(6,0)\,,(0,4)$	35. $(0,0)$
 37.	 39.

 41.	43. $\frac{4}{3}$
45. $-\frac{2}{3}$	 47.
 49.	51. 1
53. $-\frac{1}{2}$	55. undefined
57. 0	59. -6

61. $\frac{5}{2}$



63.



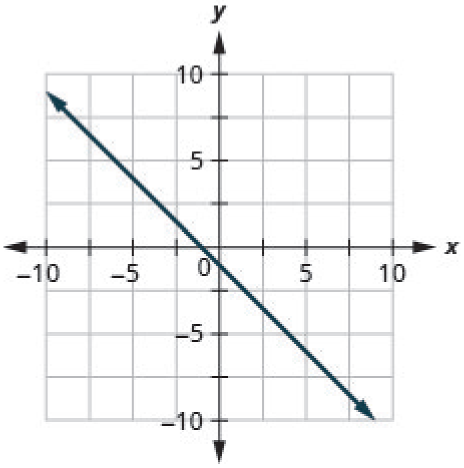
65.

67. $\frac{1}{10}$

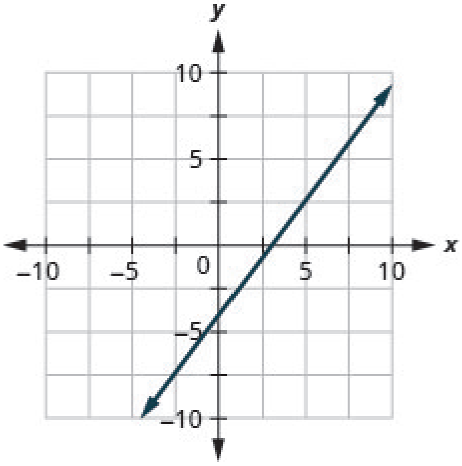
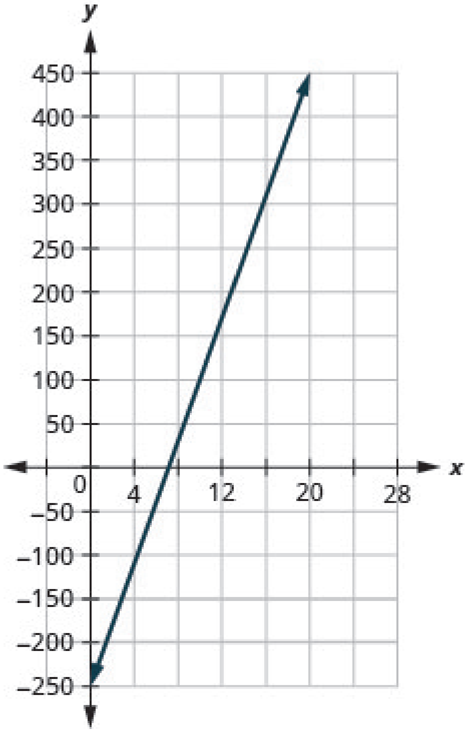
69. slope $m = -\frac{2}{3}$ and *y*-intercept $(0, 4)$

71. $\frac{5}{3}$; $(0, -6)$

73. $\frac{4}{5}$; $\left(0, -\frac{8}{5}\right)$



75.

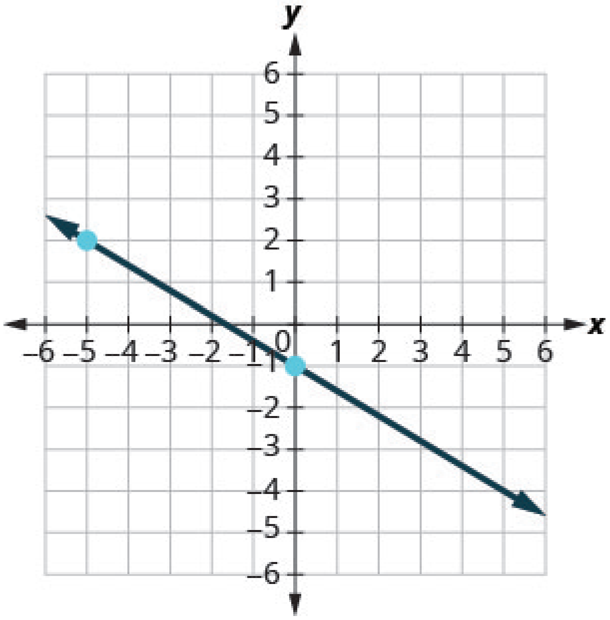
 77.	79. horizontal line
81. intercepts	83. plotting points
85. a) −?250 b) ?450 c) The slope, 35, means that Marjorie's weekly profit, *P*, increases by $35 for each additional student lesson she teaches. The *P*–intercept means that when the number of lessons is 0, Marjorie loses $250. d) 	87. not parallel
89. perpendicular	91. $y = -5x - 3$
93. $y = -2x$	95. $y = -3x + 5$
97. $y = -4$	99. $y = \frac{3}{5}x$
101. $y = -2x - 5$	103. $y = \frac{1}{2}x - \frac{5}{2}$

105. $y = 2$	107. $y = -\frac{2}{5}x + 8$
109. $y = 3$	111. $y = -\frac{3}{2}x - 6$
113. $y = 1$	

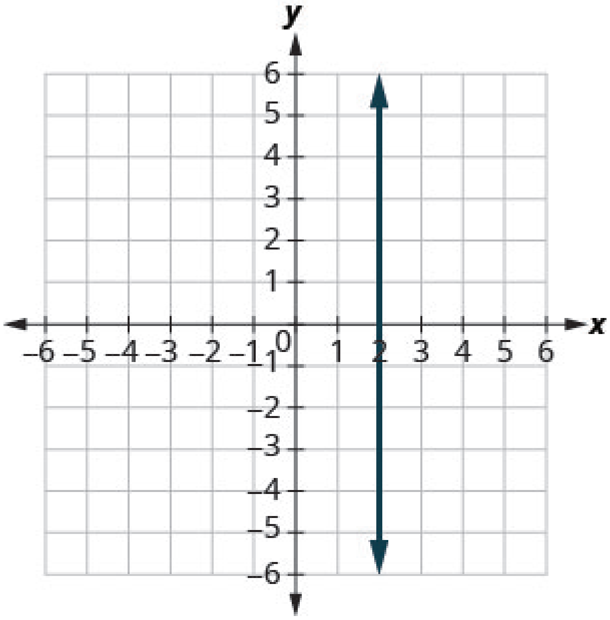
Practice Test

1. Plot each point in a rectangular coordinate system. a) $(2, 5)$ b) $(-1, -3)$ c) $(0, 2)$ d) $\left(-4, \frac{3}{2}\right)$ e) $(5, 0)$	2. Which of the given ordered pairs are solutions to the equation $3x - y = 6$? a) $(3, 3)$ b) $(2, 0)$ c) $(4, -6)$
3. Find three solutions to the linear equation $y = -2x - 4$.	4. Find the *x*– and *y*-intercepts of the equation $4x - 3y = 12$.

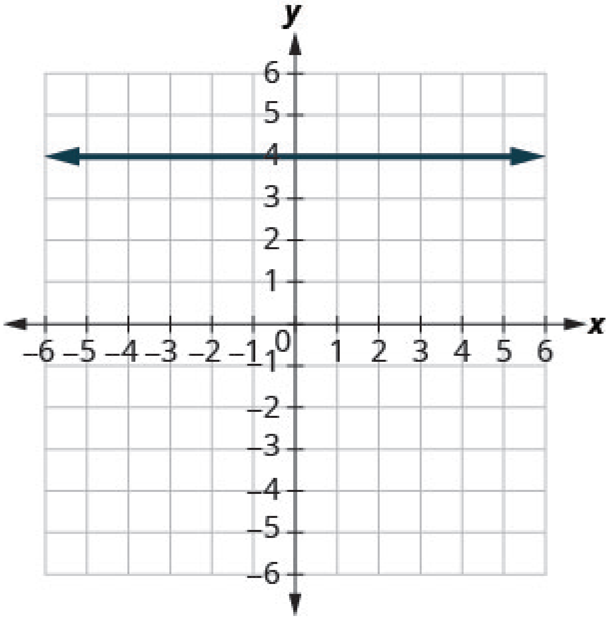
Find the slope of each line shown.



5.



6.



7.

8. Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.	9. Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

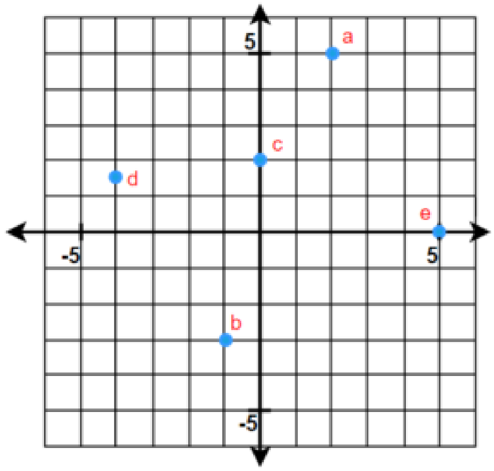
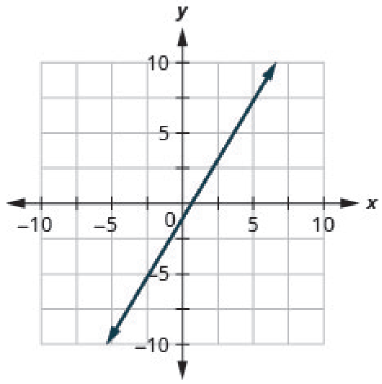
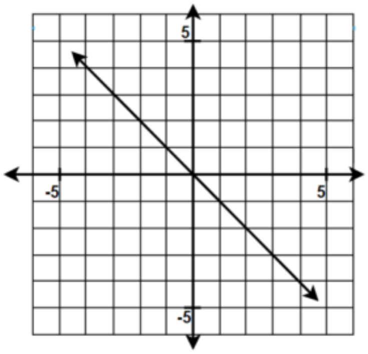
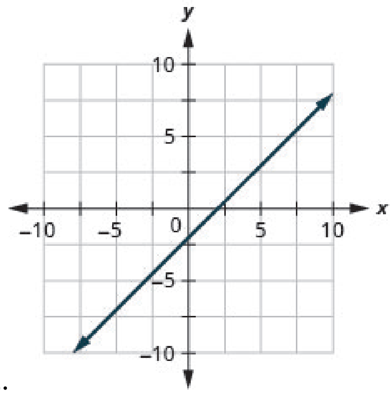
Graph the line for each of the following equations

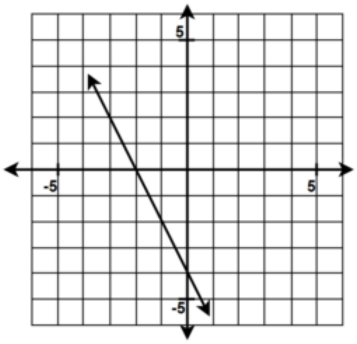
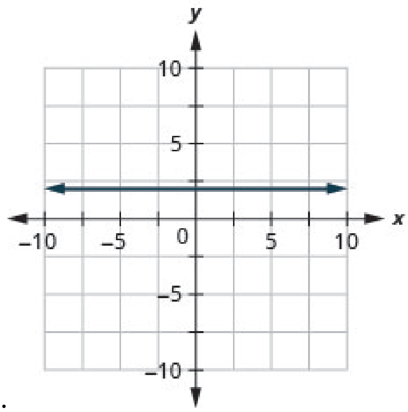
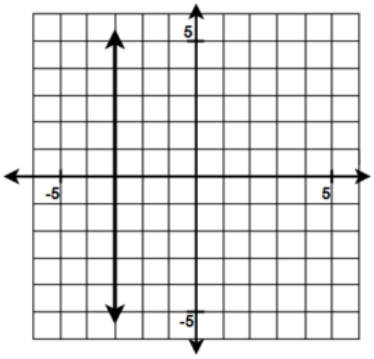
10. $y = \frac{5}{3}x - 1$	11. $y = -x$
12. $x - y = 2$	13. $4x + 2y = -8$
14. $y = 2$	15. $x = -3$

Find the equation of each line. Write the equation in slope–intercept form.

16. slope $-\frac{3}{4}$ and *y*-intercept $(0, -2)$	17. $m = 2$, point $(-3, -1)$
18. containing $(10, 1)$ and $(6, -1)$	19. parallel to the line $y = -\frac{2}{3}x - 1$, containing the point $(-3, 8)$
20. perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$	

Practice Test Answers

 1.	2. a) yes b) yes c) no
3. Answer may vary	4. $(3, 0)$, $(0, -4)$
5. m = $frac35$	6. undefined
7. m = 0	8. 1
9. y = $\frac{1}{2}$x – $\frac{5}{2}$	 10.
 11.	 12.

13. 	14. 
15. 	16. $y = -\frac{3}{4}x - 2$
17.$y = 2x + 5$	18. $y = \frac{1}{2}x - 4$
19. $y = -\frac{2}{3}x + 6$	20. $y = -\frac{4}{5}x - 5$

Attributions

This chapter has been adapted from “Review Exercises” and “Practice Test” in Chapter 4 of *Elementary Algebra* (OpenStax) by Lynn Marecek and MaryAnne Anthony-Smith, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Copyright page for more information.

CHAPTER 7 Powers, Roots, and Scientific Notation

Square roots are used to determine the time it would take for a stone falling from the edge of this cliff to hit the land below.

Suppose a stone falls from the edge of a cliff. The number of feet the stone has dropped after t seconds can be found by multiplying 16 times the square of t. But to calculate the number of seconds it would take the stone to hit the land below, we need to use a square root. In this chapter, we will introduce and apply the properties of exponents and square roots, and scientific notation.

7.1 Use Multiplication Properties of Exponents

Learning Objectives

By the end of this section, you will be able to:

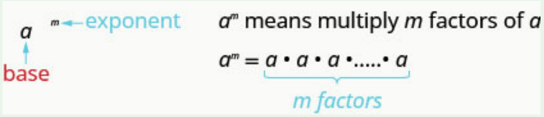
- Simplify expressions with exponents
- Simplify expressions using the Product Property for Exponents
- Simplify expressions using the Power Property for Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials

Simplify Expressions with Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means $2 \cdot 2 \cdot 2 \cdot 2$

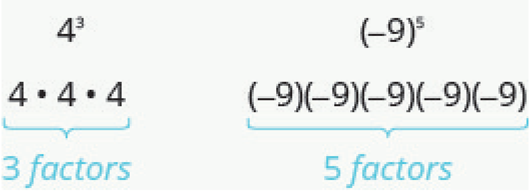
Let's review the vocabulary for expressions with exponents.

Exponential Notation



This is read a to the m^{th} power.

In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.



Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

EXAMPLE 1

Simplify: a) 4^3 b) 7^1 c) $\left(\frac{5}{6}\right)^2$ d) $(0.63)^2$.

Solution

a)	4^3
Multiply three factors of 4.	$4 \cdot 4 \cdot 4$
Simplify.	64
b)	7^1
Multiply one factor of 7.	7
c)	$\left(\frac{5}{6}\right)^2$
Multiply two factors.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$
Simplify.	$\frac{25}{36}$
d)	$(0.63)^2$
Multiply two factors.	$(0.63)(0.63)$
Simplify.	0.3969

TRY IT 1.1

Simplify: a) 6^3 b) 15^1 c) $\left(\frac{3}{7}\right)^2$ d) $(0.43)^2$.

Show answer

a) 216 b) 15 c) $\frac{9}{49}$ d) 0.1849

TRY IT 1.2

Simplify: a) 2^5 b) 21^1 c) $\left(\frac{2}{5}\right)^3$ d) $(0.218)^2$.

Show answer

a) 32b) 21 c) $\frac{8}{125}$ d) 0.047524

EXAMPLE 2

Simplify: a) $(-5)^4$ b) -5^4.

Solution

a)	$(-5)^4$
Multiply four factors of -5.	$(-5)(-5)(-5)(-5)$
Simplify.	625
b)	-5^4
Multiply four factors of 5.	-(5 · 5 · 5 · 5)
Simplify.	-625

TRY IT 2.1

Simplify: a) $(-3)^4$ b) -3^4.

Show answer

a) 81 b) -81

TRY IT 2.2

Simplify: a) $(-13)^2$ b) -13^2.

Show answer

a) 169 b) -169

Notice the similarities and differences in (Example 2) a) and (Example 2) b)! Why are the answers different? As we follow the order of operations in part a) the parentheses tell us to raise the (-5) to the 4th power. In part b) we raise just the 5 to the 4th power and then take the opposite.

Simplify Expressions Using the Product Property for Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We'll derive the properties of exponents by looking for patterns in several examples.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^3$
What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x \cdot x}_{3\ factors}}_{5\ factors}$
So, we have	x^5
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3$ is x^{2+3}, or x^5

We write:

$$\begin{gathered} x^2 \cdot x^3 \\ x^{2+3} \\ x^5 \end{gathered}$$

The base stayed the same and we added the exponents. This leads to the **Product Property for Exponents**.

Product Property for Exponents

If a is a real number, and m and n are counting numbers, then

$a^m \cdot a^n = a^{m+n}$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} 2^2 \cdot 2^3 &\stackrel{?}{=} 2^{2+3} \\ 4 \cdot 8 &\stackrel{?}{=} 2^5 \\ 32 &= 32\checkmark \end{aligned}$$

EXAMPLE 3

Simplify: $y^5 \cdot y^6$.

Solution

	$y^5 \cdot y^6$
Use the product property, a^m \cdot a^n = a^{m+n}.	y^{5+6}
Simplify.	y^{11}

TRY IT 3.1

Simplify: $b^9 \cdot b^8$.

Show answer

b^{17}

TRY IT 3.2

Simplify: $x^{12} \cdot x^4$.

Show answer

x^{16}

EXAMPLE 4

Simplify: a) $2^5 \cdot 2^9$ b) $3 \cdot 3^4$.

Solution

a.

	$2^5 \cdot 2^9$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	2^{5+9}
Simplify.	2^{14}

b.

	$3 \cdot 3^4$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	3^{1+4}
Simplify.	3^5

TRY IT 4.1

Simplify: a) $5 \cdot 5^5$ b) $4^9 \cdot 4^9$.

Show answer
a) 5^6 b) 4^{18}

TRY IT 4.2

Simplify: a) $7^6 \cdot 7^8$ b) $10 \cdot 10^{10}$.

Show answer
a) 7^{14} b) 10^{11}

EXAMPLE 5

Simplify: a) $a^7 \cdot a$ b) $x^{27} \cdot x^{13}$.

Solution

a.

	$a^7 \cdot a$
Rewrite, a = a^1.	$a^7 \cdot a^1$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	a^{7+1}
Simplify.	a^8

b.

	$x^{27} \cdot x^{13}$
Notice, the bases are the same, so add the exponents.	x^{27+13}
Simplify.	x^{40}

TRY IT 5.1

Simplify: a) $p^5 \cdot p$ b) $y^{14} \cdot y^{29}$.

Show answer
a) p^6 b) y^{43}

TRY IT 5.2

Simplify: a) $z \cdot z^7$ b) $b^{15} \cdot b^{34}$.

Show answer
a) z^8 b) b^{49}

We can extend the Product Property for Exponents to more than two factors.

EXAMPLE 6

Simplify: $d^4 \cdot d^5 \cdot d^2$.

Solution

	$d^4 \cdot d^5 \cdot d^2$
Add the exponents, since bases are the same.	d^{4+5+2}
Simplify.	d^{11}

TRY IT 6.1

Simplify: $x^6 \cdot x^4 \cdot x^8$.

Show answer
x^{18}

TRY IT 6.2

Simplify: $b^5 \cdot b^9 \cdot b^5$.

Show answer
b^{19}

Simplify Expressions Using the Power Property for Exponents

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$
What does this mean? How many factors altogether?	$x^2 \cdot x^2 \cdot x^2$ $\underbrace{\underbrace{x \cdot x}_{2\text{ factors}} \cdot \underbrace{x \cdot x}_{2\text{ factors}} \cdot \underbrace{x \cdot x}_{2\text{ factors}}}_{6\text{ factors}}$
So we have	x^6
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3$ is $x^{2\cdot 3}$ or x^6

We write:

$$\begin{gathered}\left(x^2\right)^3 \\ x^{2\cdot 3} \\ x^6\end{gathered}$$

We multiplied the exponents. This leads to the Power Property for Exponents.

Power Property for Exponents

If a is a real number, and m and n are whole numbers, then
$(a^m)^n = a^{m\cdot n}$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} 2^2\cdot 2^3 &\overset{?}{=} 2^{2+3} \\ 4\cdot 8 &\overset{?}{=} 2^5 \\ 32 &= 32\checkmark \end{aligned}$$

EXAMPLE 7

Simplify: a) $\left(y^5\right)^9$ b) $\left(4^4\right)^7$.

Solution

a)

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{m\cdot n}$.	$y^{5\cdot 9}$
Simplify.	y^{45}

b)

	$(4^4)^7$
Use the power property.	$4^{4\cdot 7}$
Simplify.	4^{28}

TRY IT 7.1

Simplify: a) $\left(b^7\right)^5$ b) $\left(5^4\right)^3$.

Show answer
a) b^{35} b) 5^{12}

TRY IT 7.2

Simplify: a) $\left(z^6\right)^9$ b) $\left(3^7\right)^7$.

Show answer
a) z^{54} b) 3^{49}

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x \cdot 2x \cdot 2x$
We group the like factors together.	$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$
How many factors of 2 and of x?	$2^3 \cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 \cdot x^3$.

We write:	$(2x)^3$
	$2^3 \cdot x^3$

The exponent applies to each of the factors! This leads to the Product to a Power Property for Exponents.

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:

$$\begin{aligned} (2 \cdot 3)^2 &\stackrel{?}{=} 2^2 \cdot 3^2 \\ 6^2 &\stackrel{?}{=} 4 \cdot 9 \\ 36 &= 36\checkmark \end{aligned}$$

EXAMPLE 8

Simplify: a) $(-9d)^2$ b) $(3mn)^3$.

Solution

a.

	$(-9d)^2$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-9)^2 d^2$
Simplify.	$81d^2$

b.

	$(3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(3)^3 m^3 n^3$
Simplify.	$27m^3 n^3$

TRY IT 8.1

Simplify: a) $(-12y)^2$ b) $(2wx)^5$.

Show answer

a) $144y^2$ b) $32w^5x^5$

TRY IT 8.2

Simplify: a) $(5wx)^3$ b) $(-3y)^3$.

Show answer

a) $125w^3x^3$ b) $-27y^3$

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let’s summarize them and then we’ll do some examples that use more than one of the properties.

Properties of Exponents

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$

All exponent properties hold true for any real numbers$m$ and n. Right now, we only use whole number exponents.

EXAMPLE 9

Simplify: a) $\left(y^3\right)^6\left(y^5\right)^4$ b) $\left(-6x^4y^5\right)^2$.

Solution

a)	$\left(y^3\right)^6\left(y^5\right)^4$
Use the Power Property.	$y^{18}\cdot y^{20}$
Add the exponents.	y^{38}
b)	$\left(-6x^4y^5\right)^2$
Use the Product to a Power Property.	$(-6)^2\left(x^4\right)^2\left(y^5\right)^2$
Use the Power Property.	$(-6)^2$
Simplify.	$36x^8y^{10}$

TRY IT 9.1

Simplify: a) $\left(a^4\right)^5\left(a^7\right)^4$ b) $\left(-2c^4d^2\right)^3$.

Show answer
a) a^{48} b) $-8c^{12}d^6$

TRY IT 9.2

Simplify: a) $\left(-3x^6y^7\right)^4$ b) $\left(q^4\right)^5\left(q^3\right)^3$.

Show answer
a) $81x^{24}y^{28}$ b) q^{29}

EXAMPLE 10

Simplify: a) $(5m)^2\left(3m^3\right)$ b) $\left(3x^2y\right)^4\left(2xy^2\right)^3$.

Solution

a)	$(5m)^2\left(3m^3\right)$
Raise $5m$ to the second power.	$5^2m^2\cdot 3m^3$
Simplify.	$25m^2\cdot 3m^3$
Use the Commutative Property.	$25\cdot 3\cdot m^2\cdot m^3$
Multiply the constants and add the exponents.	$75m^5$
b)	$\left(3x^2y\right)^4\left(2xy^2\right)^3$
Use the Product to a Power Property.	$\left(3^4x^8y^4\right)\left(2^3x^3y^6\right)$
Simplify.	$\left(81x^8y^4\right)\left(8x^3y^6\right)$
Use the Commutative Property.	$81\cdot 8\cdot x^8\cdot x^3\cdot y^4\cdot y^6$
Multiply the constants and add the exponents.	$648x^{11}y^{10}$

TRY IT 10.1

Simplify: a) $(5n)^2\left(3n^{10}\right)$ b) $\left(c^4d^2\right)^5\left(3cd^5\right)^4$.

Show answer

a) $75n^{12}$ b) $81c^{24}d^{30}$

TRY IT 10.2

Simplify: a) $\left(a^3b^2\right)^6\left(4ab^3\right)^4$ b) $(2x)^3\left(5x^7\right)$.

Show answer

a) $256a^{22}b^{24}$ b) $40x^{10}$

Multiply Monomials

A term in algebra is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. Some examples of monomial are $8, -2x^2, 4y^3$, and $11z^7$.

Monomials

A monomial is a term of the form ax^m, where a is a constant and m is a positive whole number.

Since a monomial is an algebraic expression,we can use the properties of exponents to multiply monomials.

EXAMPLE 11

Multiply: $\left(3x^2\right)\left(-4x^3\right)$.

Solution

	$\left(3x^2\right)\left(-4x^3\right)$
Use the Commutative Property to rearrange the terms.	$3\cdot(-4)\cdot x^2\cdot x^3$
Multiply.	$-12x^5$

TRY IT 11.1

Multiply: $\left(5y^7\right)\left(-7y^4\right)$.

Show answer
$-35y^{11}$

TRY IT 11.2

Multiply: $\left(-6b^4\right)\left(-9b^5\right)$.

Show answer
$54b^9$

EXAMPLE 12

Multiply: $\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$.

Solution

	$\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

TRY IT 12.1

Multiply: $\left(\frac{2}{5}a^4b^3\right)\left(15ab^3\right)$.

Show answer
$6a^5b^6$

TRY IT 12.2

Multiply: $\left(\frac{2}{3}r^5s\right)\left(12r^6s^7\right)$.

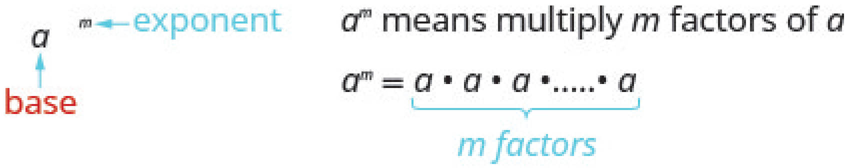
Show answer
$8r^{11}s^8$

Additional Online Resources

- Multiplication Properties of Exponents

Key Concepts

- **Exponential Notation**



- **Properties of Exponents**
    - If a, b are real numbers and m, n are whole numbers, then

$$\begin{array}{lll} \textbf{Product Property} & a^m \cdot a^n & = a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = a^m b^m \end{array}$$

Practice Makes Perfect

Simplify Expressions with Exponents

In the following exercises, simplify each expression with exponents.

1. a) 3^5 b) 9^1 c) $\left(\frac{1}{3}\right)^2$ d) $(0.2)^4$	2. a) 10^4 b) 17^1 c) $\left(\frac{2}{9}\right)^2$ d) $(0.5)^3$
3. a) 2^6 b) 14^1 c) $\left(\frac{2}{5}\right)^3$ d) $(0.7)^2$	4. a) 8^3 b) 8^1 c) $\left(\frac{3}{4}\right)^3$ d) $(0.4)^3$
5. a) $(-6)^4$ b) -6^4	6. a) $(-2)^6$ b) -2^6
7. a) $-\left(\frac{1}{4}\right)^4$ b) $\left(-\frac{1}{4}\right)^4$	8. a) $-\left(\frac{2}{3}\right)^2$ b) $\left(-\frac{2}{3}\right)^2$
9. a) -0.5^2 b) $(-0.5)^2$	10. a) -0.1^4 b) $(-0.1)^4$

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression using the Product Property for Exponents.

11. $d^3 \cdot d^6$	12. $x^4 \cdot x^2$
13. $n^{19} \cdot n^{12}$	14. $q^{27} \cdot q^{15}$
15. a) $4^5 \cdot 4^9$ b) $8^9 \cdot 8$	16. a) $3^{10} \cdot 3^6$ b) $5 \cdot 5^4$
17. a) $y \cdot y^3$ b) $z^{25} \cdot z^8$	17. a) $y \cdot y^3$ b) $z^{25} \cdot z^8$
19. $w \cdot w^2 \cdot w^3$	20. $y \cdot y^3 \cdot y^5$
21. $a^4 \cdot a^3 \cdot a^9$	22. $c^5 \cdot c^{11} \cdot c^2$
23. $m^x \cdot m^3$	24. $n^y \cdot n^2$
25. $y^a \cdot y^b$	26. $x^p \cdot x^q$

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression using the Power Property for Exponents.

27. a) $\left(m^4\right)^2$ b) $\left(10^3\right)^6$	28. a) $\left(b^2\right)^7$ b) $\left(3^8\right)^2$
29. a) $\left(y^3\right)^x$ b) $(5^x)^y$	30. a) $\left(x^2\right)^y$ b) $(7^a)^b$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression using the Product to a Power Property.

31. a) $(6a)^2$ b) $(3xy)^2$	32. a) $(5x)^2$ b) $(4ab)^2$
33. a) $(-4m)^3$ b) $(5ab)^3$	34. a) $(-7n)^3$ b) $(3xyz)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

35. a) $\left(y^2\right)^4 \cdot \left(y^3\right)^2$ b) $\left(10a^2b\right)^3$	36. a) $\left(w^4\right)^3 \cdot \left(w^5\right)^2$ b) $\left(2xy^4\right)^5$
37. a) $\left(-2r^3s^2\right)^4$ b) $\left(m^5\right)^3 \cdot \left(m^9\right)^4$	38. a) $\left(-10q^2p^4\right)^3$ b) $\left(n^3\right)^{10} cdot \left(n^5\right)^2$
39. a) $(3x)^2\,(5x)$ b) $\left(5t^2\right)^3(3t)^2$	40. a) $(2y)^3\,(6y)$ b) $\left(10k^4\right)^3\left(5k^6\right)^2$
41. a) $(5a)^2(2a)^3$ b) $\left(\frac{1}{2}y^2\right)^3\left(\frac{2}{3}y\right)^2$	42. a) $(4b)^2(3b)^3$ b) $\left(\frac{1}{2}j^2\right)^5\left(\frac{2}{5}j^3\right)^2$
43. a) $\left(\frac{2}{5}x^2y\right)^3$ b) $\left(\frac{8}{9}xy^4\right)^2$	44. a) $\left(2r^2\right)^3(4r)^2$ b) $\left(3x^3\right)^3\left(x^5\right)^4$
45. a) $\left(m^2n\right)^2\left(2mn^5\right)^4$ b) $\left(3pq^4\right)^2\left(6p^6q\right)^2$	

Multiply Monomials

In the following exercises, multiply the terms.

46. $\left(6y^7\right)\left(-3y^4\right)$	47. $\left(-10x^5\right)\left(-3x^3\right)$
48. $\left(-8u^6\right)\left(-9u\right)$	49. $\left(-6c^4\right)\left(-12c\right)$
50. $\left(\frac{1}{5}f^8\right)\left(20f^3\right)$	51. $\left(\frac{1}{4}d^5\right)\left(36d^2\right)$
52. $\left(4a^3b\right)\left(9a^2b^6\right)$	53. $\left(6m^4n^3\right)\left(7mn^5\right)$
54. $\left(\frac{4}{7}rs^2\right)\left(14rs^3\right)$	55. $\left(\frac{5}{8}x^3y\right)\left(24x^5y\right)$
56. $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$	56. $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

Mixed Practice

In the following exercises, simplify each expression.

58. $\left(x^2\right)^4\cdot\left(x^3\right)^2$	59. $\left(y^4\right)^3\cdot\left(y^5\right)^2$
60. $\left(a^2\right)^6\cdot\left(a^3\right)^8$	61. $\left(b^7\right)^5\cdot\left(b^2\right)^6$
62. $\left(2m^6\right)^3$	63. $\left(3y^2\right)^4$
64. $\left(10x^2y\right)^3$	65. $\left(2mn^4\right)^5$
66. $\left(-2a^3b^2\right)^4$	67. $\left(-10u^2v^4\right)^3$
68. $\left(\frac{2}{3}x^2y\right)^3$	69. $\left(\frac{7}{9}pq^4\right)^2$
70. $\left(8a^3\right)^2(2a)^4$	71. $\left(5r^2\right)^3(3r)^2$
72. $\left(10p^4\right)^3\left(5p^6\right)^2$	73. $\left(4x^3\right)^3\left(2x^5\right)^4$
74. $\left(\frac{1}{2}x^2y^3\right)^4\left(4x^5y^3\right)^2$	75. $\left(\frac{1}{3}m^3n^2\right)^4\left(9m^8n^3\right)^2$
76. $\left(3m^2n\right)^2\left(2mn^5\right)^4$	77. $\left(2pq^4\right)^3\left(5p^6q\right)^2$

Everyday Math

78. **Email** Kate emails a flyer to ten of her friends and tells them to forward it to ten of their friends, who forward it to ten of their friends, and so on. The number of people who receive the email on the second round is 10^2, on the third round is 10^3, as shown in the table below. How many people will receive the email on the sixth round? Simplify the expression to show the number of people who receive the email.

Round	**Number of people**
1	10
2	10^2
3	10^3
…	…
6	?

79. **Salary** Jamal’s boss gives him a 3% raise every year on his birthday. This means that each year, Jamal’s salary is 1.03 times his last year’s salary. If his original salary was $35,000, his salary after 1 year was $\$35,000\,(1.03)$, after 2 years was $\$35,000(1.03)^2$, after 3 years was $\$35,000(1.03)^3$, as shown in the table below. What will Jamal’s salary be after 10 years? Simplify the expression, to show Jamal’s salary in dollars.

Year	**Salary**
1	$\$35,000\,(1.03)$
2	$\$35,000(1.03)^2$
3	$\$35,000(1.03)^3$
…	…
10	?

80. **Clearance** A department store is clearing out merchandise in order to make room for new inventory. The plan is to mark down items by 30% each week. This means that each week the cost of an item is 70% of the previous week’s cost. If the original cost of a sofa was $1,000, the cost for the first week would be $\$1,000\,(0.70)$ and the cost of the item during the second week would be $\$1,000(0.70)^2$. Complete the table shown below. What will be the cost of the sofa during the fifth week? Simplify the expression, to show the cost in dollars.

Week	**Cost**
1	$\$1,000\,(0.70)$
2	$\$1,000(0.70)^2$
3	
…	…
5	?

81. **Depreciation** Once a new car is driven away from the dealer, it begins to lose value. Each year, a car loses 10% of its value. This means that each year the value of a car is 90% of the previous year’s value. If a new car was purchased for ?20,000, the value at the end of the first year would be $\$20,000\,(0.90)$ and the value of the car after the end of the second year would be $\$20,000(0.90)^2$. Complete the table shown below. What will be the value of the car at the end of the eighth year? Simplify the expression, to show the value in dollars.

Week	**Cost**
1	$\$20,000\,(0.90)$
2	$\$20,000(0.90)^2$
3	
4	…
8	?

Writing Exercises

82. Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

83. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

84. Jorge thinks $\left(\frac{1}{2}\right)^2$ is 1. What is wrong with his reasoning?

85. Explain why $x^3 \cdot x^5$ is x^8, and not x^{15}

Answers

2. a) 10,000 b) 17 c) $\frac{4}{81}$ d) 0.125	4. a) 512 b) 8 c) $\frac{27}{64}$ d) 0.064
6. a) 64 b) -64	8. a) $-\frac{4}{9}$ b) $\frac{4}{9}$
10. a) -0.0001 b) 0.0001	12. x^6
14. q^{42}	16. a) 3^{16} b) 5^5
18. a) w^6 b) u^{94}	20. y^9
22. c^{18}	24. n^{y+2}
26. x^{p+q}	28. a) b^{14} b) 3^{16}
30. a) x^{2y} b) 7^{ab}	32. a) $25x^2$ b) $16a^2b^2$
34. a) $-343n^3$ b) $81x^4y^4z^4$	36. a) w^{22} b) $32x^5y^{20}$
38. a) $-1000q^6p^{12}$ b) n^{40}	40. a) $48y^4$ b) $25,000k^{24}$
42. a) $432b^5$ b) $\frac{1}{200}j^{16}$	44. a) $128r^8$ b) $\frac{1}{200}j^{16}$
46. $-18y^{11}$	48. $72u^7$
50. $4f^{11}$	52. $36a^5b^7$
54. $8r^2s^5$	56. $\frac{1}{2}x^3y^3$
58. x^{14}	60. a^{36}
62. $8m^{18}$	64. $1000x^6y^3$
66. $16a^{12}b^8$	68. $\frac{8}{27}x^6y^3$
70. $1024a^{10}$	72. $25000p^{24}$
74. $x^{18}y^{18}$	76. $144m^8n^{22}$
78. $1,000,000$	80. \$168.07
82. Answers will vary.	84. Answers will vary.

Attributions

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7.2 Use Quotient Property of Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the Quotient Property for Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the quotient to a Power Property
- Simplify expressions by applying several properties

Simplify Expressions Using the Quotient Property for Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties below.

Summary of Exponent Properties for Multiplication

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. You have learned to simplify fractions by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property. This property will also help you work with algebraic fractions—which are also quotients.

Equivalent Fractions Property

If $a, b,$ and c are whole numbers where $b \neq 0, c \neq 0,$
then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$

As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents.

When the larger exponent was in the numerator, we were left with factors in the numerator.

When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1

We write:

$\frac{x^5}{x^2}$	$\frac{x^2}{x^3}$
x^{5-2}	$\frac{1}{x^{3-2}}$
x^3	$\frac{1}{x}$

This leads to the *Quotient Property for Exponents*.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then
$\frac{a^m}{a^n} = a^{m-n}, m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, n > m$

A couple of examples with numbers may help to verify this property.

$$\begin{aligned} \frac{3^4}{3^2} &= 3^{4-2} & \frac{5^2}{5^3} &= \frac{1}{5^{3-2}} \\ \frac{81}{9} &= 3^2 & \frac{25}{125} &= \frac{1}{5^1} \\ 9 &= 9\checkmark & \frac{1}{5} &= \frac{1}{5}\checkmark \end{aligned}$$

EXAMPLE 1

Simplify: a) $\frac{x^9}{x^7}$ b) $\frac{3^{10}}{3^2}$.

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

Since 9 > 7, there are more factors of *x* in the numerator.	$\frac{x^9}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

b.

Since 10 > 2, there are more factors of *x* in the numerator.	$\frac{3^{10}}{3^2}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	3^{10-2}
Simplify.	3^8

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

TRY IT 1.1

Simplify: a) $\frac{x^{15}}{x^{10}}$ b) $\frac{6^{14}}{6^5}$.

Show answer

a) x^5 b) 6^9

TRY IT 1.2

Simplify: a) $\frac{y^{43}}{y^{37}}$ b) $\frac{10^{15}}{10^{7}}$.

Show answer
a) y^6 b) 10^8

EXAMPLE 2

Simplify: a) $\frac{b^8}{b^{12}}$ b) $\frac{7^3}{7^5}$.

Solution
To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

Since 12 > 8, there are more factors of *b* in the denominator.	$\frac{b^8}{b^{12}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{b^{12-8}}$
Simplify.	$\frac{1}{b^4}$

b.

Since 5 > 3, there are more factors of 3 in the denominator.	$\frac{7^3}{7^5}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{7^{5-3}}$
Simplify.	$\frac{1}{7^2}$
Simplify.	$\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

TRY IT 2.1

Simplify: a) $\frac{x^{18}}{x^{22}}$ b) $\frac{12^{15}}{12^{30}}$.

Show answer

a) $\frac{1}{x^4}$ b) $\frac{1}{12^{15}}$

TRY IT 2.2

Simplify: a) $\frac{m^7}{m^{15}}$ b) $\frac{9^8}{9^{19}}$.

Show answer

a) $\frac{1}{m^8}$ b) $\frac{1}{9^{11}}$

Notice the difference in the two previous examples:

- If we start with more factors in the numerator, we will end up with factors in the numerator.
- If we start with more factors in the denominator, we will end up with factors in the denominator.

The first step in simplifying an expression using the Quotient Property for Exponents is to determine whether the exponent is larger in the numerator or the denominator.

EXAMPLE 3

Simplify: a) $\frac{a^5}{a^9}$ b) $\frac{x^{11}}{x^7}$.

Solution

a. Is the exponent of a larger in the numerator or denominator? Since 9 > 5, there are more a's in the denominator and so we will end up with factors in the denominator.

	$\frac{a^5}{a^9}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{a^{9-5}}$
Simplify.	$\frac{1}{a^4}$

b. Notice there are more factors of x in the numerator, since 11 > 7. So we will end up with factors in the numerator.

	$\frac{x^{11}}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	x^{11-7}
Simplify.	x^4

TRY IT 3.1

Simplify: a) $\frac{b^{19}}{b^{11}}$ b) $\frac{z^5}{z^{11}}$.

Show answer

a) b^8 b) $\frac{1}{z^6}$

TRY IT 3.2

Simplify: a) $\frac{p^9}{p^{17}}$ b) $\frac{w^{13}}{w^9}$.

Show answer

a) $\frac{1}{p^8}$ b) w^4

Simplify Expressions with an Exponent of Zero

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

$$\frac{2}{2} = 1\frac{17}{17} = 1\frac{-43}{-43} = 1$$

In words, a number divided by itself is 1. So, $\frac{x}{x} = 1$, for any x $(x \neq 0)$, since any number divided by itself is 1

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1

	$\frac{8}{8} = 1$
Write 8 as 2^3.	$\frac{2^3}{2^3} = 1$
Subtract exponents.	$2^{3-3} = 1$
Simplify.	$2^0 = 1$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:

$$\frac{a^m}{a^m} \qquad \frac{a^m}{a^m}$$

$$a^{m-m} \qquad \frac{\overbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}^{m \text{ factors}}}{\underbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}_{m \text{ factors}}}$$

$$a^0 \qquad 1$$

We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Zero Exponent

If a is a non-zero number, then $a^0 = 1$.

Any nonzero number raised to the zero power is 1

In this text, we assume any variable that we raise to the zero power is not zero.

EXAMPLE 4

Simplify: a) 9^0 b) n^0.

Solution
The definition says any non-zero number raised to the zero power is 1

a) Use the definition of the zero exponent.	9^0 1
b) Use the definition of the zero exponent.	n^0 1

TRY IT 4.1

Simplify: a) 15^0 b) m^0.

Show answer
a) 1 b) 1

TRY IT 4.2

Simplify: a) k^0 b) 29^0.

Show answer
a) 1 b) 1

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

What about raising an expression to the zero power? Let's look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the product to a power rule.	2^0x^0
Use the zero exponent property.	$1 \cdot 1$
Simplify.	1

This tells us that any nonzero expression raised to the zero power is one.

EXAMPLE 5

Simplify: a) $(5b)^0$ b) $\left(-4a^2b\right)^0$.

Solution

a)	$(5b)^0$
Use the definition of the zero exponent.	1
b)	$\left(-4a^2b\right)^0$
Use the definition of the zero exponent.	1

TRY IT 5.1

Simplify: a) $(11z)^0$ b) $\left(-11pq^3\right)^0$.

Show answer

a) 1 b) 1

TRY IT 5.2

Simplify: a) $(-6d)^0$ b) $\left(-8m^2n^3\right)^0$.

Show answer

a) 1 b) 1

Simplify Expressions Using the Quotient to a Power Property

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$
This means:	$\frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y}$
Multiply the fractions.	$\frac{x \cdot x \cdot x}{y \cdot y \cdot y}$
Write with exponents.	$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We write:	$\left(\frac{x}{y}\right)^3$
	$\frac{x^3}{y^3}$

This leads to the *Quotient to a Power Property for Exponents*.

Quotient to a Power Property for Exponents

If a and b are real numbers, $b \neq 0$, and m is a counting number, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:

$$\begin{aligned} \left(\frac{2}{3}\right)^3 &= \frac{2^3}{3^3} \\ \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} &= \frac{8}{27} \\ \frac{8}{27} &= \frac{8}{27}\checkmark \end{aligned}$$

EXAMPLE 6

Simplify: a) $\left(\frac{3}{7}\right)^2$ b) $\left(\frac{b}{3}\right)^4$ c) $\left(\frac{k}{j}\right)^3$.

Solution

a)

	$\left(\frac{3}{7}\right)^2$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{3^2}{7^2}$
Simplify.	$\frac{9}{49}$

b)

	$\left(\frac{b}{3}\right)^4$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

c)

	$\left(\frac{k}{j}\right)^3$
Raise the numerator and denominator to the third power.	$\frac{k^3}{j^3}$

TRY IT 6.1

Simplify: a) $\left(\frac{5}{8}\right)^2$ b) $\left(\frac{p}{10}\right)^4$ c) $\left(\frac{m}{n}\right)^7$.

Show answer

a) $\frac{25}{64}$ b) $\frac{p^4}{10,000}$ c) $\frac{m^7}{n^7}$

TRY IT 6.2

Simplify: a) $\left(\frac{1}{3}\right)^3$ b) $\left(\frac{-2}{q}\right)^3$ c) $\left(\frac{w}{x}\right)^4$.

Show answer

a) $\frac{1}{27}$ b) $\frac{-8}{q^3}$ c) $\frac{w^4}{x^4}$

Simplify Expressions by Applying Several Properties

We'll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Summary of Exponent Properties

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{b^m} = a^{m-n}, a \neq 0, m > n$ $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	$a^o = 1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

EXAMPLE 7

Simplify: $\frac{\left(y^4\right)^2}{y^6}$.

Solution

	$\frac{\left(y^4\right)^2}{y^6}$
Multiply the exponents in the numerator.	$\frac{y^8}{y^6}$
Subtract the exponents.	y^2

TRY IT 7.1

Simplify: $\frac{\left(m^5\right)^4}{m^7}$.

Show answer

m^{13}

TRY IT 7.2

Simplify: $\frac{\left(k^2\right)^6}{k^7}$.

Show answer

k^5

EXAMPLE 8

Simplify: $\frac{b^{12}}{\left(b^2\right)^6}$.

Solution

	$\frac{b^{12}}{\left(b^2\right)^6}$
Multiply the exponents in the numerator.	$\frac{b^{12}}{b^{12}}$
Subtract the exponents.	b^0
Simplify.	1

TRY IT 8.1

Simplify: $\frac{n^{12}}{\left(n^3\right)^4}$.

Show answer

1

TRY IT 8.2

Simplify: $\dfrac{x^{15}}{\left(x^3\right)^5}$.

Show answer
1

EXAMPLE 9

Simplify: $\left(\dfrac{y^9}{y^4}\right)^2$.

Solution

	$\left(\dfrac{y^9}{y^4}\right)^2$
Remember parentheses come before exponents. Notice the bases are the same, so we can simplify inside the parentheses. Subtract the exponents.	$\left(y^5\right)^2$
Multiply the exponents.	y^{10}

TRY IT 9.1

Simplify: $\left(\dfrac{r^5}{r^3}\right)^4$.

Show answer
r^8

TRY IT 9.2

Simplify: $\left(\frac{v^6}{v^4}\right)^3$.

Show answer
v^6

EXAMPLE 10

Simplify: $\left(\frac{j^2}{k^3}\right)^4$.

Solution
Here we cannot simplify inside the parentheses first, since the bases are not the same.

	$\left(\frac{j^2}{k^3}\right)^4$
Raise the numberator and denominator to the third power using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	
Use the Power Property and simplify.	

TRY IT 10.1

Simplify: $\left(\frac{a^3}{b^2}\right)^4$.

Show answer
$\frac{a^{12}}{b^8}$

TRY IT 10.2

Simplify: $\left(\frac{q^7}{r^5}\right)^3$.

Show answer

$\frac{q^{21}}{r^{15}}$

EXAMPLE 11

Simplify: $\left(\frac{2m^2}{5n}\right)^4$.

Solution

	$\left(\frac{2m^2}{5n}\right)^4$
Raise the numberator and denominator to the fourth power, using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{\left(2m^2\right)^4}{(5n)^4}$
Raise each factor to the fourth power.	$\frac{\left(2m^2\right)^4}{(5n)^4}$
Use the Power Property and simplify.	$\frac{16m^8}{625n^4}$

TRY IT 11.1

Simplify: $\left(\frac{7x^3}{9y}\right)^2$.

Show answer

$\frac{49x^6}{81y^2}$

TRY IT 11.2

Simplify: $\left(\frac{3x^4}{7y}\right)^2$.

Show answer

$\frac{9x^8}{49y^2}$

EXAMPLE 12

Simplify: $\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$.

Solution

	$\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$
Use the Power Property, $\left(a^m\right)^n = a^{m\cdot n}$.	$\frac{\left(x^{12}\right)\left(x^{10}\right)}{\left(x^{30}\right)}$
Add the exponents in the numerator.	$\frac{x^{22}}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{x^8}$

TRY IT 12.1

Simplify: $\frac{\left(a^2\right)^3\left(a^2\right)^4}{\left(a^4\right)^5}$.

Show answer

$\frac{1}{a^6}$

TRY IT 12.2

Simplify: $\frac{\left(p^3\right)^4\left(p^5\right)^3}{\left(p^7\right)^6}$.

Show answer

$\frac{1}{p^{15}}$

EXAMPLE 13

Simplify: $\frac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$.

Solution

	$\frac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\frac{(10)^2\left(p^3\right)^2}{(5)^3(p)^3(2)^4\left(p^5\right)^4}$
Use the Power Property, $\left(a^m\right)^n = a^{m\cdot n}$.	$\frac{100p^6}{125p^3 \cdot 16p^{20}}$
Add the exponents in the denominator.	$\frac{100p^6}{125 \cdot 16p^{23}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{100}{125 \cdot 16p^{17}}$
Simplify.	$\frac{1}{20p^{17}}$

TRY IT 13.1

Simplify: $\frac{\left(3r^3\right)^2\left(r^3\right)^7}{\left(r^3\right)^3}$.

Show answer

$9r^{18}$

TRY IT 13.2

Simplify: $\frac{\left(2x^4\right)^5}{\left(4x^3\right)^2\left(x^3\right)^5}$.

Show answer

$\frac{2}{x}$

Divide Monomials

You have now been introduced to all the properties of exponents and used them to simplify expressions. Next, you'll see how to use these properties to divide monomials. Later, you'll use them to divide polynomials.

EXAMPLE 14

Find the quotient: $56x^7 \div 8x^3$.

Solution

	$56x^7 \div 8x^3$
Rewrite as a fraction.	$\frac{56x^7}{8x^3}$
Use fraction multiplication.	$\frac{56}{8} \cdot \frac{x^7}{x^3}$
Simplify and use the Quotient Property.	$7x^4$

TRY IT 14.1

Find the quotient: $42y^9 \div 6y^3$.

Show answer
$7y^6$

TRY IT 14.2

Find the quotient: $48z^8 \div 8z^2$.

Show answer
$6z^6$

EXAMPLE 15

Find the quotient: $\frac{45a^2b^3}{-5ab^5}$.

Solution

	$\dfrac{45a^2b^3}{-5ab^5}$
Use fraction multiplication.	$\dfrac{45}{-5}\cdot\dfrac{a^2}{a}\cdot\dfrac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9\cdot a\cdot\dfrac{1}{b^2}$
Multiply.	$-\dfrac{9a}{b^2}$

TRY IT 15.1

Find the quotient: $\dfrac{-72a^7b^3}{8a^{12}b^4}$.

Show answer

$-\dfrac{9}{a^5b}$

TRY IT 15.2

Find the quotient: $\dfrac{-63c^8d^3}{7c^{12}d^2}$.

Show answer

$\dfrac{-9d}{c^4}$

EXAMPLE 16

Find the quotient: $\dfrac{24a^5b^3}{48ab^4}$.

Solution

	$\frac{24a^5b^3}{48ab^4}$
Use fraction multiplication.	$\frac{24}{48} \cdot \frac{a^5}{a} \cdot \frac{b^3}{b^4}$
Simplify and use the Quotient Property.	$\frac{1}{2} \cdot a^4 \cdot \frac{1}{b}$
Multiply.	$\frac{a^4}{2b}$

TRY IT 16.1

Find the quotient: $\frac{16a^7b^6}{24ab^8}$.

Show answer

$\frac{2a^6}{3b^2}$

TRY IT 16.2

Find the quotient: $\frac{27p^4q^7}{-45p^{12}q}$.

Show answer

$-\frac{3q^6}{5p^8}$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

EXAMPLE 17

Find the quotient: $\dfrac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Be very careful to simplify $\dfrac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\dfrac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$\dfrac{2y^6}{3x^4}$

TRY IT 17.1

Find the quotient: $\dfrac{28x^5y^{14}}{49x^9y^{12}}$.

Show answer

$\dfrac{4y^2}{7x^4}$

TRY IT 17.2

Find the quotient: $\dfrac{30m^5n^{11}}{48m^{10}n^{14}}$.

Show answer

$\dfrac{5}{8m^5n^3}$

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we'll first find the product of two monomials in the numerator before we simplify the fraction. This follows the order of operations. Remember, a fraction bar is a grouping symbol.

EXAMPLE 18

Find the quotient: $\dfrac{\left(6x^2y^3\right)\left(5x^3y^2\right)}{\left(3x^4y^5\right)}$.

Solution

	$\dfrac{\left(6x^2y^3\right)\left(5x^3y^2\right)}{\left(3x^4y^5\right)}$
Simplify the numerator.	$\dfrac{30x^5y^5}{3x^4y^5}$
Simplify.	$10x$

TRY IT 18.1

Find the quotient: $\dfrac{\left(6a^4b^5\right)\left(4a^2b^5\right)}{12a^5b^8}$.

Show answer
$2ab^2$

TRY IT 18.2

Find the quotient: $\dfrac{\left(-12x^6y^9\right)\left(-4x^5y^8\right)}{-12x^{10}y^{12}}$.

Show answer
$-4xy^5$

Additional Online Resources

- Rational Expressions
- Dividing Monomials

- Dividing Monomials 2

Key Concepts

- **Quotient Property for Exponents:**
    - If a is a real number, $a \neq 0$, and m, n are whole numbers, then: $\frac{a^m}{a^n} = a^{m-n}, m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{m-n}}, n > m$
- **Zero Exponent**
    - If a is a non-zero number, then $a^0 = 1$.
- **Quotient to a Power Property for Exponents**:
    - If a and b are real numbers, $b \neq 0$, and m is a counting number, then: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
    - To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
    - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n$	$= a^{m+n}$
Power Property	$(a^m)^n$	$= a^{m \cdot n}$
Product to a Power	$(ab)^m$	$= a^m b^m$
Quotient Property	$\frac{a^m}{b^m} =$	$a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n} =$	$\frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	a^o	$= 1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$= \frac{a^m}{b^m}, b \neq 0$

Practice Makes Perfect

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

1. a) $\frac{x^{18}}{x^3}$ b) $\frac{5^{12}}{5^3}$	2. a) $\frac{y^{20}}{y^{10}}$ b) $\frac{7^{16}}{7^2}$
3. a) $\frac{p^{21}}{p^7}$ b) $\frac{4^{16}}{4^4}$	4. a) $\frac{u^{24}}{u^3}$ b) $\frac{9^{15}}{9^5}$
5. a) $\frac{q^{18}}{q^{36}}$ b) $\frac{10^2}{10^3}$	6. a) $\frac{t^{10}}{t^{40}}$ b) $\frac{8^3}{8^5}$
7. a) $\frac{b}{b^9}$ b) $\frac{4}{4^6}$	8. a) $\frac{x}{x^7}$ b) $\frac{10}{10^3}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

9. a) 20^0 b) b^0	10. a) 13^0 b) k^0
11. a) -27^0 b) $-\left(27^0\right)$	12. a) -15^0 b) $-\left(15^0\right)$
13. a) $(25x)^0$ b) $25x^0$	14. a) $(6y)^0$ b) $6y^0$
15. a) $(12x)^0$ b) $\left(-56p^4q^3\right)^0$	16. a) $7y^0(17y)^0$ b) $\left(-93c^7d^{15}\right)^0$
17. a) $12n^0 - 18m^0$ b) $(12n)^0 - (18m)^0$	18. a) $15r^0 - 22s^0$ b) $(15r)^0 - (22s)^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

19. a) $\left(\frac{3}{4}\right)^3$ b) $\left(\frac{p}{2}\right)^5$ c) $\left(\frac{x}{y}\right)^6$	20. a) $\left(\frac{2}{5}\right)^2$ b) $\left(\frac{x}{3}\right)^4$ c) $\left(\frac{a}{b}\right)^5$
21. a) $\left(\frac{a}{3b}\right)^4$ b) $\left(\frac{5}{4m}\right)^2$	22. a) $\left(\frac{x}{2y}\right)^3$ b) $\left(\frac{10}{3q}\right)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

23. $\frac{\left(a^2\right)^3}{a^4}$

24. $\frac{\left(p^3\right)^4}{p^5}$

25. $\frac{\left(y^3\right)^4}{y^{10}}$

26. $\frac{\left(x^4\right)^5}{x^{15}}$

27. $\frac{u^6}{\left(u^3\right)^2}$

28. $\frac{v^{20}}{\left(v^4\right)^5}$

29. $\frac{m^{12}}{\left(m^8\right)^3}$

30. $\frac{n^8}{\left(n^6\right)^4}$

31. $\left(\frac{p^9}{p^3}\right)^5$

32. $\left(\frac{q^8}{q^2}\right)^3$

33. $\left(\frac{r^2}{r^6}\right)^3$

34. $\left(\frac{m^4}{m^7}\right)^4$

35. $\left(\frac{p}{r^{11}}\right)^2$

36. $\left(\frac{a}{b^6}\right)^3$

37. $\left(\frac{w^5}{x^3}\right)^8$

38. $\left(\frac{y^4}{z^{10}}\right)^5$

39. $\left(\frac{2j^3}{3k}\right)^4$

40. $\left(\frac{3m^5}{5n}\right)^3$

41. $\left(\frac{3c^2}{4d^6}\right)^3$

42. $\left(\frac{5u^7}{2v^3}\right)^4$

43. $\left(\frac{k^2k^8}{k^3}\right)^2$

44. $\left(\frac{j^2j^5}{j^4}\right)^3$

45. $\frac{\left(t^2\right)^5\left(t^4\right)^2}{\left(t^3\right)^7}$

46. $\frac{\left(q^3\right)^6\left(q^2\right)^3}{\left(q^4\right)^8}$

47. $\frac{\left(-2p^2\right)^4\left(3p^4\right)^2}{\left(-6p^3\right)^2}$

48. $\frac{\left(-2k^3\right)^2\left(6k^2\right)^4}{\left(9k^4\right)^2}$

49. $\frac{\left(-4m^3\right)^2\left(5m^4\right)^3}{\left(-10m^6\right)^3}$

50. $\frac{\left(-10n^2\right)^3\left(4n^5\right)^2}{\left(2n^8\right)^2}$

Divide Monomials

In the following exercises, divide the monomials.

51. $56b^8 \div 7b^2$	52. $63v^{10} \div 9v^2$
53. $-88y^{15} \div 8y^3$	54. $-72u^{12} \div 12u^4$
55. $\frac{45a^6b^8}{-15a^{10}b^2}$	56. $\frac{54x^9y^3}{-18x^6y^{15}}$
57. $\frac{15r^4s^9}{18r^9s^2}$	58. $\frac{20m^8n^4}{30m^5n^9}$
59. $\frac{18a^4b^8}{-27a^9b^5}$	60. $\frac{45x^5y^9}{-60x^8y^6}$
61. $\frac{64q^{11}r^9s^3}{48q^6r^8s^5}$	62. $\frac{65a^{10}b^8c^5}{42a^7b^6c^8}$
63. $\frac{\left(10m^5n^4\right)\left(5m^3n^6\right)}{25m^7n^5}$	64. $\frac{\left(-18p^4q^7\right)\left(-6p^3q^8\right)}{-36p^{12}q^{10}}$
65. $\frac{\left(6a^4b^3\right)\left(4ab^5\right)}{(12a^2b)(a^3b)}$	66. $\frac{\left(4u^2v^5\right)\left(15u^3v\right)}{(12u^3v)(u^4v)}$

Mixed Practice

67.

a) $24a^5 + 2a^5$
b) $24a^5 - 2a^5$
c) $24a^5\dot{2}a^5$
d) $24a^5 \div 2a^5$

69.

a) $p^4 \cdot p^6$
b) $\left(p^4\right)^6$

70.

a) $q^5 \cdot q^3$
b) $\left(q^5\right)^3$

71.

a) $\frac{y^3}{y}$
b) $\frac{y}{y^3}$

72.

a) $\frac{z^6}{z^5}$
b) $\frac{z^5}{z^6}$

73. $\left(8x^5\right)(9x) \div 6x^3$

74. $(4y)\left(12y^7\right) \div 8y^2$

75. $\frac{27a^7}{3a^3} + \frac{54a^9}{9a^5}$

76. $\frac{32c^{11}}{4c^5} + \frac{42c^9}{6c^3}$

77. $\frac{32y^5}{8y^2} - \frac{60y^{10}}{5y^7}$

78. $\frac{48x^6}{6x^4} - \frac{35x^9}{7x^7}$

79. $\frac{63r^6s^3}{9r^4s^2} - \frac{72r^2s^2}{6s}$

80. $\frac{56y^4z^5}{7y^3z^3} - \frac{45y^2z^2}{5y}$

Everyday Math

81. **Memory** One megabyte is approximately 10^6 bytes. One gigabyte is approximately 10^9 bytes. How many megabytes are in one gigabyte?

82. **Memory** One gigabyte is approximately 10^9 bytes. One terabyte is approximately 10^{12} bytes. How many gigabytes are in one terabyte?

Writing Exercises

83. Jennifer thinks the quotient $\frac{a^{24}}{a^{6}}$ simplifies to a^4. What is wrong with her reasoning?	84. Maurice simplifies the quotient $\frac{d^7}{d}$ by writing $\frac{\cancel{d}^7}{\cancel{d}} = 7$. What is wrong with his reasoning?
85. When Drake simplified -3^0 and $(-3)^0$ he got the same answer. Explain how using the Order of Operations correctly gives different answers.	86. Robert thinks x^0 simplifies to 0. What would you say to convince Robert he is wrong?

Answers

2. a) y^{10} b) 7^{14}	4. a) u^{21} b) 9^{10}
6. a) $\frac{1}{t^{30}}$ b) $\frac{1}{64}$	8. a) $\frac{1}{x^6}$ b) $\frac{1}{100}$
10. a) 1 b) 1	12. a) -1 b) -1
14. a) 1 b) 6	16. a) 7 b) 1
18. a) -7 b) 0	20. a) $\frac{4}{25}$ b) $\frac{x^4}{81}$ c) $\frac{a^5}{b^5}$
22. a) $\frac{x^3}{8y^3}$ b) $\frac{10{,}000}{81q^4}$	24. p^7
26. x^5	28. 1
30. $\frac{1}{n^{12}}$	32. q^{18}
34. $\frac{1}{m^{12}}$	36. $\frac{a^3}{b^{18}}$
38. $\frac{y^{20}}{z^{50}}$	40. $\frac{27m^{15}}{125n^3}$
42. $\frac{625u^{28}}{16v^{12}}$	44. j^9
46. $\frac{1}{q^8}$	48. $64k^6$
50. $-4,000$	52. $7v^8$
54. $-6u^8$	56. $-\frac{3x^3}{y^{12}}$
58. $\frac{-2m^3}{3n^5}$	60. $\frac{-3y^3}{4x^3}$
62. $\frac{65a^3b^2}{42c^3}$	64. $\frac{-3q^5}{p^5}$
66. $\frac{5v^4}{u^2}$	68. a) $18n^{10}$ b) $12n^{10}$ c) $45n^{20}$ d) 5
70. a) q^8 b) q^{15}	72. a) z b) $\frac{1}{z}$
74. $6y^6$	76. $15c^6$
78. $3x^2$	80. $-yz^2$
82. 10^3	84. Answers will vary.

86. Answers will vary.	

Attributions

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7.3 Integer Exponents and Scientific Notation

Learning Objectives

By the end of this section, you will be able to:

- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents introduced earlier in this chapter, has two forms depending on whether the exponent is larger in the numerator or the denominator.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and $m \text{ and } n$ are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n}, m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, n > m$$

What if we just subtract exponents regardless of which is larger?

Let’s consider $\frac{x^2}{x^5}$.

We subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^2}{x^5}$$
$$x^{2-5}$$
$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}$$

$$\frac{1}{x^3}$$

This implies that $x^{-3}=\frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Negative Exponent

If n is an integer and $a\neq 0$, then $a^{-n}=\frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

EXAMPLE 1

Simplify: a) 4^{-2} b) 10^{-3}.

Solution

a)	4^{-2}
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{4^2}$
Simplify.	$\frac{1}{16}$
b)	10^{-3}
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

TRY IT 1.1

Simplify: a) 2^{-3} b) 10^{-7}.

Show answer
a) $\frac{1}{8}$ b) $\frac{1}{10^7}$

TRY IT 1.2

Simplify: a) 3^{-2} b) 10^{-4}.

Show answer
a) $\frac{1}{9}$ b) $\frac{1}{10{,}000}$

In (Example 1) we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^n}$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1\cdot\frac{a^n}{1}$
Multiply.	a^n

This leads to the Property of Negative Exponents.

Property of Negative Exponents

If n is an integer and $a\neq 0$, then $\frac{1}{a^{-n}}=a^n$.

EXAMPLE 2

Simplify: a) $\frac{1}{y^{-4}}$ b) $\frac{1}{3^{-2}}$.

Solution

a)	$\frac{1}{y^{-4}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	y^4
b)	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	3^2
Simplify.	9

TRY IT 2.1

Simplify: a) $\frac{1}{p^{-8}}$ b) $\frac{1}{4^{-3}}$.

Show answer

a) p^8 b) 64

TRY IT 2.2

Simplify: a) $\frac{1}{q^{-7}}$ b) $\frac{1}{2^{-4}}$.

Show answer

a) q^7 b) 16

Suppose now we have a fraction raised to a negative exponent. Let’s use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.	
This tells us that:	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Quotient to a Negative Exponent Property

If a and b are real numbers, $a \neq 0, b \neq 0$, and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

EXAMPLE 3

Simplify: a) $\left(\frac{5}{7}\right)^{-2}$ b) $\left(-\frac{2x}{y}\right)^{-3}$.

Solution

a)	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(\frac{7}{5}\right)^{2}$
Simplify.	$\frac{49}{25}$
b)	$\left(-\frac{2x}{y}\right)^{-3}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(-\frac{y}{2x}\right)^{3}$
Simplify.	$-\frac{y^3}{8x^3}$

TRY IT 3.1

Simplify: a) $\left(\frac{2}{3}\right)^{-4}$ b) $\left(-\frac{6m}{n}\right)^{-2}$.

Show answer

a) $\frac{81}{16}$ b) $\frac{n^2}{36m^2}$

TRY IT 3.2

Simplify: a) $\left(\frac{3}{5}\right)^{-3}$ b) $\left(-\frac{a}{2b}\right)^{-4}$.

Show answer

a) $\frac{125}{27}$ b) $\frac{16b^4}{a^4}$

When simplifying an expression with exponents, we must be careful to correctly identify the base.

EXAMPLE 4

Simplify: a) $(-3)^{-2}$ b) -3^{-2} c) $\left(-\frac{1}{3}\right)^{-2}$ d) $-\left(\frac{1}{3}\right)^{-2}$.

Solution

a) Here the exponent applies to the base -3.	$(-3)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\frac{1}{(-3)^{-2}}$
Simplify.	$\frac{1}{9}$
b) The expression -3^{-2} means "find the opposite of 3^{-2}." Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.	-3^{-2}
Rewrite as a product with -1.	$-1 \cdot 3^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \frac{1}{3^2}$
Simplify.	$-\frac{1}{9}$
c) Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.	$\left(-\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\left(-\frac{3}{1}\right)^{2}$
Simplify.	9
d) The expression $-\left(\frac{1}{3}\right)^{-2}$ means "find the opposite of $\left(\frac{1}{3}\right)^{-2}$." Here the exponent applies to the base $\left(\frac{1}{3}\right)$.	
Rewrite as a product with -1.	$-1 \cdot \left(\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \left(\frac{3}{1}\right)^{2}$
Simplify.	-9

TRY IT 4.1

Simplify: a) $(-5)^{-2}$ b) -5^{-2} c) $\left(-\frac{1}{5}\right)^{-2}$ d) $-\left(\frac{1}{5}\right)^{-2}$.

Show answer

a) $\frac{1}{25}$ b) $-\frac{1}{25}$ c) 25 d) -25

TRY IT 4.2

Simplify: a) $(-7)^{-2}$ b) -7^{-2}, c) $\left(-\frac{1}{7}\right)^{-2}$ d) $-\left(\frac{1}{7}\right)^{-2}$.

Show answer
a) $\frac{1}{49}$ b) $-\frac{1}{49}$ c) 49 d) -49

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

EXAMPLE 5

Simplify: a) $4 \cdot 2^{-1}$ b) $(4 \cdot 2)^{-1}$.

Solution

a) Do exponents before multiplication.	$4 \cdot 2^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$4 \cdot \frac{1}{2^1}$
Simplify.	2
b)	$(4 \cdot 2)^{-1}$
Simplify inside the parentheses first.	$(8)^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{8^1}$
Simplify.	$\frac{1}{8}$

TRY IT 5.1

Simplify: a) $6 \cdot 3^{-1}$ b) $(6 \cdot 3)^{-1}$.

Show answer
a) 2 b) $\frac{1}{18}$

TRY IT 5.2

Simplify: a) $8 \cdot 2^{-2}$ b) $(8 \cdot 2)^{-2}$.

Show answer

a) 2 b) $\frac{1}{16}$

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

EXAMPLE 6

Simplify: a) x^{-6} b) $\left(u^4\right)^{-3}$.

Solution

a)	x^{-6}
Use the definition of a negative exponent $a^{-n} = \frac{1}{a^n}$	$\frac{1}{x^6}$
b)	$\left(u^4\right)^{-3}$
Use the definition of a negative exponent $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(u^4)^3}$
Simplify.	$\frac{1}{u^{12}}$

TRY IT 6.1

Simplify: a) y^{-7} b) $\left(z^3\right)^{-5}$.

Show answer

a) $\frac{1}{y^7}$ b) $\frac{1}{z^{15}}$

TRY IT 6.2

Simplify: a) p^{-9} b) $\left(q^4\right)^{-6}$.

Show answer
a) $\frac{1}{p^9}$ b) $\frac{1}{q^{24}}$

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

EXAMPLE 7

Simplify: a) $5y^{-1}$ b) $(5y)^{-1}$ c) $(-5y)^{-1}$.

Solution

a) Notice the exponent applies to just the base.	$5y^{-1}$
Take the reciprocal of y and change the sign of the exponent.	$5 \cdot \frac{1}{y^1}$
Simplify.	$\frac{5}{y}$
b) Her the parentheses make the exponent apply to the base.	$(5y)^{-1}$
Take the reciprocal of $5y$ and change the sign of the exponent.	$\frac{1}{(5y)^1}$
Simplify.	$\frac{1}{5y}$
c) The base here is $-5y$.	$(-5y)^{-1}$
Take the reciprocal of $-5y$ and change the sign of the exponent.	$\frac{1}{(-5y)^1}$
Simplify.	$\frac{1}{-5y}$
Use $\frac{a}{-b} = -\frac{a}{b}$	$-\frac{1}{5y}$

TRY IT 7.1

Simplify: a) $8p^{-1}$ b) $(8p)^{-1}$ c) $(-8p)^{-1}$.

Show answer

a) $\frac{8}{p}$ b) $\frac{1}{8p}$ c) $-\frac{1}{8p}$

TRY IT 7.2

Simplify: a) $11q^{-1}$ b) $(11q)^{-1}-(11q)^{-1}$ c) $(-11q)^{-1}$.

Show answer

a) $\frac{1}{11q}$ b) $\frac{1}{11q}-\frac{1}{11q}$ c) $-\frac{1}{11q}$

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n} = a^{m-n}$, for $a \neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Summary of Exponent Properties

If a and b are real numbers, and m and n are integers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{a^n}$	$=$	$a^{m-n}, a \neq 0$
Zero Exponent Property	a^0	$=$	$1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m},\ b \neq 0$
Properties of Negative Exponents	a^{-n}	$=$	$\frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n}$	$=$	$\left(\frac{b}{a}\right)^n$

EXAMPLE 8

Simplify: a) $x^{-4} \cdot x^{6}$ b) $y^{-6} \cdot y^{4}$ c) $z^{-5} \cdot z^{-3}$.

Solution

a.

	$x^{-4} \cdot x^{6}$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	{x}^{-4 +6}
Simplify	x^{2}

b.

	$y^{-6} \cdot y^{4}$
Notice the same bases, so add the exponents.	y^{-6+4}
Simplify.	y^{-2}
Use the definition of a negative exponent, $\frac{1}{a^n}$.	$\frac{1}{y^2}$

c.

	$z^{-5} \cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Take the reciprocal and change the sign of the exponent, using the definition of a negative exponent.	$\frac{1}{z^8}$

TRY IT 8.1

Simplify: a) $x^{-3} \cdot x^{7}$ b) $y^{-7} \cdot y^{2}$ c) $z^{-4} \cdot z^{-5}$.

Show answer

a) x^4 b) $\frac{1}{y^5}$ c) $\frac{1}{z^9}$

TRY IT 8.2

Simplify: a) $a^{-1} \cdot a^{6}$ b) $b^{-8} \cdot b^{4}$ c) $c^{-8} \cdot c^{-7}$.

Show answer
a) a^5 b) $\frac{1}{b^4}$ c) $\frac{1}{c^{15}}$

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

EXAMPLE 9

Simplify: $\left(m^4n^{-3}\right)\left(m^{-5}n^{-2}\right)$.

Solution

	$\left(m^4n^{-3}\right)\left(m^{-5}n^{-2}\right)$
Use the Commutative Property to get like bases together.	$m^4m^{-5}\cdot n^{-2}n^{-3}$
Add the exponents for each base.	$m^{-1}\cdot n^{-5}$
Take the reciprocals and change the signs of the exponents.	$\frac{1}{m^1}\cdot\frac{1}{n^5}$
Simplify.	$\frac{1}{mn^5}$

TRY IT 9.1

Simplify: $\left(p^6q^{-2}\right)\left(p^{-9}q^{-1}\right)$.

Show answer
$\frac{1}{p^3q^3}$

TRY IT 9.2

Simplify: $\left(r^5s^{-3}\right)\left(r^{-7}s^{-5}\right)$.

Show answer
$\frac{1}{r^2s^8}$

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

EXAMPLE 10

Simplify: $\left(2x^{-6}y^{8}\right)\left(-5x^{5}y^{-3}\right)$.

Solution

	$\left(2x^{-6}y^{8}\right)\left(-5x^{5}y^{-3}\right)$
Rewrite with the like bases together.	$2(-5)\cdot\left(x^{-6}x^{5}\right)\cdot\left(y^{8}y^{-3}\right)$
Multiply the coefficients and add the exponents of each variable.	$-10\cdot x^{-1}\cdot y^{5}$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^{n}}$.	$-10\cdot\frac{1}{x^{1}}\cdot y^{5}$
Simplify.	$\frac{-10y^{5}}{x}$

TRY IT 10.1

Simplify: $\left(3u^{-5}v^{7}\right)\left(-4u^{4}v^{-2}\right)$.

Show answer

$-\frac{12v^{5}}{u}$

TRY IT 10.2

Simplify: $\left(-6c^{-6}d^{4}\right)\left(-5c^{-2}d^{-1}\right)$.

Show answer

$\frac{30d^{3}}{c^{8}}$

In the next two examples, we'll use the Power Property and the Product to a Power Property.

EXAMPLE 11

Simplify: $\left(6k^{3}\right)^{-2}$.

Solution

	$\left(6k^3\right)^{-2}$
Use the product to a Power Property, $(ab)^m = a^m b^m$.	$(6)^{-2}\left(k^3\right)^{-2}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a Negative Exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{6^2} \cdot \frac{1}{k^6}$
Simplify.	$\frac{1}{36k^6}$

TRY IT 11.1

Simplify: $\left(-4x^4\right)^{-2}$.

Show answer

$\frac{1}{16x^8}$

TRY IT 11.2

Simplify: $\left(2b^3\right)^{-4}$.

Show answer

$\frac{1}{16b^{12}}$

EXAMPLE 12

Simplify: $\left(5x^{-3}\right)^2$.

Solution

	$\left(5x^{-3}\right)^2$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2\left(x^{-3}\right)^2$
Simplify and multiply the exponents of x using the Power Property, $\left(a^m\right)^n = a^{m \cdot n}$.	$25 \cdot x^{-6}$
Rewrite by using the Definition of a Negative Exponent, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

TRY IT 12.1

Simplify: $\left(8a^{-4}\right)^2$.

Show answer
$\frac{64}{a^8}$

TRY IT 12.2

Simplify: $\left(2c^{-4}\right)^3$.

Show answer
$\frac{8}{c^{12}}$

To simplify a fraction, we use the Quotient Property and subtract the exponents.

EXAMPLE 13

Simplify: $\frac{r^5}{r^{-4}}$.

Solution

	$\frac{r^5}{r^{-4}}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$r^{5-(-4)}$
Simplify.	r^9

TRY IT 13.1

Simplify: $\frac{x^8}{x^{-3}}$.

Show answer
x^{11}

TRY IT 13.2

Simplify: $\frac{y^8}{y^{-6}}$.

Show answer
y^{13}

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1,000$ and 0.004 means $4 \times \frac{1}{1,000}$.

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

$4,000$	0.004
$4 \times 1,000$	$4 \times \frac{1}{1,000}$
4×10^3	$4 \times \frac{1}{10^3}$
	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

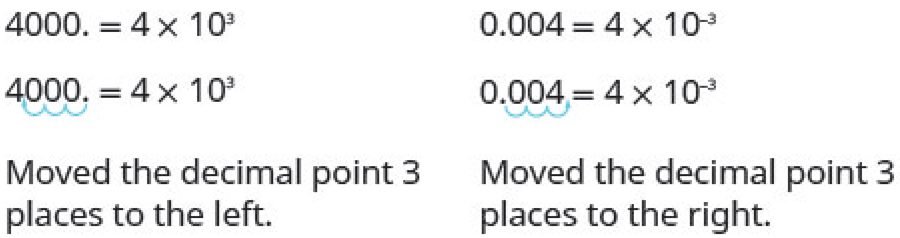
Scientific Notation

A number is expressed in scientific notation when it is of the form

$a \times 10^n$ where $1 \leq a < 10$ and n is an integer

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.



$4000. = 4 \times 10^3$

$4\underbrace{000.} = 4 \times 10^3$

Moved the decimal point 3 places to the left.

$0.004 = 4 \times 10^{-3}$

$0.\underbrace{004} = 4 \times 10^{-3}$

Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10

The power of 10 is positive when the number is larger than 1: $4,000 = 4 \times 10^3$
The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$

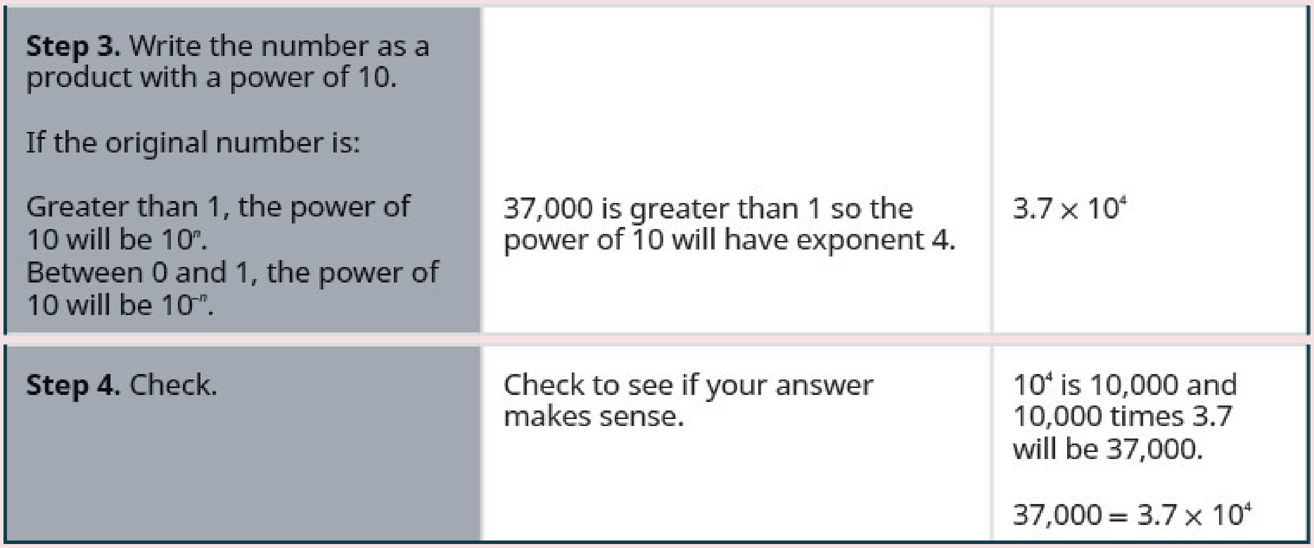
EXAMPLE 14

How to Convert from Decimal Notation to Scientific Notation

Write in scientific notation: 37,000.

Solution

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	Remember, there is a decimal at the end of 37,000. Move the decimal after the 3. 3.700 is between 1 and 10.	37,000.
Step 2. Count the number of decimal places, *n*, that the decimal point was moved.	The decimal point was moved 4 places to the left.	



Step 3. Write the number as a product with a power of 10. If the original number is: Greater than 1, the power of 10 will be 10^n. Between 0 and 1, the power of 10 will be 10^{-n}.	37,000 is greater than 1 so the power of 10 will have exponent 4.	3.7×10^4
Step 4. Check.	Check to see if your answer makes sense.	10^4 is 10,000 and 10,000 times 3.7 will be 37,000. $37{,}000 = 3.7 \times 10^4$

TRY IT 14.1

Write in scientific notation: $96,000$.

Show answer
9.6×10^4

TRY IT 14.2

Write in scientific notation: $48,300$.

Show answer
4.83×10^4

HOW TO: Convert from decimal notation to scientific notation

1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
2. Count the number of decimal places, *n*, that the decimal point was moved.
3. Write the number as a product with a power of 10.
   If the original number is:
   - greater than 1, the power of 10 will be 10^n.

  - between 0 and 1, the power of 10 will be 10^{-n}.

4. Check.

EXAMPLE 15

Write in scientific notation: 0.0052.

Solution
The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10

	 0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	 0.0052
Count the number of decimal places the point was moved.	 3 places
Write as a product with a power of 10.	 5.2×10^{-3}
Check.	
$5.2 \ \times \ 10^{-3}$ $5.2 \ \times \ \frac{1}{10^3}$ $5.2 \ \times \ \frac{1}{1000}$ $5.2 \ \times \ 0.001$	
0.0052	

TRY IT 15.1

Write in scientific notation: 0.0078.

Show answer
$7.8 \ \times \ 10^{-3}$

TRY IT 15.2

Write in scientific notation: 0.0129.

Show answer

1.29×10^{-2}

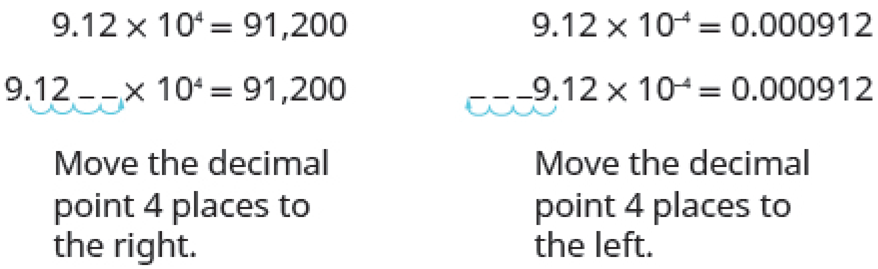
Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let’s look at two numbers written in scientific notation and see.

$$\begin{array}{cc} 9.12 \times 10^{4} & 9.12 \times 10^{-4} \\ 9.12 \times 10,000 & 9.12 \times 0.0001 \\ 91,200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$$9.12 \times 10^{4} = 91,200 \qquad 9.12 \times 10^{-4} = 0.000912$$



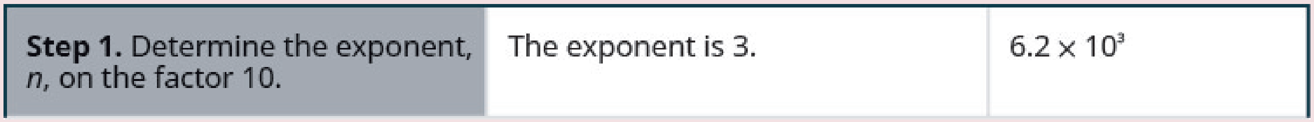
In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

EXAMPLE 16

How to Convert Scientific Notation to Decimal Form

Convert to decimal form: 6.2×10^{3}.

Solution



Step 1. Determine the exponent, *n*, on the factor 10.	The exponent is 3.	6.2×10^{3}

Step 2. Move the decimal *n* places, adding zeros if needed. If the exponent is positive, move the decimal point *n* places to the right. If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	The exponent is positive, so move the decimal point 3 places to the right. We need to add 2 zeros as placeholders.	 6.200. 6,200
Step 3. Check to see if your answer makes sense.		10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$

TRY IT 16.1

Convert to decimal form: 1.3×10^3.

Show answer
1,300

TRY IT 16.2

Convert to decimal form: 9.25×10^4.

Show answer
92,500

The steps are summarized below.

HOW TO: Convert scientific notation to decimal form.

To convert scientific notation to decimal form:

1. Determine the exponent, n, on the factor 10.
2. Move the decimal n places, adding zeros if needed.

  - If the exponent is positive, move the decimal point n places to the right.
  - If the exponent is negative, move the decimal point $|n|$ places to the left.
3. Check.

EXAMPLE 17

Convert to decimal form: 8.9×10^{-2}.

Solution

	8.9×10^{-2}
Determine the exponent, *n*, on the factor 10.	The exponent is -2.
Since the exponent is negative, move the decimal point 2 places to the left.	
Add zeros as needed for placeholders.	$8.9 \times 10^{-2} = 0.089$

TRY IT 17.1

Convert to decimal form: 1.2×10^{-4}.

Show answer
0.00012

TRY IT 17.2

Convert to decimal form: 7.5×10^{-2}.

Show answer
0.075

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets.

Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

EXAMPLE 18

Multiply. Write answers in decimal form: $\left(4 \times 10^{5}\right)\left(2 \times 10^{-7}\right)$.

Solution

	$\left(4 \times 10^{5}\right)\left(2 \times 10^{-7}\right)$
Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^{5} \cdot 10^{-7}$
Multiply.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

TRY IT 18.1

Multiply $\left(3 \times 10^{6}\right)\left(2 \times 10^{-8}\right)$. Write answers in decimal form.

Show answer
0.06

TRY IT 18.2

Multiply $\left(3 \times 10^{-2}\right)\left(3 \times 10^{-1}\right)$. Write answers in decimal form.

Show answer
0.009

EXAMPLE 19

Divide. Write answers in decimal form: $\frac{9 \times 10^{3}}{3 \times 10^{-2}}$.

Solution

	$\frac{9\times10^3}{3\times10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{9}{3}\times\frac{10^3}{10^{-2}}$
Divide.	3×10^5
Change to decimal form by moving the decimal five places right.	$300,000$

TRY IT 19.1

Divide $\frac{8\times10^4}{2\times10^{-1}}$. Write answers in decimal form.

Show answer
400,000

TRY IT 19.2

Divide $\frac{8\times10^2}{4\times10^{-2}}$. Write answers in decimal form.

Show answer
20,000

Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- Negative Exponents
- Scientific Notation
- Scientific Notation 2

Key Concepts

- **Property of Negative Exponents**
  - If n is a positive integer and $a\neq0$, then $\frac{1}{a^{-n}}=a^n$

- **Quotient to a Negative Exponent**
    - If a, b are real numbers, $b \neq 0$ and n is an integer , then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$
- **To convert a decimal to scientific notation:**
    1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
    2. Count the number of decimal places, n, that the decimal point was moved.
    3. Write the number as a product with a power of 10. If the original number is:
        - greater than 1, the power of 10 will be 10^{n}
        - between 0 and 1, the power of 10 will be 10^{-n}
    4. Check.
- **To convert scientific notation to decimal form:**
    1. Determine the exponent, n, on the factor 10.
    2. Move the decimal $n$places, adding zeros if needed.
        - If the exponent is positive, move the decimal point n places to the right.
        - If the exponent is negative, move the decimal point $|n|$ places to the left.
    3. Check

Practice Makes Perfect

Use the Definition of a Negative Exponent

In the following exercises, simplify.

1. a) 3^{-4} b) 10^{-2}

2. a) 4^{-2} b) 10^{-3}

3. a) 2^{-8} b) 10^{-2}

4. a) 5^{-3} b) 10^{-5}

5. a) $\frac{1}{c^{-5}}$ b) $\frac{1}{5^{-2}}$

6. a) $\frac{1}{c^{-5}}$ b) $\frac{1}{3^{-2}}$

7. a) $\frac{1}{t^{-9}}$ b) $\frac{1}{10^{-4}}$

8. a) $\frac{1}{q^{-10}}$ b) $\frac{1}{10^{-3}}$

9. a) $\left(\frac{3}{10}\right)^{-2}$ b) $\left(-\frac{2}{cd}\right)^{-3}$

10. a) $\left(\frac{5}{8}\right)^{-2}$ b) $\left(-\frac{3m}{n}\right)^{-2}$

11. a) $\left(\frac{7}{2}\right)^{-3}$ b) $\left(-\frac{3}{xy^2}\right)^{-3}$

12. a) $\left(\frac{4}{9}\right)^{-3}$ b) $\left(-\frac{u^2}{2v}\right)^{-5}$

13.

a) $(-7)^{-2}$
b) -7^{-2}
c) $\left(-\frac{1}{7}\right)^{-2}$
d) $-\left(\frac{1}{7}\right)^{-2}$

14.

a) $(-5)^{-2}$
b) -5^{-2}
c) $\left(-\frac{1}{5}\right)^{-2}$
d) $-\left(\frac{1}{5}\right)^{-2}$

15.

a) -5^{-3}
b) $\left(-\frac{1}{5}\right)^{-3}$
c) $-\left(\frac{1}{5}\right)^{-3}$
d) $(-5)^{-3}$

16.

a) -3^{-3}
b) $\left(-\frac{1}{3}\right)^{-3}$
c) $\left(\frac{1}{3}\right)^{-3}$
d) $(-3)^{-3}$

17. a) $2 \cdot 5^{-1}$ b) $(2 \cdot 5)^{-1}$

18. a) $3 \cdot 5^{-1}$ b) $(3 \cdot 5)^{-1}$

19. a) $3 \cdot 4^{-2}$ b) $(3 \cdot 4)^{-2}$

20. a) $4 \cdot 5^{-2}$ b) $(4 \cdot 5)^{-2}$

21. a) b^{-5} b) $\left(k^2\right)^{-5}$

22. a) m^{-4} b) $\left(x^3\right)^{-4}$

23. a) s^{-8} b) $\left(a^9\right)^{-10}$

24. a) p^{-10} b) $\left(q^6\right)^{-8}$

25. a) $6r^{-1}$ b) $(6r)^{-1}$ c) $(-6r)^{-1}$	26. a) $7n^{-1}$ b) $(7n)^{-1}$ c) $(-7n)^{-1}$
27. a) $(2q)^{-4}$ b) $2q^{-4}$ c) $-2q^{-4}$	28. a) $(3p)^{-2}$ b) $3p^{-2}$ c) $-3p^{-2}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

29. a) $s^3 \cdot s^{-7}$ b) $q^{-8} \cdot q^3$ c) $y^{-2} \cdot y^{-5}$	30. a) b^4b^{-8} b) $r^{-2}r^5$ c) $x^{-7}x^{-3}$
31. a) $y^5 \cdot y^{-5}$ b) $y \cdot y^5$ c) $y \cdot y^{-5}$	32. a) $a^3 \cdot a^{-3}$ b) $a \cdot a^3$ c) $a \cdot a^{-3}$
33. $x^4 \cdot x^{-2} \cdot x^{-3}$	34. $\left(w^4x^{-5}\right)\left(w^{-2}x^{-4}\right)$
35. $\left(m^3n^{-3}\right)\left(m^{-5}n^{-1}\right)$	36. $\left(uv^{-2}\right)\left(u^{-5}v^{-3}\right)$
37. $\left(pq^{-4}\right)\left(p^{-6}q^{-3}\right)$	38. $\left(-6c^{-3}d^9\right)\left(2c^4d^{-5}\right)$
39. $\left(-2j^{-5}k^8\right)\left(7j^2k^{-3}\right)$	40. $\left(-4r^{-2}s^{-8}\right)\left(9r^4s^3\right)$
41. $\left(-5m^4n^6\right)\left(8m^{-5}n^{-3}\right)$	42. $\left(5x^2\right)^{-2}$
43. $\left(4y^3\right)^{-3}$	44. $\left(3z^{-3}\right)^2$
45. $\left(2p^{-5}\right)^2$	46. $\frac{t^9}{t^{-3}}$
47. $\frac{n^5}{n^{-2}}$	48. $\frac{x^{-7}}{x^{-3}}$
49. $\frac{y^{-5}}{y^{-10}}$	

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

50. 57,000	51. 340,000
52. 8,750,000	53. 1,290,000
54. 0.026	55. 0.041
56. 0.00000871	57. 0.00000103

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

58. 5.2×10^2	59. 8.3×10^2
60. 7.5×10^6	61. 1.6×10^{10}
62. 2.5×10^{-2}	63. 3.8×10^{-2}
64. 4.13×10^{-5}	65. 1.93×10^{-5}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply. Write your answer in decimal form.

66. $\left(3 \times 10^{-5}\right)\left(3 \times 10^{9}\right)$	67. $\left(2 \times 10^{2}\right)\left(1 \times 10^{-4}\right)$
68. $\left(7.1 \times 10^{-2}\right)\left(2.4 \times 10^{-4}\right)$	69. $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$

In the following exercises, divide. Write your answer in decimal form.

70. $\frac{7 \times 10^{-3}}{1 \times 10^{-7}}$	71. $\frac{5 \times 10^{-2}}{1 \times 10^{-10}}$
72. $\frac{6 \times 10^{4}}{3 \times 10^{-2}}$	73. $\frac{8 \times 10^{6}}{4 \times 10^{-1}}$

Everyday Math

74. The population of the United States on July 1, 2010 was about 34,000,000. Write the number in scientific notation.	75. The population of the world on July 1, 2010 was more than 6,850,000,000. Write the number in scientific notation
76. The average width of a human hair is 0.0018 centimetres. Write the number in scientific notation.	77. The probability of winning the 2010 Megamillions lottery was about 0.0000000057. Write the number in scientific notation.
78. In 2010, the number of Facebook users each day who changed their status to 'engaged' was 2×10^4. Convert this number to decimal form.	79. At the start of 2012, the US federal budget had a deficit of more than $?1.5 \times 10^{13}$. Convert this number to decimal form.
80. The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}. Convert this number to decimal form.	81. The width of a proton is 1×10^{-5} of the width of an atom. Convert this number to decimal form.
82. **Health care costs** The Centers for Medicare and Medicaid projects that American consumers will spend more than \$4 trillion on health care by 2017 a. Write 4 trillion in decimal notation. b. Write 4 trillion in scientific notation.	83. **Coin production** In 1942, the U.S. Mint produced 154,500,000 nickels. Write 154,500,000 in scientific notation.
84. **Distance** The distance between Earth and one of the brightest stars in the night star is 33.7 light years. One light year is about 6,000,000,000,000 (6 trillion), miles. a) Write the number of miles in one light year in scientific notation. b)Use scientific notation to find the distance between Earth and the star in miles. Write the answer in scientific notation.	85. **Debt** At the end of fiscal year 2019 the gross Canadian federal government debt was estimated to be approximately \$685,450,000,000 (\$685.45 billion), according to the Federal Budget. The population of Canada was approximately 37,590,000 people at the end of fiscal year 2019 a) Write the debt in scientific notation. b) Write the population in scientific notation. c) Find the amount of debt per person by using scientific notation to divide the debt by the population. Write the answer in scientific notation.

Writing Exercises.

86. a) Explain the meaning of the exponent in the expression 2^3. b) Explain the meaning of the exponent in the expression 2^{-3}.	87. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Answers

1. a) $\frac{1}{81}$ b) $\frac{1}{100}$	3. a) $\frac{1}{256}$ b) $\frac{1}{100}$	5. a) c^5 b) 25
7. a) t^9 b) 10000	9. a) $\frac{100}{9}$ b) $-\frac{c^3d^3}{8}$	11. a) $\frac{8}{343}$ b) $-\frac{x^3y^6}{27}$
13. a) $\frac{1}{49}$ b) $-\frac{1}{49}$c) 49 d) -49	15. a) $-\frac{1}{125}$ b) -125 c) -125d) $-\frac{1}{125}$	17. a) $\frac{2}{5}$ b) $\frac{1}{10}$
19. a) $\frac{3}{16}$ b) $\frac{1}{144}$	21. a)$\frac{1}{b^5}$ b) $\frac{1}{k^{10}}$	23. a) $\frac{1}{s^8}$ b) $\frac{1}{a^{90}}$
25. a) $\frac{6}{r}$ b) $\frac{1}{6r}$c)$-\frac{1}{6r}$	27. a)$\frac{1}{16q^4}$ b) $\frac{2}{q^4}$ c) $-\frac{2}{q^4}$	29. a) $\frac{1}{s^4}$ b) $\frac{1}{q^5}$ c) $\frac{1}{y^7}$
31. a) 1 b) y^6 c) $\frac{1}{y^4}$	33. $\frac{1}{x}$	35. $\frac{1}{m^2n^4}$
37. $\frac{1}{p^5q^7}$	39. $-\frac{14k^5}{j^3}$	41. $-\frac{40n^3}{m}$
43. $\frac{1}{64y^9}$	45. $\frac{4}{p^{10}}$	47. n^7
49. y^5	51. 3.4×10^5	53. 1.29×10^6
55. 4.1×10^{-2}	57. 1.03×10^{-6}	59. 830
61. 16,000,000,000	63. 0.038	65. 0.0000193
67. 0.02	69. 5.6×10^{-6}	71. 500,000,000
73. 20,000,000	75. 6.85×10^9.	77. 5.7×10^{-10}
79. 15,000,000,000,000	81. 0.00001	83. 1.545×10^8
85. a) 1.86×10^{13} b) 3×10^8 c) 6.2×10^4	87. Answers will vary	

Attributions

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7.4 Simplify and Use Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Estimate square roots
- Approximate square roots
- Simplify variable expressions with square roots
- Use square roots in applications

Simplify Expressions with Square Roots

To start this section, we need to review some important vocabulary and notation.

Remember that when a number n is multiplied by itself, we can write this as n^2, which we read aloud as “n squared.” For example, 8^2 is read as “8 squared.”

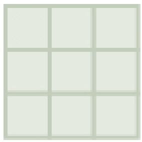
We call 64 the *square* of 8 because $8^2 = 64$. Similarly, 121 is the square of 11, because $11^2 = 121$.

Square of a Number

If $n^2 = m$, then m is the square of n.

Modeling Squares

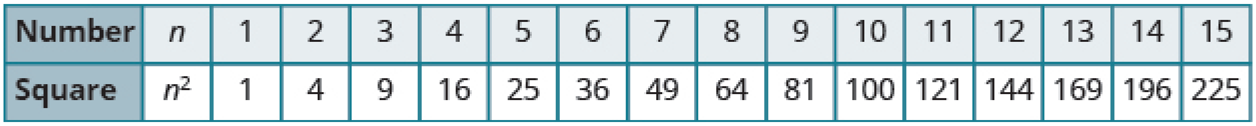
Do you know why we use the word *square*? If we construct a square with three tiles on each side, the total number of tiles would be nine.



This is why we say that the square of three is nine.
$3^2 = 9$

The number 9 is called a perfect square because it is the square of a whole number.

The chart shows the squares of the counting numbers 1 through 15. You can refer to it to help you identify the perfect squares.



Number	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Perfect Squares

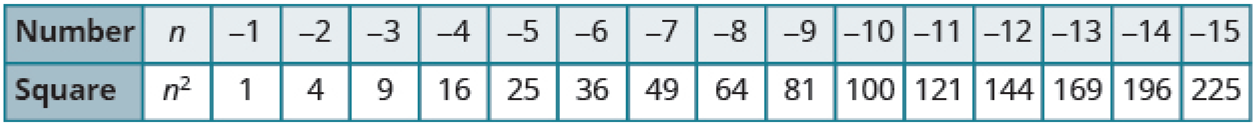
A perfect square is the square of a whole number.

What happens when you square a negative number?

$$\begin{aligned} (-8)^2 &= (-8)(-8) \\ &= 64 \end{aligned}$$

When we multiply two negative numbers, the product is always positive. So, the square of a negative number is always positive.

The chart shows the squares of the negative integers from -1 to -15.



Number	n	–1	–2	–3	–4	–5	–6	–7	–8	–9	–10	–11	–12	–13	–14	–15
Square	n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Did you notice that these squares are the same as the squares of the positive numbers?

Square Roots

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We can also say that 10 is a square root of 100.

Square Root of a Number

A number whose square is m is called a square root of m.

If $n^2 = m$, then n is a square root of m.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100.

So, every positive number has two square roots: one positive and one negative.

What if we only want the positive square root of a positive number? The *radical sign*, $\sqrt{\ }$, stands for the positive square root. The positive square root is also called the principal square root.

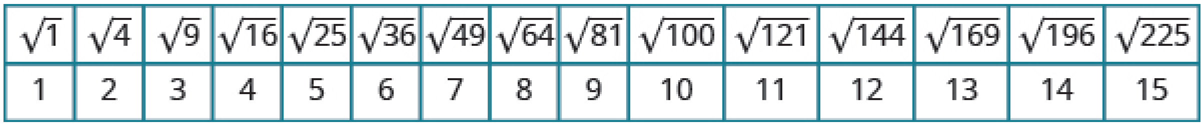
Square Root Notation

$\sqrt{m}$ is read as “the square root of m.”

If $m = n^2$, then $\sqrt{m} = n$ for n ≥ 0.

We can also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

The chart shows the square roots of the first 15 perfect square numbers.



$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

EXAMPLE 1

Simplify: a) $\sqrt{25}$ b) $\sqrt{121}$.

Solution

a)	
	$\sqrt{25}$
Since $5^2 = 25$	5

b)	
	$\sqrt{121}$
Since $11^2 = 121$	-11

TRY IT 1.1

Simplify: a) $\sqrt{36}$ b) $\sqrt{169}$.

Show answer

a. 6
b. 13

TRY IT 1.2

Simplify: a) $\sqrt{16}$ b) $\sqrt{196}$.

Show answer

a. 4
b. 14

Every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$.

EXAMPLE 2

Simplify. a) $-\sqrt{9}$ b) $-\sqrt{144}$.

Solution

a)	
	$-\sqrt{9}$
The negative is in front of the radical sign.	-3

b)	
	$-\sqrt{144}$
The negative is in front of the radical sign.	-12

TRY IT 2.1

Simplify: a) $-\sqrt{4}$ b) $-\sqrt{225}$.

Show answer

a. −2
b. −15

TRY IT 2.2

Simplify: a) $-\sqrt{81}$ b) $-\sqrt{64}$.

Show answer

a. −9
b. −8

Square Root of a Negative Number

Can we simplify $\sqrt{-25}$? Is there a number whose square is $-25?$
$(\quad)^2 = -25?$

None of the numbers that we have dealt with so far have a square that is -25. Why? Any positive number squared is positive, and any negative number squared is also positive. In the next chapter we will see that all the numbers we work with are called the real numbers. So we say there is no real number equal to $\sqrt{-25}$. If we are asked to find the square root of any negative number, we say that the solution is not a real number.

EXAMPLE 3

Simplify: a) $\sqrt{-169}$ b) $-\sqrt{121}$.

Solution

a) There is no real number whose square is -169. Therefore, $\sqrt{-169}$ is not a real number.

b) The negative is in front of the radical sign, so we find the opposite of the square root of 121.

	$-\sqrt{121}$
The negative is in front of the radical.	-11

TRY IT 3.1

Simplify: a) $\sqrt{-196}$ b) $-\sqrt{81}$.

Show answer

- a. not a real number
- b. −9

TRY IT 3.2

Simplify: a) $\sqrt{-49}$ b) $-\sqrt{121}$.

Show answer

- a. −7
- b. not a real number

Square Roots and the Order of Operations

When using the order of operations to simplify an expression that has square roots, we treat the radical sign as a grouping symbol. We simplify any expressions under the radical sign before performing other operations.

EXAMPLE 4

Simplify: a) $\sqrt{25}+\sqrt{144}$ b) $\sqrt{25+144}$.

Solution

a) Use the order of operations.	
	$\sqrt{25} + \sqrt{144}$
Simplify each radical.	$5 + 12$
Add.	17

b) Use the order of operations.	
	$\sqrt{25 + 144}$
Add under the radical sign.	$\sqrt{169}$
Simplify.	13

TRY IT 4.1

Simplify: a) $\sqrt{9} + \sqrt{16}$ b) $\sqrt{9 + 16}$.

Show answer

a. 7
b. 5

TRY IT 4.2

Simplify: a) $\sqrt{64 + 225}$ b) $\sqrt{64} + \sqrt{225}$.

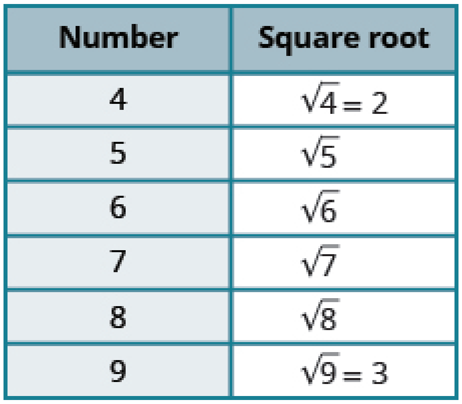
Show answer

a. 17
b. 23

Notice the different answers in parts a) and b) of (Example 4). It is important to follow the order of operations correctly. In a), we took each square root first and then added them. In b), we added under the radical sign first and then found the square root.

Estimate Square Roots

So far we have only worked with square roots of perfect squares. The square roots of other numbers are not whole numbers.



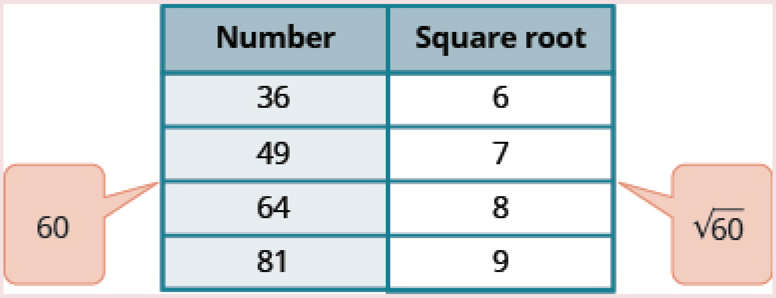
Number	Square root
4	$\sqrt{4} = 2$
5	$\sqrt{5}$
6	$\sqrt{6}$
7	$\sqrt{7}$
8	$\sqrt{8}$
9	$\sqrt{9} = 3$

We might conclude that the square roots of numbers between 4 and 9 will be between 2 and 3, and they will not be whole numbers. Based on the pattern in the table above, we could say that $\sqrt{5}$ is between 2 and 3. Using inequality symbols, we write
$2 < \sqrt{5} < 3$

EXAMPLE 5

Estimate $\sqrt{60}$ between two consecutive whole numbers.

Solution
Think of the perfect squares closest to 60. Make a small table of these perfect squares and their squares roots.



Number	Square root
36	6
49	7
64	8
81	9

Locate 60 between two consecutive perfect squares.	$49 < 60 < 64$
$\sqrt{60}$ is between their square roots.	$7 < \sqrt{60} < 8$

TRY IT 5.1

Estimate $\sqrt{38}$ between two consecutive whole numbers.

Show answer

$6 < \sqrt{38} < 7$

TRY IT 5.2

Estimate $\sqrt{84}$ between two consecutive whole numbers.

Show answer

$9 < \sqrt{84} < 10$

Approximate Square Roots with a Calculator

The square roots of numbers that are not perfect squares are not whole numbers, they are irrational numbers. Its decimal form does not stop and does not repeat. Are irrational numbers real numbers? Yes, they are. When we put together the irrational numbers and rational numbers, we get the set of real numbers.

Let's see how we can use calculator to find the approximate square roots of those irrational numbers.

There are mathematical methods to approximate square roots, but it is much more convenient to use a calculator to find square roots. Find the $\sqrt{\ }$ or $\sqrt{x}$ key on your calculator. You will to use this key to approximate square roots. When you use your calculator to find the square root of a number that is not a perfect square, the answer that you see is not the exact number. It is an approximation, to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read *approximately*.

Suppose your calculator has a 10-digit display. Using it to find the square root of 5 will give 2.236067977. This is the approximate square root of 5. When we report the answer, we should use the "approximately equal to" sign instead of an equal sign.
$\sqrt{5} \approx 2.236067978$. The square root of 5 is the example of irrational number and its approximation displays nine digits after the decimal place.

You will seldom use this many digits for applications in algebra. So, if you wanted to round $\sqrt{5}$ to two decimal places, you would write
$\sqrt{5} \approx 2.24$

How do we know these values are approximations and not the exact values? Look at what happens when we square them.

$$\begin{aligned} 2.236067978^2 &= 5.000000002 \\ 2.24^2 &= 5.0176 \end{aligned}$$

The squares are close, but not exactly equal, to 5.

EXAMPLE 6

Round $\sqrt{17}$ to two decimal places using a calculator.

Solution

	$\sqrt{17}$
Use the calculator square root key.	4.123105626
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

TRY IT 6.1

Round $\sqrt{11}$ to two decimal places.

Show answer

≈ 3.32

TRY IT 6.2

Round $\sqrt{13}$ to two decimal places.

Show answer

≈ 3.61

Simplify Variable Expressions with Square Roots

Expressions with square root that we have looked at so far have not had any variables. What happens when we have to find a square root of a variable expression?

Consider $\sqrt{9x^2}$, where $x \geq 0$. Can you think of an expression whose square is $9x^2$?

$$\begin{array}{rcl} (?)^2 & = & 9x^2 \\ (3x)^2 & = & 9x^2 \qquad \text{so } \sqrt{9x^2} = 3x \end{array}$$

When we use a variable in a square root expression, for our work, we will assume that the variable represents a non-negative number. In every example and exercise that follows, each variable in a square root expression is greater than or equal to zero.

EXAMPLE 7

Simplify: $\sqrt{x^2}$.

Solution

Think about what we would have to square to get x^2. Algebraically, $(?)^2 = x^2$

	$\sqrt{x^2}$
Since $(x)^2 = x^2$	x

TRY IT 7.1

Simplify: $\sqrt{y^2}$.

Show answer

y

TRY IT 7.2

Simplify: $\sqrt{m^2}$.

Show answer

m

EXAMPLE 8

Simplify: $\sqrt{16x^2}$.

Solution

	$\sqrt{16x^2}$
Since $(4x)^2 = 16x^2$	$4x$

TRY IT 8.1

Simplify: $\sqrt{64x^2}$.

Show answer
$8x$

TRY IT 8.2

Simplify: $\sqrt{169y^2}$.

Show answer
$13y$

EXAMPLE 9

Simplify: $-\sqrt{81y^2}$.

Solution

	$-\sqrt{81y^2}$
Since $(9y)^2 = 81y^2$	$-9y$

TRY IT 9.1

Simplify: $-\sqrt{121y^2}$.

Show answer

$-11y$

TRY IT 9.2

Simplify: $-\sqrt{100p^2}$.

Show answer

$-10p$

EXAMPLE 10

Simplify: $\sqrt{36x^2y^2}$.

Solution

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$	$6xy$

TRY IT 10.1

Simplify: $\sqrt{100a^2b^2}$.

Show answer

$10ab$

TRY IT 10.2

Simplify: $\sqrt{225m^2n^2}$.

Show answer
15*mn*

Use Square Roots in Applications

As you progress through your college courses, you’ll encounter several applications of square roots. Once again, if we use our strategy for applications, it will give us a plan for finding the answer!

HOW TO: Use a strategy for applications with square roots.

1. Identify what you are asked to find.
2. Write a phrase that gives the information to find it.
3. Translate the phrase to an expression.
4. Simplify the expression.
5. Write a complete sentence that answers the question.

Square Roots and Area

We have solved applications with area before. If we were given the length of the sides of a square, we could find its area by squaring the length of its sides. Now we can find the length of the sides of a square if we are given the area, by finding the square root of the area.

If the area of the square is A square units, the length of a side is $\sqrt{A}$ units. See the table below.

Area (square units)	**Length of side (units)**
9	$\sqrt{9}=3$
144	$\sqrt{144}=12$
A	$\sqrt{A}$

EXAMPLE 11

Mike and Lychelle want to make a square patio. They have enough concrete for an area of 200 square feet. To the nearest tenth of a foot, how long can a side of their square patio be?

Solution
We know the area of the square is 200 square feet and want to find the length of the side. If the area of the square is A square units, the length of a side is $\sqrt{A}$ units.

What are you asked to find?	The length of each side of a square patio
Write a phrase.	The length of a side
Translate to an expression.	$\sqrt{A}$
Evaluate $\sqrt{A}$ when $A = 200$.	$\sqrt{200}$
Use your calculator.	$14.142135...$
Round to one decimal place.	14.1 feet
Write a sentence.	Each side of the patio should be 14.1 feet.

TRY IT 11.1

Katie wants to plant a square lawn in her front yard. She has enough sod to cover an area of 370 square feet. To the nearest tenth of a foot, how long can a side of her square lawn be?

Show answer
19.2 feet

TRY IT 11.2

Sergio wants to make a square mosaic as an inlay for a table he is building. He has enough tile to cover an area of 2704 square centimetres. How long can a side of his mosaic be?

Show answer
52 centimetres

Square Roots and Gravity

Another application of square roots involves gravity. On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by evaluating the expression $\frac{\sqrt{h}}{4}$. For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by evaluating $\frac{\sqrt{64}}{4}$.

	$\frac{\sqrt{64}}{4}$
Take the square root of 64.	$\frac{8}{4}$
Simplify the fraction.	2

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

EXAMPLE 12

Christy dropped her sunglasses from a bridge 400 feet above a river. How many seconds does it take for the sunglasses to reach the river?

Solution

What are you asked to find?	The number of seconds it takes for the sunglasses to reach the river
Write a phrase.	The time it will take to reach the river
Translate to an expression.	$\frac{\sqrt{h}}{4}$
Evaluate $\frac{\sqrt{h}}{4}$ when $h = 400$.	$\frac{\sqrt{400}}{4}$
Find the square root of 400.	$\frac{20}{4}$
Simplify.	5
Write a sentence.	It will take 5 seconds for the sunglasses to reach the river.

TRY IT 12.1

A helicopter drops a rescue package from a height of 1296 feet. How many seconds does it take for the package to reach the ground?

Show answer

9 seconds

TRY IT 12.2

A window washer drops a squeegee from a platform 196 feet above the sidewalk. How many seconds does it take for the squeegee to reach the sidewalk?

Show answer
3.5 seconds

Square Roots and Accident Investigations

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes. According to some formulas, if the length of the skid marks is d feet, then the speed of the car can be found by evaluating $\sqrt{24d}$.

EXAMPLE 13

After a car accident, the skid marks for one car measured 190 feet. To the nearest tenth, what was the speed of the car (in mph) before the brakes were applied?

Solution

What are you asked to find?	The speed of the car before the brakes were applied
Write a phrase.	The speed of the car
Translate to an expression.	$\sqrt{24d}$
Evaluate $\sqrt{24d}$ when $d = 190.$	$\sqrt{24 \cdot 190}$
Multiply.	$\sqrt{4,560}$
Use your calculator.	$67.527772...$
Round to tenths.	67.5
Write a sentence.	The speed of the car was approximately 67.5 miles per hour.

TRY IT 13.1

An accident investigator measured the skid marks of a car and found their length was 76 feet. To the nearest tenth, what was the speed of the car before the brakes were applied?

Show answer
42.7 mph

TRY IT 13.2

The skid marks of a vehicle involved in an accident were 122 feet long. To the nearest tenth, how fast had the vehicle been going before the brakes were applied?

Show answer
54.1 mph

The *Links to Literacy* activity “Sea Squares” will provide you with another view of the topics covered in this section.

ACCESS ADDITIONAL ONLINE RESOURCES

- Introduction to Square Roots
- Estimating Square Roots with a Calculator

Key Concepts

- **Square Root Notation** $\sqrt{m}$ is read ‘the square root of m’
  If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$. 

- **Use a strategy for applications with square roots.**
  - Identify what you are asked to find.
  - Write a phrase that gives the information to find it.
  - Translate the phrase to an expression.
  - Simplify the expression.
  - Write a complete sentence that answers the question.

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

1. $\sqrt{36}$	2. $\sqrt{4}$
3. $\sqrt{64}$	4. $\sqrt{144}$
5. $-\sqrt{4}$	6. $-\sqrt{100}$
7. $-\sqrt{1}$	8. $-\sqrt{121}$
9. $\sqrt{-121}$	10. $\sqrt{-36}$
11. $\sqrt{-9}$	12. $\sqrt{-49}$
13. $\sqrt{9+16}$	14. $\sqrt{25+144}$
15. $\sqrt{9}+\sqrt{16}$	16. $\sqrt{25}+\sqrt{144}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

17. $\sqrt{70}$	18. $\sqrt{55}$
19. $\sqrt{200}$	20. $\sqrt{172}$

Approximate Square Roots with a Calculator

In the following exercises, use a calculator to approximate each square root and round to two decimal places.

21. $\sqrt{19}$	22. $\sqrt{21}$
23. $\sqrt{53}$	24. $\sqrt{47}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify. (Assume all variables are greater than or equal to zero.)

25. $\sqrt{y^2}$	26. $\sqrt{b^2}$
27. $\sqrt{49x^2}$	28. $\sqrt{100y^2}$
29. $-\sqrt{64a^2}$	30. $-\sqrt{25x^2}$
31. $\sqrt{144x^2y^2}$	32. $\sqrt{196a^2b^2}$

Use Square Roots in Applications

In the following exercises, solve. Round to one decimal place.

33. **Landscaping** Reid wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. How long can a side of his garden be?

34. **Landscaping** Tasha wants to make a square patio in her yard. She has enough concrete to pave an area of 130 square feet. How long can a side of her patio be?

35. **Gravity** An airplane dropped a flare from a height of $1,024$ feet above a lake. How many seconds did it take for the flare to reach the water?

36. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. How many seconds did it take for the cell phone to reach the ground?

37. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, $4,000$ feet above the Colorado River. How many seconds did it take for the hammer to reach the river?

38. **Accident investigation** The skid marks from a car involved in an accident measured 54 feet. What was the speed of the car before the brakes were applied?

39. **Accident investigation** The skid marks from a car involved in an accident measured 216 feet. What was the speed of the car before the brakes were applied?

40. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. What was the speed of the vehicle before the brakes were applied?

41. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 117 feet. What was the speed of the vehicle before the brakes were applied?

Everyday Math

42. **Decorating** Denise wants to install a square accent of designer tiles in her new shower. She can afford to buy 625 square centimetres of the designer tiles. How long can a side of the accent be?

43. **Decorating** Morris wants to have a square mosaic inlaid in his new patio. His budget allows for $2,025$ tiles. Each tile is square with an area of one square inch. How long can a side of the mosaic be?

Writing Exercises

44. Why is there no real number equal to $\sqrt{-64}$?

45. What is the difference between 9^2 and $\sqrt{9}$?

Answers

1. 6	3. 8	5. -2
7. -1	9. not a real number	11. not a real number
13. 5	15. 7	17. $8 < \sqrt{70} < 9$
19. $14 < \sqrt{200} < 15$	21. 4.36	23. 7.28
25. y	27. $7x$	29. $-8a$
31. $12xy$	33. 8.7 feet	35. 8 seconds
37. 15.8 seconds	39. 72 mph	41. 53.0 mph
43. 45 inches	45. Answers will vary. 9^2 reads: “nine squared” and means nine times itself. The expression $\sqrt{9}$ reads: “the square root of nine” which gives us the number such that if it were multiplied by itself would give you the number inside of the square root.	

Attributions

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7.5 Simplify Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify square roots
- Use the Quotient Property to simplify square roots

In the last section, we estimated the square root of a number between two consecutive whole numbers. We can say that $\sqrt{50}$ is between 7 and 8. This is fairly easy to do when the numbers are small enough that we can use in (Simplify and Use Square Roots).

But what if we want to estimate $\sqrt{500}$? If we simplify the square root first, we'll be able to estimate it easily. There are other reasons, too, to simplify square roots as you'll see later in this chapter.

A square root is considered *simplified* if its radicand contains no perfect square factors.

Simplified Square Root

$\sqrt{a}$ is considered simplified if a has no perfect square factors.

So $\sqrt{31}$ is simplified. But $\sqrt{32}$ is not simplified, because 16 is a perfect square factor of 32

Use the Product Property to Simplify Square Roots

The properties we will use to simplify expressions with square roots are similar to the properties of exponents. We know that $(ab)^m = a^m b^m$. The corresponding property of square roots says that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

Product Property of Square Roots

If *a*, *b* are non-negative real numbers, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

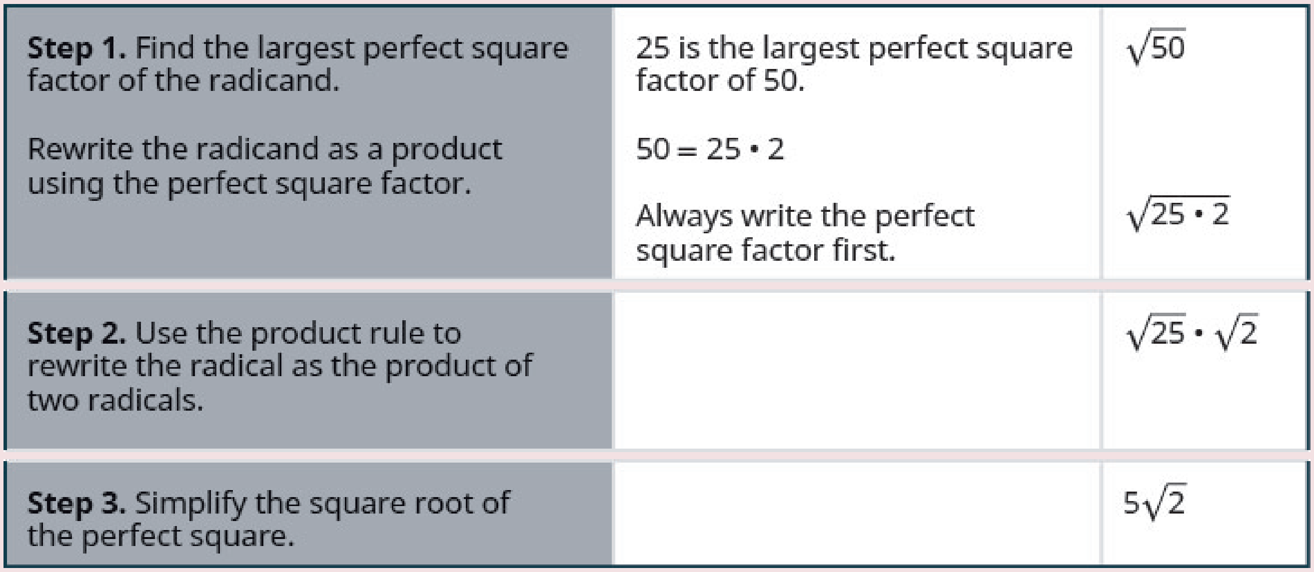
We use the Product Property of Square Roots to remove all perfect square factors from a radical. We will show how to do this in (Example 1).

EXAMPLE 1

How To Use the Product Property to Simplify a Square Root

Simplify: $\sqrt{50}$.

Solution



Step 1. Find the largest perfect square factor of the radicand.	25 is the largest perfect square factor of 50.	$\sqrt{50}$
Rewrite the radicand as a product using the perfect square factor.	$50 = 25 \cdot 2$ Always write the perfect square factor first.	$\sqrt{25 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{25} \cdot \sqrt{2}$
Step 3. Simplify the square root of the perfect square.		$5\sqrt{2}$

TRY IT 1.1

Simplify: $\sqrt{48}$.

Show answer

$4\sqrt{3}$

TRY IT 1.2

Simplify: $\sqrt{45}$.

Show answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{50}$ is $5\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

HOW TO: Simplify a square root using the product property.

1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect-square factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the square root of the perfect square.

EXAMPLE 2

Simplify: $\sqrt{500}$.

Solution

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals.	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

TRY IT 2.1

Simplify: $\sqrt{288}$.

Show answer

$12\sqrt{2}$

TRY IT 2.2

Simplify: $\sqrt{432}$.

Show answer

$12\sqrt{3}$

We could use the simplified form $10\sqrt{5}$ to estimate $\sqrt{500}$. We know 5 is between 2 and 3, and $\sqrt{500}$ is $10\sqrt{5}$. So $\sqrt{500}$ is between 20 and 30.

The next example is much like the previous examples, but with variables.

EXAMPLE 3

Simplify: $\sqrt{x^3}$.

Solution

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$x\sqrt{x}$

TRY IT 3.1

Simplify: $\sqrt{b^5}$.

Show answer

$b^2\sqrt{b}$

TRY IT 3.2

Simplify: $\sqrt{p^9}$.

Show answer

$p^4\sqrt{p}$

We follow the same procedure when there is a coefficient in the radical, too.

EXAMPLE 4

Simplify: $\sqrt{25y^5}$.

Solution

	$\sqrt{25y^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{25y^4 \cdot y}$
Rewrite the radical as the product of two radicals.	$\sqrt{25y^4} \cdot \sqrt{y}$
Simplify.	$5y^2\sqrt{y}$

TRY IT 4.1

Simplify: $\sqrt{16x^7}$.

Show answer
$4x^3\sqrt{x}$

TRY IT 4.2

Simplify: $\sqrt{49v^9}$.

Show answer
$7v^4\sqrt{v}$

In the next example both the constant and the variable have perfect square factors.

EXAMPLE 5

Simplify: $\sqrt{72n^7}$.

Solution

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6n^3\sqrt{2n}$

TRY IT 5.1

Simplify: $\sqrt{32y^5}$.

Show answer
$4y^2\sqrt{2y}$

TRY IT 5.2

Simplify: $\sqrt{75a^9}$.

Show answer
$5a^4\sqrt{3a}$

EXAMPLE 6

Simplify: $\sqrt{63u^3v^5}$.

Solution

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Simplify.	$3uv^2\sqrt{7uv}$

TRY IT 6.1

Simplify: $\sqrt{98a^7b^5}$.

Show answer

$7a^3b^2\sqrt{2ab}$

TRY IT 6.2

Simplify: $\sqrt{180m^9n^{11}}$.

Show answer

$6m^4n^5\sqrt{5mn}$

We have seen how to use the Order of Operations to simplify some expressions with radicals. To simplify $\sqrt{25}+\sqrt{144}$ we must simplify each square root separately first, then add to get the sum of 17

The expression $\sqrt{17}+\sqrt{7}$ cannot be simplified—to begin we'd need to simplify each square root, but neither 17 nor 7 contains a perfect square factor.

In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer.

EXAMPLE 7

Simplify: $3+\sqrt{32}$.

Solution

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16\cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16}\cdot\sqrt{2}$
Simplify.	$3+4\sqrt{2}$

The terms are not like and so we cannot add them. Trying to add an integer and a radical is like trying to add an integer and a variable—they are not like terms!

TRY IT 7.1

Simplify: $5+\sqrt{75}$.

Show answer
$5+5\sqrt{3}$

TRY IT 7.2

Simplify: $2+\sqrt{98}$.

Show answer
$2+7\sqrt{2}$

The next example includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

EXAMPLE 8

Simplify: $\frac{4-\sqrt{48}}{2}$.

Solution

	$\frac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\frac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\frac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\frac{4\left(1-\sqrt{3}\right)}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\frac{\cancel{2}\cdot 2\left(1-\sqrt{3}\right)}{\cancel{2}}$
Simplify.	$2\left(1-\sqrt{3}\right)$

TRY IT 8.1

Simplify: $\frac{10-\sqrt{75}}{5}$.

Show answer

$2-\sqrt{3}$

TRY IT 8.2

Simplify: $\frac{6-\sqrt{45}}{3}$.

Show answer

$2-\sqrt{5}$

Use the Quotient Property to Simplify Square Roots

Whenever you have to simplify a square root, the first step you should take is to determine whether the radicand is a perfect square. A *perfect square fraction* is a fraction in which both the numerator and the denominator are perfect squares.

EXAMPLE 9

Simplify: $\sqrt{\frac{9}{64}}$.

Solution

	$\sqrt{\frac{9}{64}}$
Since $\left(\frac{3}{8}\right)^2=\frac{9}{64}$	$\frac{3}{8}$

TRY IT 9.1

Simplify: $\sqrt{\frac{25}{16}}$.

Show answer
$\frac{5}{4}$

TRY IT 9.2

Simplify: $\sqrt{\frac{49}{81}}$.

Show answer
$\frac{7}{9}$

If the numerator and denominator have any common factors, remove them. You may find a perfect square fraction!

EXAMPLE 10

Simplify: $\sqrt{\frac{45}{80}}$.

Solution

	$\sqrt{\frac{45}{80}}$
Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\frac{5\cdot 9}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\frac{9}{16}}$
$\text{Simplify}\left(\frac{3}{4}\right)^2 = \frac{9}{16}$	$\frac{3}{4}$

TRY IT 10.1

Simplify: $\sqrt{\frac{75}{48}}$.

Show answer

$\frac{5}{4}$

TRY IT 10.2

Simplify: $\sqrt{\frac{98}{162}}$.

Show answer

$\frac{7}{9}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$.

EXAMPLE 11

Simplify: $\sqrt{\frac{m^6}{m^4}}$.

Solution

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	m

TRY IT 11.1

Simplify: $\sqrt{\frac{a^8}{a^6}}$.

Show answer
a

TRY IT 11.2

Simplify: $\sqrt{\frac{x^{14}}{x^{10}}}$.

Show answer
x^2

EXAMPLE 12

Simplify: $\sqrt{\frac{48p^7}{3p^3}}$.

Solution

	$\sqrt{\frac{48p^7}{3p^3}}$
Simplify the fraction inside the radical first.	$\sqrt{16p^4}$
Simplify.	$4p^2$

TRY IT 12.1

Simplify: $\sqrt{\frac{75x^5}{3x}}$.

Show answer
$5x^2$

TRY IT 12.2

Simplify: $\sqrt{\frac{72z^{12}}{2z^{10}}}$.

Show answer

$6z$

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

We can use a similar property to simplify a square root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect square we simplify the numerator and denominator separately.

Quotient Property of Square Roots

If *a*, *b* are non-negative real numbers and $b \neq 0$, then

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

EXAMPLE 13

Simplify: $\sqrt{\frac{21}{64}}$.

Solution

	$\sqrt{\frac{21}{64}}$
We cannot simplify the fraction inside the radical. Rewrite using the quotient property.	$\frac{\sqrt{21}}{\sqrt{64}}$
Simplify the square root of 64. The numerator cannot be simplified.	$\frac{\sqrt{21}}{8}$

TRY IT 13.1

Simplify: $\sqrt{\frac{19}{49}}$.

Show answer

$\frac{\sqrt{19}}{7}$

TRY IT 13.2

Simplify: $\sqrt{\frac{28}{81}}$.

Show answer

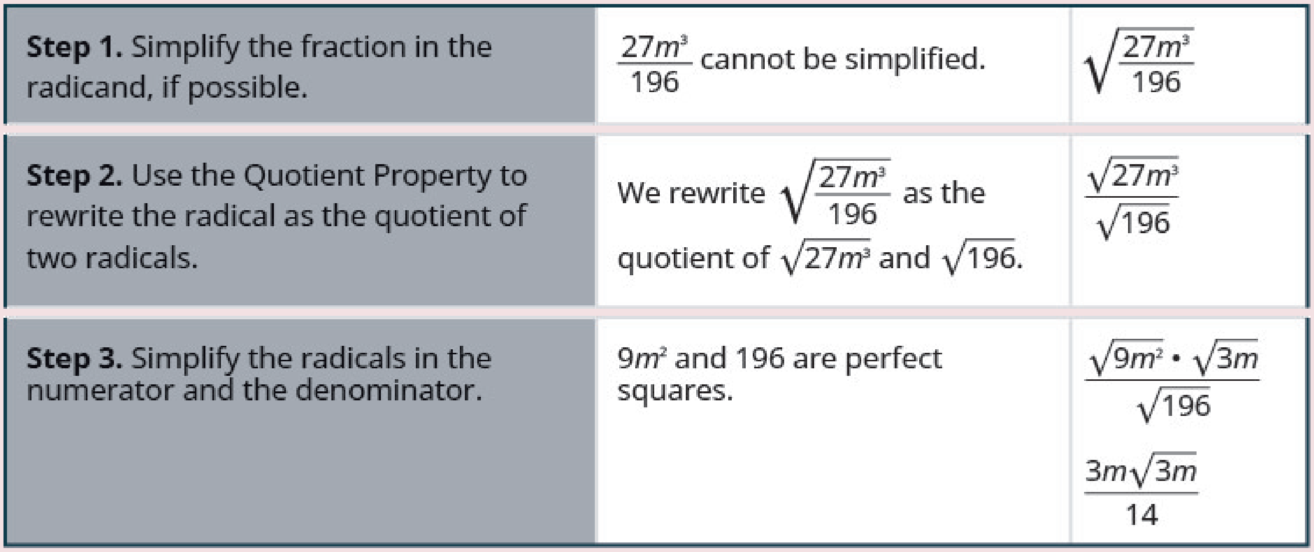
$\frac{2\sqrt{7}}{9}$

EXAMPLE 14

How to Use the Quotient Property to Simplify a Square Root

Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution



Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

TRY IT 14.1

Simplify: $\sqrt{\frac{24p^3}{49}}$.

Show answer

$\frac{2p\sqrt{6p}}{7}$

TRY IT 14.2

Simplify: $\sqrt{\frac{48x^5}{100}}$.

Show answer

$\frac{2x^2\sqrt{3x}}{5}$

HOW TO: Simplify a square root using the quotient property.

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

EXAMPLE 15

Simplify: $\sqrt{\frac{45x^5}{y^4}}$.

Solution

	$\sqrt{\frac{45x^5}{y^4}}$
We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.	$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$
Simplify.	$\frac{3x^2\sqrt{5x}}{y^2}$

TRY IT 15.1

Simplify: $\sqrt{\frac{80m^3}{n^6}}$.

Show answer

$\frac{4m\sqrt{5m}}{n^3}$

TRY IT 15.2

Simplify: $\sqrt{\frac{54u^7}{v^8}}$.

Show answer

$\frac{3u^3\sqrt{6u}}{v^4}$

Be sure to simplify the fraction in the radicand first, if possible.

EXAMPLE 16

Simplify: $\sqrt{\frac{81d^9}{25d^4}}$.

Solution

	$\sqrt{\frac{81d^9}{25d^4}}$
Simplify the fraction in the radicand.	$\sqrt{\frac{81d^5}{25}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{81d^5}}{\sqrt{25}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{81d^4}\cdot\sqrt{d}}{5}$
Simplify.	$\frac{9d^2\sqrt{d}}{5}$

TRY IT 16.1

Simplify: $\sqrt{\frac{64x^7}{9x^3}}$.

Show answer

$\frac{8x^2}{3}$

TRY IT 16.2

Simplify: $\sqrt{\frac{16a^9}{100a^5}}$.

Show answer

$\frac{2a^2}{5}$

EXAMPLE 17

Simplify: $\sqrt{\frac{18p^5q^7}{32pq^2}}$.

Solution

	$\sqrt{\frac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\frac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\frac{3p^2q^2\sqrt{q}}{4}$

TRY IT 17.1

Simplify: $\sqrt{\frac{50x^5y^3}{72x^4y}}$.

Show answer

$\frac{5y\sqrt{x}}{6}$

TRY IT 17.2

Simplify: $\sqrt{\frac{48m^7n^2}{125m^5n^9}}$.

Show answer

$\frac{4m\sqrt{3}}{5n^3\sqrt{5n}}$

Key Concepts

- **Simplified Square Root** $\sqrt{a}$ is considered simplified if a has no perfect-square factors.
- **Product Property of Square Roots** If a, b are non-negative real numbers, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$
- **Simplify a Square Root Using the Product Property** To simplify a square root using the Product Property:

   1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.
   2. Use the product rule to rewrite the radical as the product of two radicals.
   3. Simplify the square root of the perfect square.

- **Quotient Property of Square Roots** If a, b are non-negative real numbers and $b \neq 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- **Simplify a Square Root Using the Quotient Property** To simplify a square root using the Quotient Property:

   1. Simplify the fraction in the radicand, if possible.
   2. Use the Quotient Rule to rewrite the radical as the quotient of two radicals.
   3. Simplify the radicals in the numerator and the denominator.

Practice Makes Perfect

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

1. $\sqrt{27}$	2. $\sqrt{80}$
3. $\sqrt{125}$	4. $\sqrt{96}$
5. $\sqrt{200}$	6. $\sqrt{147}$
7. $\sqrt{450}$	8. $\sqrt{252}$
9. $\sqrt{800}$	10. $\sqrt{288}$
11. $\sqrt{675}$	12. $\sqrt{1250}$
13. $\sqrt{x^7}$	14. $\sqrt{y^{11}}$
15. $\sqrt{p^3}$	16. $\sqrt{q^5}$
17. $\sqrt{m^{13}}$	18. $\sqrt{n^{21}}$
19. $\sqrt{r^{25}}$	20. $\sqrt{s^{33}}$
21. $\sqrt{49n^{17}}$	22. $\sqrt{25m^9}$
23. $\sqrt{81r^{15}}$	24. $\sqrt{100s^{19}}$
25. $\sqrt{98m^5}$	26. $\sqrt{32n^{11}}$
27. $\sqrt{125r^{13}}$	28. $\sqrt{80s^{15}}$
29. $\sqrt{200p^{13}}$	30. $\sqrt{128q^3}$
31. $\sqrt{242m^{23}}$	32. $\sqrt{175n^{13}}$
33. $\sqrt{147m^7n^{11}}$	34. $\sqrt{48m^7n^5}$
35.$\sqrt{75r^{13}s^9}$	36. $\sqrt{96r^3s^3}$
37. $\sqrt{300p^9q^{11}}$	38. $\sqrt{192q^3r^7}$
39. $\sqrt{242m^{13}n^{21}}$	40. $\sqrt{150m^9n^3}$
41. $5+\sqrt{12}$	42. $8+\sqrt{96}$
43. $1+\sqrt{45}$	44. $3+\sqrt{125}$
45. $\frac{10-\sqrt{24}}{2}$	46. $\frac{8-\sqrt{80}}{4}$
47. $\frac{3+\sqrt{90}}{3}$	48. $\frac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

49. $\sqrt{\frac{49}{64}}$	50. $\sqrt{\frac{100}{36}}$
51. $\sqrt{\frac{121}{16}}$	52. $\sqrt{\frac{144}{169}}$
53. $\sqrt{\frac{72}{98}}$	54. $\sqrt{\frac{75}{12}}$
55. $\sqrt{\frac{9}{25}}$	56. $\sqrt{\frac{300}{243}}$
57. $\sqrt{\frac{x^{10}}{x^6}}$	58. $\sqrt{\frac{p^{20}}{p^{10}}}$
59. $\sqrt{\frac{y^4}{y^8}}$	60. $\sqrt{\frac{q^8}{q^{14}}}$
61. $\sqrt{\frac{200x^7}{2x^3}}$	62. $\sqrt{\frac{98y^{11}}{2y^5}}$
63. $\sqrt{\frac{96p^9}{6p}}$	64. $\sqrt{\frac{108q^{10}}{3q^2}}$
65. $\sqrt{\frac{36}{35}}$	66. $\sqrt{\frac{144}{65}}$
67. $\sqrt{\frac{20}{81}}$	68. $\sqrt{\frac{21}{196}}$
69. $\sqrt{\frac{96x^7}{121}}$	70. $\sqrt{\frac{108y^4}{49}}$
71. $\sqrt{\frac{300m^5}{64}}$	72. $\sqrt{\frac{125n^7}{169}}$
73. $\sqrt{\frac{98r^5}{100}}$	74. $\sqrt{\frac{180s^{10}}{144}}$
75. $\sqrt{\frac{28q^6}{225}}$	76. $\sqrt{\frac{150r^3}{256}}$
77. $\sqrt{\frac{75r^9}{s^8}}$	78. $\sqrt{\frac{72x^5}{y^6}}$
79. $\sqrt{\frac{28p^7}{q^2}}$	80. $\sqrt{\frac{45r^3}{s^{10}}}$
81. $\sqrt{\frac{100x^5}{36x^3}}$	82. $\sqrt{\frac{49r^{12}}{16r^6}}$
83. $\sqrt{\frac{121p^5}{81p^2}}$	84. $\sqrt{\frac{25r^8}{64r}}$
85. $\sqrt{\frac{32x^5y^3}{18x^3y}}$	86. $\sqrt{\frac{75r^6s^8}{48rs^4}}$

87. $\sqrt{\frac{27p^2q}{108p^5q^3}}$

88. $\sqrt{\frac{50r^5s^2}{128r^2s^5}}$

Everyday Math

89.

a) Elliott decides to construct a square garden that will take up 288 square feet of his yard. Simplify $\sqrt{288}$ to determine the length and the width of his garden. Round to the nearest tenth of a foot.

b) Suppose Elliott decides to reduce the size of his square garden so that he can create a 5-foot-wide walking path on the north and east sides of the garden. Simplify $\sqrt{288}-5$ to determine the length and width of the new garden. Round to the nearest tenth of a foot.

90.

a) Melissa accidentally drops a pair of sunglasses from the top of a roller coaster, 64 feet above the ground. Simplify $\sqrt{\frac{64}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

b) Suppose the sunglasses in the previous example were dropped from a height of 144 feet. Simplify $\sqrt{\frac{144}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

Writing Exercises

91. Explain why $\sqrt{x^4}=x^2$. Then explain why $\sqrt{x^{16}}=x^8$.

92. Explain why $7+\sqrt{9}$ is not equal to $\sqrt{7+9}$.

Answers

1. $3\sqrt{3}$	3. $5\sqrt{5}$	5. $10\sqrt{2}$
7. $15\sqrt{2}$	9. $20\sqrt{2}$	11. $15\sqrt{3}$
13. $x^3\sqrt{x}$	15. $p\sqrt{p}$	17. $m^6\sqrt{m}$
19. $r^{12}\sqrt{r}$	21. $7n^8\sqrt{n}$	23. $9r^7\sqrt{r}$
25. $7m^2\sqrt{2m}$	27. $5r^6\sqrt{5r}$	29. $10p^6\sqrt{2p}$
31. $11m^{11}\sqrt{2m}$	33. $7m^3n^5\sqrt{3mn}$	35. $5r^6s^4\sqrt{3rs}$ 70)
37. $10p^4q^5\sqrt{3pq}$	39. $11m^6n^{10}\sqrt{2mn}$	41. $5+2\sqrt{3}$
43. $1+3\sqrt{5}$	45. $5-2\sqrt{6}$	47. $1+\sqrt{10}$
49. $\frac{7}{8}$	51. $\frac{11}{4}$	53. $\frac{6}{7}$
55. $\frac{3}{5}$	57. x^2	59. $\frac{1}{y^2}$
61. $10x^2$	63. $4p^4$	65. $\frac{6}{\sqrt{35}}$
67. $\frac{2\sqrt{5}}{9}$	69. $\frac{4x^3\sqrt{6x}}{11}$	71. $\frac{10m^2\sqrt{3m}}{8}$
73. $\frac{7r^2\sqrt{2r}}{10}$	75. $\frac{2q^3\sqrt{7}}{15}$	77. $\frac{5r^4\sqrt{3r}}{s^4}$
79. $\frac{4p^3\sqrt{7p}}{q}$	81. $\frac{5x}{3}$	83. $\frac{11p\sqrt{p}}{9}$
85. $\frac{4xy}{3}$	87. $\frac{1}{2pq\sqrt{p}}$	89. a)17.0 feetb)15.0 feet
91. Answers will vary.		

Attributions

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7.6 Chapter Review

Review Exercises

Simplify Expressions with Exponents

In the following exercises, simplify.

1. 17^1	2. 10^4
3. $(0.5)^3$	4. $\left(\frac{2}{9}\right)^2$
5. -2^6	6. $(-2)^6$

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression.

7. $p^{15} \cdot p^{16}$	8. $x^4 \cdot x^3$
9. $8 \cdot 8^5$	10. $4^{10} \cdot 4^6$
11. $y^c \cdot y^3$	12. $n \cdot n^2 \cdot n^4$

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression.

13. $\left(5^3\right)^2$	14. $\left(m^3\right)^5$
15. $(3^r)^s$	16. $\left(y^4\right)^x$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression.

17. $(-5y)^3$	18 $(4a)^2$
19. $(10xyz)^3$	20. $(2mn)^5$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

21. $\left(4a^3b^2\right)^3$	22. $\left(p^2\right)^5 \cdot \left(p^3\right)^6$
23. $\left(2q^3\right)^4(3q)^2$	24. $(5x)^2\,(7x)$
25. $\left(\frac{2}{5}m^2n\right)^3$	26. $\left(\frac{1}{3}x^2\right)^2\left(\frac{1}{2}x\right)^3$

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

27. $\frac{10^{25}}{10^5}$	28. $\frac{u^{24}}{u^6}$
29. $\frac{v^{12}}{v^{48}}$	30. $\frac{3^4}{3^6}$
31. $\frac{5}{5^8}$	32. $\frac{x}{x^5}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

33. x^0	34. 75^0
35. $\left(-12^0\right)(-12)^0$	36. -12^0
37. $(25x)^0$	38. $25x^0$
39. $(19n)^0-(25m)^0$	40. $19n^0-25m^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

41. $\left(\frac{m}{3}\right)^4$	42. $\left(\frac{2}{5}\right)^3$
43. $\left(\frac{x}{2y}\right)^6$	44. $\left(\frac{r}{s}\right)^8$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

45. $\frac{n^{10}}{\left(n^5\right)^2}$	46. $\frac{\left(x^3\right)^5}{x^9}$
47. $\left(\frac{r^8}{r^3}\right)^4$	48. $\left(\frac{q^6}{q^8}\right)^3$
49. $\left(\frac{3x^4}{2y^2}\right)^5$	50. $\left(\frac{c^2}{d^5}\right)^9$
51. $\frac{\left(3n^2\right)^4\left(-5n^4\right)^3}{\left(-2n^5\right)^2}$	52. $\left(\frac{v^3v^9}{v^6}\right)^4$

Divide Monomials

In the following exercises, divide the monomials.

53. $\frac{64a^5b^9}{-16a^{10}b^3}$	54. $-65y^{14}\div 5y^2$
55. $\frac{\left(8p^6q^2\right)\left(9p^3q^5\right)}{16p^8q^7}$	56. $\frac{144x^{15}y^8z^3}{18x^{10}y^2z^{12}}$

Use the Definition of a Negative Exponent

In the following exercises, simplify.

57. $(-5)^{-3}$	58. 9^{-2}
59. $(6u)^{-3}$	60. $3\cdot 4^{-3}$
61. $\left(\frac{3}{4}\right)^{-2}$	62. $\left(\frac{2}{5}\right)^{-1}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

63. $q^{-6} \cdot q^{-5}$	64. $p^{-2} \cdot p^{8}$
65. $\left(y^{8}\right)^{-1}$	66. $\left(c^{-2}d\right)\left(c^{-3}d^{-2}\right)$
67. $\frac{a^{8}}{a^{12}}$	68. $\left(q^{-4}\right)^{-3}$
69. $\frac{r^{-2}}{r^{-3}}$	70. $\frac{n^{5}}{n^{-4}}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

71. 0.00429	72. 8,500,000
73. In 2015, the population of the world was about 7,200,000,000 people.	74. The thickness of a dime is about 0.053 inches.

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

75. 1.5 ×10^{10}	76. 3.8 ×10^{5}
77. 5.5 ×10^{-1}	78. 9.1 ×10^{-7}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply and write your answer in decimal form.

79. 3.5 ×10^{-2})(6.2 × 10^{-1})	80. 2 ×10^{5}) (4 ×10^{-3})

In the following exercises, divide and write your answer in decimal form.

81. $\frac{9 \times 10^{-5}}{3 \times 10^{2}}$	82. $\frac{8 \times 10^{5}}{4 \times 10^{-1}}$

Simplify Expressions with Square Roots

In the following exercises, simplify.

83. $\sqrt{144}$	84. $\sqrt{64}$
85. $-\sqrt{81}$	86. $-\sqrt{25}$
87. $\sqrt{-36}$	88. $\sqrt{-9}$
89. $\sqrt{64+225}$	90. $\sqrt{64}+\sqrt{225}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

91. $\sqrt{155}$	92. $\sqrt{28}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

93. $\sqrt{57}$	94. $\sqrt{15}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify. (Assume all variables are greater than or equal to zero.)

95. $\sqrt{64b^2}$	96. $\sqrt{q^2}$
97. $\sqrt{225m^2n^2}$	98. $-\sqrt{121a^2}$
99. $\sqrt{49y^2}$	100. $-\sqrt{100q^2}$
101. $\sqrt{121c^2d^2}$	102. $\sqrt{4a^2b^2}$

Use Square Roots in Applications

In the following exercises, solve. Round to one decimal place.

103.**Landscaping** Janet wants to plant a square flower garden in her yard. She has enough topsoil to cover an area of 30 square feet. How long can a side of the flower garden be?

104. **Art** Diego has 225 square inch tiles. He wants to use them to make a square mosaic. How long can each side of the mosaic be?

105. **Accident investigation** The skid marks of a car involved in an accident were 216 feet. How fast had the car been going before applying the brakes?

106. **Gravity** A hiker dropped a granola bar from a lookout spot 576 feet above a valley. How long did it take the granola bar to reach the valley floor?

Review Exercise Answers

1. 17	3. 0.125
5. -64	7. p^{31}
9. 8^6	11. y^{c+3}
13. 5^6	15. 3^{rs}
17. $-125y^3$	19. $1000x^3y^3z^3$
21. $64a^9b^6$	23. $48q^{14}$
25. $\frac{8}{125}m^6n^3$	27. 10^{20}
29. $\frac{1}{v^{36}}$	31. $\frac{1}{5^7}$
33. 1	35. 1
37. 1	39. 0
41. $\frac{m^4}{81}$	43. $\frac{x^6}{64y^6}$
45. 1	47. r^{20}
49. $\frac{343x^{20}}{32y^{10}}$	51. $-\frac{10{,}125n^{10}}{4}$
53. $-\frac{4b^6}{a^5}$	55. $\frac{9p}{2}$
57. $-\frac{1}{125}$	59.$\frac{1}{216u^3}$
61. $\frac{16}{9}$	63. $\frac{1}{q^{11}}$
65. $\frac{1}{y^8}$	67. $\frac{1}{a^4}$
69. r	71. 4.29×10^{-3}
73. 7.2×10^9	75. $15,000,000,000$
77. 0.55	79. 0.0217
81. 0.0000003	83. 12
85. −9	87. not a real number
89. 17	91. $12 < \sqrt{155} < 13$
93. 7.55	95. 8b
97. 15mn	99. 7y
101. 11cd	103. 5.5 feet
105. 72 mph	

Practice Test

In the following exercises, simplify each expression.

1. $\left(-\frac{2}{5}\right)^3$	2. $u \cdot u^4$
3. $\left(4a^3b^5\right)^2$	4. $\frac{n^{-2}}{n^{-10}}$
5. $\frac{3^8}{3^{10}}$	6. $\left(\frac{v^2v^6}{v^4}\right)^2$
7. $\left(87x^{15}y^3z^{22}\right)^0$	8. $\left(\frac{m^4 \cdot m}{m^3}\right)^6$
9. $\frac{80c^8d^2}{16cd^{10}}$	10. 5^{-2}
11. $q^{-4} \cdot q^{-5}$	12. $(4m)^{-3}$
13. $\frac{8.4 \times 10^{-3}}{4 \times 10^3}$	14. $\left(3.4 \times 10^9\right)\left(2.2 \times 10^{-5}\right)$
15. $\sqrt{81}$	16. $-\sqrt{49}$
17. $\sqrt{-16}$	18. $\sqrt{b^2}$
19. $-\sqrt{64a^2}$	20. $-\sqrt{144q^2}$
21. Convert 83,000,000 to scientific notation.	22. Convert 6.91×10^{-5} to decimal form.

Practice Test Answers

1. $-\frac{8}{125}$	2. u^5
3. $16a^6b^{10}$	4. n^8
5. $\frac{1}{9}$	6. v^8
7. 1	8. m^{12}
9. $\frac{5c^7}{d^8}$	10. $\frac{1}{25}$
11. $\frac{1}{q}$	12. $\frac{1}{64m^3}$
13. 2.1×10^{-6}	14. 7.48×10^4
15. 9	16. -7
17. not a real number	18. b
19. $-8a$	20. $-12q$
21. 8.3×10^7	22. 0.0000691

CHAPTER 8 Polynomials

Architects use polynomials to design curved shapes such as this suspension bridge, the Silver Jubilee bridge in Halton, England.

We have seen that the graphs of linear equations are straight lines. Graphs of other types of equations, called polynomial equations, are curves, like the outline of this suspension bridge. Architects use polynomials to design the shape of a bridge like this and to draw the blueprints for it. Engineers use polynomials to calculate the stress on the bridge’s supports to ensure they are strong enough for the intended load. In this chapter, you will explore operations with and properties of polynomials.

8.1 Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a *term* is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. Some examples of monomial are $8, -2x^2, 4y^3$, and $11z^7$.

Monomials

A monomial is a term of the form ax^m, where a is a constant and m is a positive whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a polynomial. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a trinomial has exactly three terms. There are no special names for polynomials with more than three terms.

Polynomials

polynomial—A monomial, or two or more monomials combined by addition or subtraction, is a polynomial.

- **monomial**—A polynomial with exactly one term is called a monomial.
- **binomial**—A polynomial with exactly two terms is called a binomial.

- **trinomial**—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	-13
Binomial	$a+7$	$4b-5$	y^2-16	$3x^3-9x^2$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the “family” of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

EXAMPLE 1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

a. $4y^2-8y-6$
b. $-5a^4b^2$
c. $2x^5-5x^3-9x^2+3x+4$
d. $13-5m^3$
e. q

Solution

	Polynomial	**Number of terms**	**Type**
a)	$4y^2-8y-6$	3	Trinomial
b)	$-5a^4b^2$	1	Monomial
c)	$2x^5-5x^3-9x^2+3x+4$	5	Polynomial
d)	$13-5m^3$	2	Binomial
e)	q	1	Monomial

TRY IT 1.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

a) $5b$ b) $8y^3 - 7y^2 - y - 3$ c) $-3x^2 - 5x + 9$ d) $81 - 4a^2$e)$-5x^6$

Show answer

a) monomial b) polynomial c) trinomial d) binomial e) monomial

TRY IT 1.2

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

a) $27z^3 - 8$ b) $12m^3 - 5m^2 - 2m$ c) $\frac{5}{6}$ d) $8x^4 - 7x^2 - 6x - 5$ e) $-n^4$

Show answer

a) binomial b) trinomial c) monomial d) polynomial e) monomial

Determine the Degree of Polynomials

The degree of a polynomial and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The degree of a constant is 0—it has no variable.

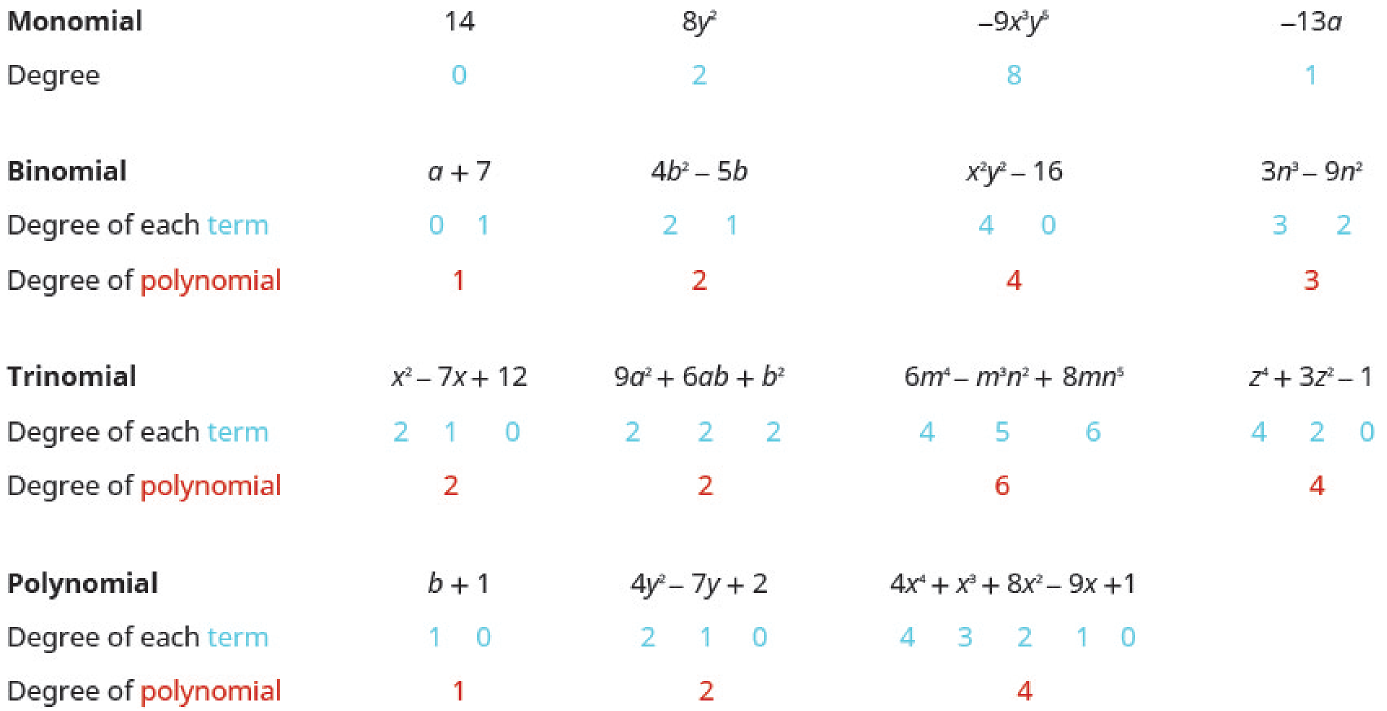
Degree of a Polynomial

The degree of a term is the sum of the exponents of its variables.

The degree of a constant is 0.

The degree of a polynomial is the highest degree of all its terms.

Let’s see how this works by looking at several polynomials. We’ll take it step by step, starting with monomials, and then progressing to polynomials with more terms.



Monomial	14	$8y^2$	$-9x^3y^5$	$-13a$
Degree	0	2	8	1
Binomial	$a+7$	$4b^2-5b$	x^2y^2-16	$3n^3-9n^2$
Degree of each term	0 1	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-7x+12$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

EXAMPLE 2

Find the degree of the following polynomials.

a. $10y$
b. $4x^3-7x+5$
c. -15
d. $-8b^2+9b-2$
e. $8xy^2+2y$

Solution

a) The exponent of y is one. $y = y^1$	$10y$ The degree is 1.
b) The highest degree of all the terms is 3.	$4x^3 - 7x + 5$ The degree is 3.
c) The degree of a constant is 0.	-15 The degree is 0.
d) The highest degree of all the terms is 2.	$-8b^2 + 9b - 2$ The degree is 2.
e) The highest degree of all the terms is 3.	$8xy^2 + 2y$ The degree is 3.

EXAMPLE 2.1

Find the degree of the following polynomials:

a) $-15b$ b) $10z^4 + 4z^2 - 5$ c) $12c^5d^4 + 9c^3d^9 - 7$ d) $3x^2y - 4x$e)-9

Show answer
a) 1 b) 4 c) 12 d) 3 e) 0

TRY IT 2.2

Find the degree of the following polynomials:

a) 52 b) $a^4b - 17a^4$ c) $5x + 6y + 2z$ d) $3x^2 - 5x + 7$e)$-a^3$

Show answer
a) 0 b) 5 c) 1 d) 2 e) 3

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

EXAMPLE 3

Add: $25y^2 + 15y^2$.

Solution

	$25y^2 + 15y^2$
Combine like terms.	$40y^2$

TRY IT 3.1

Add: $12q^2 + 9q^2$.

Show answer

$21q^2$

TRY 3.2

Add: $-15c^2 + 8c^2$.

Show answer

$-7c^2$

EXAMPLE 4

Subtract: $16p - (-7p)$.

Solution

	$16p - (-7p)$
Combine like terms.	$23p$

TRY IT 4.1

Subtract: $8m - (-5m)$.

Show answer
$13m$

TRY IT 4.2

Subtract: $-15z^3 - \left(-5z^3\right)$.

Show answer
$-10z^3$

Remember that like terms must have the same variables with the same exponents.

EXAMPLE 5

Simplify: $c^2 + 7d^2 - 6c^2$.

Solution

	$c^2 + 7d^2 - 6c^2$
Combine like terms.	$-5c^2 + 7d^2$

TRY IT 5.1

Add: $8y^2 + 3z^2 - 3y^2$.

Show answer
$5y^2 + 3z^2$

TRY IT 5.2

Add: $3m^2 + n^2 - 7m^2$.

Show answer
$-4m^2 + n^2$

EXAMPLE 6

Simplify: $u^2v + 5u^2 - 3v^2$.

Solution

	$u^2v + 5u^2 - 3v^2$
There are no like terms to combine.	$u^2v + 5u^2 - 3v^2$

TRY IT 6.1

Simplify: $m^2n^2 - 8m^2 + 4n^2$.

Show answer
There are no like terms to combine.

TRY IT 6.2

Simplify: $pq^2 - 6p - 5q^2$.

Show answer
There are no like terms to combine.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

EXAMPLE 7

Find the sum: $\left(5y^2-3y+15\right)+\left(3y^2-4y-11\right)$.

Solution

Identify like terms.	$(\underline{\underline{5y^2}}-\underline{3y}+\underline{1}5)+(\underline{\underline{3y^2}}-\underline{4y}-\underline{1}1)$
Rearrange to get the like terms together.	$\underline{\underline{5y^2+3y^2}}-\underline{3y-4y}+\underline{1}5-\underline{1}1$
Combine like terms.	$8y^2-7y+4$

TRY IT 7.1

Find the sum: $\left(7x^2-4x+5\right)+\left(x^2-7x+3\right)$.

Show answer

$8x^2-11x+1$

TRY IT 7.2

Find the sum: $\left(14y^2+6y-4\right)+\left(3y^2+8y+5\right)$.

Show answer

$17y^2+14y+1$

EXAMPLE 8

Find the difference: $\left(9w^2-7w+5\right)-\left(2w^2-4\right)$.

Solution

	$(9w^2 - 7w + 5) - (2w^2 - 4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}} - \underline{7w} + \underset{\sim}{5} - \underline{\underline{2w^2}} + \underset{\sim}{4}$
Rearrange the terms.	$\underline{\underline{9w^2 - 2w^2}} - \underline{7w} + \underset{\sim}{5} + \underset{\sim}{4}$
Combine like terms.	$7w^2 - 7w + 9$

TRY IT 8.1

Find the difference: $\left(8x^2 + 3x - 19\right) - \left(7x^2 - 14\right)$.

Show answer
$15x^2 + 3x - 5$

TRY IT 8.2

Find the difference: $\left(9b^2 - 5b - 4\right) - \left(3b^2 - 5b - 7\right)$.

Show answer
$6b^2 + 3$

EXAMPLE 9

Subtract: $\left(c^2 - 4c + 7\right)$ from $\left(7c^2 - 5c + 3\right)$.

Solution

	Subtract $(c^2 - 4c + 7)$ from $(7c^2 - 5c + 3)$.
	$(7c^2 - 5c + 3) - (c^2 - 4c + 7)$
Distribute and identify like terms.	$\underline{\underline{7c^2}} - \underline{5c} + \underset{\sim}{3} - \underline{\underline{c^2}} + \underline{4c} - \underset{\sim}{7}$
Rearrange the terms.	$\underline{\underline{7c^2 - c^2}} - \underline{5c + 4c} + \underset{\sim}{3} - \underset{\sim}{7}$
Combine like terms.	$6c^2 - c - 4$

TRY IT 9.1

Subtract: $\left(5z^2 - 6z - 2\right)$ from $\left(7z^2 + 6z - 4\right)$.

Show answer
$2z^2 + 12z - 2$

TRY IT 9.2

Subtract: $\left(x^2 - 5x - 8\right)$ from $\left(6x^2 + 9x - 1\right)$.

Show answer
$5x^2 + 14x + 7$

EXAMPLE 10

Find the sum: $\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$.

Solution

	$\left(u^2-6uv+5v^2\right)+\left(3u^2+2uv\right)$
Distribute.	$u^2-6uv+5v^2+3u^2+2uv$
Rearrange the terms, to put like terms together.	$u^2+3u^2-6uv+2uv+5v^2$
Combine like terms.	$4u^2-4uv+5v^2$

EXAMPLE 10.1

Find the sum: $\left(3x^2-4xy+5y^2\right)+\left(2x^2-xy\right)$.

Show answer

$5x^2-5xy+5y^2$

EXAMPLE 10.2

Find the sum: $\left(2x^2-3xy-2y^2\right)+\left(5x^2-3xy\right)$.

Show answer

$7x^2-6xy-2y^2$

EXAMPLE 11.1

Find the difference: $\left(p^2+q^2\right)-\left(p^2+10pq-2q^2\right)$.

Solution

	$\left(p^2+q^2\right)-\left(p^2+10pq-2q^2\right)$
Distribute.	$p^2+q^2-p^2-10pq+2q^2$
Rearrange the terms, to put like terms together.	$p^2-p^2-10pq+q^2+2q^2$
Combine like terms.	$-10pq^2+3q^2$

TRY IT 11.1

Find the difference: $\left(a^2+b^2\right)-\left(a^2+5ab-6b^2\right)$.

Show answer
$-5ab-5b^2$

TRY IT 11.2

Find the difference: $\left(m^2+n^2\right)-\left(m^2-7mn-3n^2\right)$.

Show answer
$4n^2+7mn$

EXAMPLE 12

Simplify: $\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$.

Solution

	$\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$
Distribute.	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms, to put like terms together.	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

TRY IT 12.1

Simplify: $\left(x^3-x^2y\right)-\left(xy^2+y^3\right)+\left(x^2y+xy^2\right)$.

Show answer
x^3-y^3

TRY IT 12.2

Simplify: $\left(p^3 - p^2q\right) + \left(pq^2 + q^3\right) - \left(p^2q + pq^2\right)$.

Show answer

$p^3 - 2p^2q + q^3$

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a polynomial. We will substitute the given value for the variable and then simplify using the order of operations.

EXAMPLE 13

Evaluate $5x^2 - 8x + 4$ when

a. $x = 4$
b. $x = -2$
c. $x = 0$

Solution

a) $x = 4$	
	$5x^2 - 8x + 4$
Substitute 4 for x.	$5(4)^2 - 8(4) + 4$
Simplify the exponents.	$5 \cdot 16 - 8(4) + 4$
Multiply.	$80 - 32 + 4$
Simplify.	52

b) $x = -2$	
	$5x^2 - 8x + 4$
Substitute -2 for x.	$5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$5 \cdot 4 - 8(-2) + 4$
Multiply.	$20 + 16 + 4$
Simplify.	40

c) $x = 0$	
	$5x^2 - 8x + 4$
Substitute 0 for x.	$5(0)^2 - 8(0) + 4$
Simplify the exponents.	$5 \cdot 0 - 8(0) + 4$
Multiply.	$0 + 0 + 4$
Simplify.	4

TRY IT 13.1

Evaluate: $3x^2 + 2x - 15$ when

a. $x = 3$
b. $x = -5$
c. $x = 0$

Show answer
a) 18 b) 50 c) -15

TRY IT 13.2

Evaluate: $5z^2 - z - 4$ when

a. $z = -2$

b. $z = 0$
c. $z = 2$

Show answer

a) 18 b) -4 c) 14

EXAMPLE 14

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250 foot tall building. Find the height after $t = 2$ seconds.

Solution

	$-16t^2 + 250$
Substitute $t = 2$.	$-16(2)^2 + 250$
Simplify.	$-16 \cdot 4 + 250$
Simplify.	$-64 + 250$
Simplify.	186
	After 2 seconds the height of the ball is 186 feet.

TRY IT 14.1

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 0$ seconds.

Show answer

250

TRY IT 14.2

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 3$ seconds.

Show answer

106

EXAMPLE 15

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 4$ feet and $y = 6$ feet.

Solution

	$6x^2 + 15xy$
Substitute $x = 4, y = 6$.	$6(4)^2 + 15(4)(6)$
Simplify.	$6 \cdot 16 + 15(4)(6)$
Simplify.	$96 + 360$
Simplify.	456
	The cost of producing the box is $456.

TRY IT 15.1

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 6$ feet and $y = 4$ feet.

Show answer
$576

TRY IT 15.2

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 5$ feet and $y = 8$ feet.

Show answer
$750

Access these online resources for additional instruction and practice with adding and subtracting polynomials.

- Add and Subtract Polynomials 1
- Add and Subtract Polynomials 2
- Add and Subtract Polynomial 3
- Add and Subtract Polynomial 4

Key Concepts

- **Monomials**
  - A monomial is a term of the form ax^m, where a is a constant and m is a whole number
- **Polynomials**
  - **polynomial—**A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial—**A polynomial with exactly one term is called a monomial.
  - **binomial—**A polynomial with exactly two terms is called a binomial.
  - **trinomial—**A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Glossary

binomial
A binomial is a polynomial with exactly two terms.
degree of a constant
The degree of any constant is 0.
degree of a polynomial
The degree of a polynomial is the highest degree of all its terms.
degree of a term
The degree of a term is the exponent of its variable.
monomial
A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.
polynomial
A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

standard form

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

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Practice Makes Perfect

Identify Polynomials, Monomials, Binomials, and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

1. a) $81b^5 - 24b^3 + 1$ b) $5c^3 + 11c^2 - c - 8$ c) $\frac{14}{15}y + \frac{1}{7}$ d) 5 e) $4y + 17$	2. a) $x^2 - y^2$ b) $-13c^4$ c) $x^2 + 5x - 7$ d) $x^2y^2 - 2xy + 8$ e) 19
3. a) $8 - 3x$ b) $z^2 - 5z - 6$ c) $y^3 - 8y^2 + 2y - 16$ d) $81b^5 - 24b^3 + 1$ e) -18	4. a) $11y^2$ b) -73 c) $6x^2 - 3xy + 4x - 2y + y^2$ d) $4y + 17$ e) $5c^3 + 11c^2 - c - 8$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

5. a) $6a^2 + 12a + 14$ b) $18xy^2z$ c) $5x + 2$ d) $y^3 - 8y^2 + 2y - 16$ e) -24	6. a) $9y^3 - 10y^2 + 2y - 6$ b) $-12p^4$ c) $a^2 + 9a + 18$ d) $20x^2y^2 - 10a^2b^2 + 30$ e) 17
7. a) $14 - 29x$ b) $z^2 - 5z - 6$ c) $y^3 - 8y^2 + 2y - 16$ d) $23ab^2 - 14$ e) -3	8. a) $62y^2$ b) 15 c) $6x^2 - 3xy + 4x - 2y + y^2$ d) $10 - 9x$ e) $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

9. $7x^2 + 5x^2$	10. $4y^3 + 6y^3$
11. $-12w + 18w$	12. $-3m + 9m$
13. $4\text{a} - 9a$	14. $-y - 5y$
15. $28x - (-12x)$	16. $13z - (-4z)$
17. $-5b - 17b$	18. $-10x - 35x$
19. $12a + 5b - 22a$	20. $14\text{x} - 3y - 13x$
21. $2a^2 + b^2 - 6a^2$	22. $5u^2 + 4v^2 - 6u^2$
23. $xy^2 - 5x - 5y^2$	24. $pq^2 - 4p - 3q^2$
25. $a^2b - 4a - 5ab^2$	26. $x^2y - 3x + 7xy^2$
27. $12\text{a} + 8b$	28. $19\text{y} + 5z$
29. Add: $4a, -3b, -8a$	30. Add: $4x, 3y, -3x$
31. Subtract $5x^6\text{from} - 12x^6$.	32. Subtract $2p^4\text{from} - 7p^4$.

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

33. $(5y^2+12y+4)+(6y^2-8y+7)$	34. $(4y^2+10y+3)+(8y^2-6y+5)$
35. $(x^2+6x+8)+(-4x^2+11x-9)$	36. $(y^2+9y+4)+(-2y^2-5y-1)$
37. $(8x^2-5x+2)+(3x^2+3)$	38. $(7x^2-9x+2)+(6x^2-4)$
39. $(5a^2+8)+(a^2-4a-9)$	40. $(p^2-6p-18)+(2p^2+11)$
41. $(4m^2-6m-3)-(2m^2+m-7)$	42. $(3b^2-4b+1)-(5b^2-b-2)$
43. $(a^2+8a+5)-(a^2-3a+2)$	44. $(b^2-7b+5)-(b^2-2b+9)$
45. $(12s^2-15s)-(s-9)$	46. $(10r^2-20r)-(r-8)$
47. Subtract $(9x^2+2)$ from $(12x^2-x+6)$.	48. Subtract $(5y^2-y+12)$ from $(10y^2-8y-20)$.
49. Subtract $(7w^2-4w+2)$ from $(8w^2-w+6)$.	50. Subtract $(5x^2-x+12)$ from $(9x^2-6x-20)$.
51. Find the sum of $(2p^3-8)$ and $(p^2+9p+18)$.	52. Find the sum of $(q^2+4q+13)$ and $(7q^3-3)$.
53. Find the sum of $(8a^3-8a)$ and $(a^2+6a+12)$.	54. Find the sum of $(b^2+5b+13)$ and $(4b^3-6)$.
55. Find the difference of (w^2+w-42) and $(w^2-10w+24)$.	56. Find the difference of $(z^2-3z-18)$ and $(z^2+5z-20)$.
57. Find the difference of $(c^2+4c-33)$ and $(c^2-8c+12)$.	58. Find the difference of $(t^2-5t-15)$ and $(t^2+4t-17)$.
59. $(7x^2-2xy+6y^2)+(3x^2-5xy)$	60. $(-5x^2-4xy-3y^2)+(2x^2-7xy)$
61. $(7m^2+mn-8n^2)+(3m^2+2mn)$	62. $(2r^2-3rs-2s^2)+(5r^2-3rs)$
63. $(a^2-b^2)-(a^2+3ab-4b^2)$	64. $(m^2+2n^2)-(m^2-8mn-n^2)$
65. $(u^2-v^2)-(u^2-4uv-3v^2)$	66. $(j^2-k^2)-(j^2-8jk-5k^2)$
67. $(p^3-3p^2q)+(2pq^2+4q^3)-(3p^2q+pq^2)$	68. $(a^3-2a^2b)+(ab^2+b^3)-(3a^2b+4ab^2)$
69. $(x^3-x^2y)-(4xy^2-y^3)+(3x^2y-xy^2)$	70. $(x^3-2x^2y)-(xy^2-3y^3)-(x^2y-4xy^2)$

Evaluate a Polynomial for a Given Value

In the following exercises, evaluate each polynomial for the given value.

71. Evaluate $8y^2 - 3y + 2$ when:

a) $y = 5$
b) $y = -2$
c) $y = 0$

72. Evaluate $5y^2 - y - 7$ when:

a) $y = -4$
b) $y = 1$
c) $y = 0$

73. Evaluate $4 - 36x$ when:

a) $x = 3$
b) $x = 0$
c) $x = -1$

74. Evaluate $16 - 36x^2$ when:

a) $x = -1$
b) $x = 0$
c) $x = 2$

75. A painter drops a brush from a platform 75 feet high. The polynomial $-16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

76. A girl drops a ball off a cliff into the ocean. The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall cliff. Find the height after $t = 2$ seconds.

77. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of *p* dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

78. A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of *p* dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

Everyday Math

79. **Fuel Efficiency** The fuel efficiency (in miles per gallon) of a car going at a speed of x miles per hour is given by the polynomial $-\frac{1}{150}x^2 + \frac{1}{3}x$. Find the fuel efficiency when $x = 30\text{mph}$.

80. **Stopping Distance** The number of feet it takes for a car traveling at x miles per hour to stop on dry, level concrete is given by the polynomial $0.06x^2 + 1.1x$. Find the stopping distance when $x = 40\text{mph}$.

81. **Rental Cost** The cost to rent a rug cleaner for d days is given by the polynomial $5.50d + 25$. Find the cost to rent the cleaner for 6 days.

82. **Height of Projectile** The height (in feet) of an object projected upward is given by the polynomial $-16t^2 + 60t + 90$ where t represents time in seconds. Find the height after $t = 2.5$ seconds.

83. **Temperature Conversion** The temperature in degrees Fahrenheit is given by the polynomial $\frac{9}{5}c + 32$ where c represents the temperature in degrees Celsius. Find the temperature in degrees Fahrenheit when $c = 65°$.

Writing Exercises

84. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.	85. Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.
86. Ariana thinks the sum $6y^2 + 5y^4$ is $11y^6$. What is wrong with her reasoning?	87. Jonathan thinks that $\frac{1}{3}$ and $\frac{1}{x}$ are both monomials. What is wrong with his reasoning?

Answers

1. a) trinomial b) polynomial c) binomial d) monomial e) binomial	3. a) binomial b) trinomial c) polynomial d) trinomial e) monomial
5. a) 2 b) 4 c) 1 d) 3 e) 0	7. a) 1 b) 2 c) 3 d) 3 e) 0
9. $12x^2$	11. $6w$
13. $-5a$	15. $40x$
17. $-22b$	19. $-10a + 5b$
21. $-4a^2 + b^2$	21. $-4a^2 + b^2$
25. $a^2b - 4a - 5ab^2$	27. $12\mathrm{a} + 8b$
29. $-4a - 3b$	31. $-17x^6$
33. $11y^2 + 4y + 11$	35. $-3x^2 + 17x - 1$
37. $11x^2 - 5x + 5$	39. $6a^2 - 4a - 1$
41. $2m^2 - 7m + 4$	43. $11a + 3$
45. $12s^2 - 14s + 9$	47. $3x^2 - x + 4$
49. $w^2 + 3w + 4$	51. $2p^3 + p^2 + 9p + 10$
51. $2p^3 + p^2 + 9p + 10$	55. $11w - 64$
57. $12c - 45$	59. $10x^2 - 7xy + 6y^2$
61. $10m^2 + 3mn - 8n^2$	63. $-3ab + 3b^2$
65. $4uv + 2v^2$	67. $p^3 - 6p^2q + pq^2 + 4q^3$
69. $x^3 + 2x^2y - 5xy^2 + y^3$	71. a) 187 b) 46 c) 2
73. a) −104 b) 4 c) 40	75. 11
77. $10,800	77. $10,800
81. $58	83. 149
85. Answers will vary.	87. Answers will vary.

Attributions

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8.2 Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2\left(x-3\right)$. You multiplied both terms in the parentheses, x and 3, by 2, to get $2x-6$. With this chapter's new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2

Multiplying a binomial by a monomial is nothing new for you! Here's an example:

EXAMPLE 1

Multiply: $4\left(x+3\right)$.

Solution

	$4(x+3)$
Distribute.	$4\cdot x+4\cdot 3$
Simplify.	$4x+12$

TRY IT 1.1

Multiply: $5\left(x+7\right)$.

Show answer
$5x+35$

TRY IT 1.2

Multiply: $3\left(y+13\right)$.

Show answer
$3y+39$

EXAMPLE 2

Multiply: $y\left(y-2\right)$.

Solution

	$y(y-2)$
Distribute.	$y\cdot y-y\cdot 2$
Simplify.	y^2-2y

TRY IT 2.1

Multiply: $x\left(x-7\right)$.

Show answer
x^2-7x

TRY IT 2.2

Multiply: $d\left(d-11\right)$.

Show answer

$d^2 - 11d$

EXAMPLE 3

Multiply: $7x\left(2x + y\right)$.

Solution

	$7x(2x + y)$
Distribute.	$7x \cdot 2x + 7x \cdot y$
Simplify.	$14x^2 + 7xy$

TRY IT 3.1

Multiply: $5x\left(x + 4y\right)$.

Show answer

$5x^2 + 20xy$

TRY IT 3.2

Multiply: $2p\left(6p + r\right)$.

Show answer

$12p^2 + 2pr$

EXAMPLE 4

Multiply: $-2y\left(4y^2 + 3y - 5\right)$.

Solution

	$-2y(4y^2 + 3y - 5)$
Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Simplify.	$-8y^3 - 6y^2 + 10y$

TRY IT 4.1

Multiply: $-3y\left(5y^2 + 8y - 7\right)$.

Show answer

$-15y^3 - 24y^2 + 21y$

TRY IT 4.2

Multiply: $4x^2\left(2x^2 - 3x + 5\right)$.

Show answer

$8x^4 - 24x^3 + 20x^2$

EXAMPLE 5

Multiply: $2x^3\left(x^2 - 8x + 1\right)$.

Solution

	$2x^3(x^2 - 8x + 1)$
Distribute.	$2x^3 \cdot x^2 + (2x^3) \cdot (-8x) + (2x^3) \cdot 1$
Simplify.	$2x^5 - 16x^4 + 2x^3$

TRY IT 5.1

Multiply: $4x\left(3x^2-5x+3\right)$.

Show answer

$12x^3-20x^2+12x$

TRY IT 5.2

Multiply: $-6a^3\left(3a^2-2a+6\right)$.

Show answer

$-18a^5 \div 12a^4-36a^3$

EXAMPLE 6

Multiply: $(x+3)\,p$.

Solution

The monomial is the second factor.	$(x+3)p$
Distribute.	$x \cdot p + 3 \cdot p$
Simplify.	$xp + 3p$

TRY IT 6.1

Multiply: $(x+8)\,p$.

Show answer

$xp+8p$

TRY IT 6.2

Multiply: $(a+4)\,p$.

Show answer

$ap+4p$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at the table below, where we multiplied a binomial by a monomial.

	$(x+3)p$
We distributed the *p* to get:	$xp+3p$
What if we have (*x* + 7) instead of *p*?	$(x+3)(x+7)$
Distribute (*x* + 7).	$x(x+7)+3(x+7)$
Distribute again.	$x^2+7x+3x+21$
Combine like terms.	$x^2+10x+21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications.

EXAMPLE 7

Multiply: $(y+5)\,(y+8)$.

Solution

	$(y+5)(y+8)$
Distribute (y + 8).	$y(y+8)+5(y+8)$
Distribute again	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

TRY IT 7.1

Multiply: $(x+8)(x+9)$.

Show answer

$x^2+17x+72$

TRY IT 7.2

Multiply: $(5x+9)(4x+3)$.

Show answer

$20x^2+51x+27$

EXAMPLE 8

Multiply: $(2y+5)(3y+4)$.

Solution

	$(2y+5)(3y+4)$
Distribute (3y + 4).	$2y(3y+4)+5(3y+4)$
Distribute again	$6y^2+8y+15y+20$
Combine like terms.	$6y^2+23y+20$

TRY IT 8.1

Multiply: $(3b+5)\,(4b+6)$.

Show answer

$12b^2+38b+30$

TRY IT 8.2

Multiply: $(a+10)\,(a+7)$.

Show answer

$a^2+17a+70$

EXAMPLE 9

Multiply: $(4y+3)\,(2y-5)$.

Solution

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

TRY IT 9.1

Multiply: $(5y+2)(6y-3)$.

Show answer
$30y^2-3y-6$

TRY IT 9.2

Multiply: $(3c+4)(5c-2)$.

Show answer
$15c^2+14c-8$

EXAMPLE 10

Multiply: $(x+2)(x-y)$.

Solution

	$(x-2)(x-y)$
Distribute.	$x(x-y)-2(x-y)$
Distribute again.	$x^2-xy-2x+2y$
There are no like terms to combine.	

TRY IT 10.1

Multiply: $(a+7)(a-b)$.

Show answer
$a^2 - ab + 7a - 7b$

TRY IT 10.2

Multiply: $(x+5)(x-y)$.

Show answer
$x^2 - xy + 5x - 5y$

Multiply a Binomial by a Binomial Using the FOIL Method

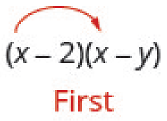
Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes, like in the above example, there are no like terms to combine.

Let’s look at the last example again and pay particular attention to how we got the four terms.

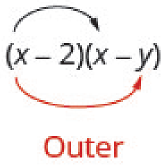
$$(x-2)(x-y)$$
$$x^2 - xy - 2x + 2y$$

Where did the first term, x^2, come from?

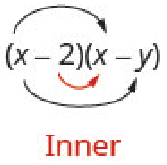
It is the product of x and x, the *first* terms in $(x - 2)$ and $(x - y)$.



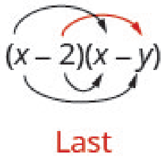
The next term, $-xy$, is the product of x and $-y$, the two *outer* terms.



The third term, $-2x$, is the product of -2 and x, the two *inner* terms.



And the last term, $+2y$, came from multiplying the two *last* terms, -2 and $-y$.

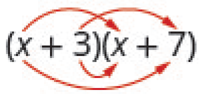


We abbreviate “First, Outer, Inner, Last” as FOIL. The letters stand for ‘**F**irst, **O**uter, **I**nner, **L**ast’. The word FOIL is easy to remember and ensures we find all four products.

$$(x - 2)(x - y)$$

$$\begin{array}{cccc} x^2 & -\,xy & -\,2x & +\,2y \\ F & O & I & L \end{array}$$

Let’s look at $(x + 3)\,(x + 7)$.

Distibutive Property	FOIL
$(x + 3)(x + 7)$	
$x(x + 7) + 3(x + 7)$	
$x^2 + 7x + 3x + 21$ $F \quad O \quad I \quad L$	$x^2 + 7x + 3x + 21$ $F \quad O \quad I \quad L$
$x^2 + 10x + 21$	$x^2 + 10x + 21$

Notice how the terms in third line fit the FOIL pattern.

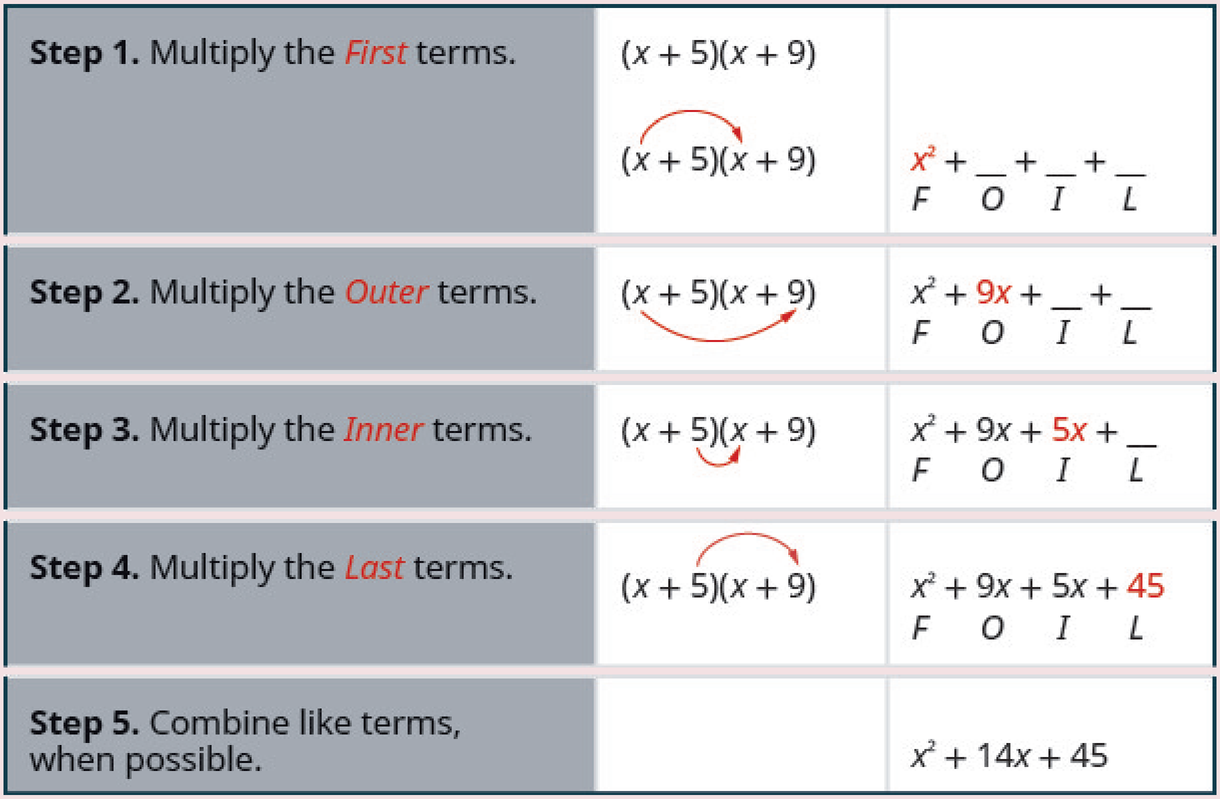
Now we will do an example where we use the FOIL pattern to multiply two binomials.

EXAMPLE 11

How to Multiply a Binomial by a Binomial using the FOIL Method

Multiply using the FOIL method: $(x+5)\,(x+9)$.

Solution



Step 1. Multiply the *First* terms.	$(x+5)(x+9)$ $(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$
Step 2. Multiply the *Outer* terms.	$(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{9x} + \underset{I}{\_} + \underset{L}{\_}$
Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{9x} + \underset{I}{5x} + \underset{L}{\_}$
Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{9x} + \underset{I}{5x} + \underset{L}{45}$
Step 5. Combine like terms, when possible.		$x^2 + 14x + 45$

TRY IT 11.1

Multiply using the FOIL method: $(x+6)\,(x+8)$.

Show answer

$x^2 + 14x + 48$

TRY IT 11.2

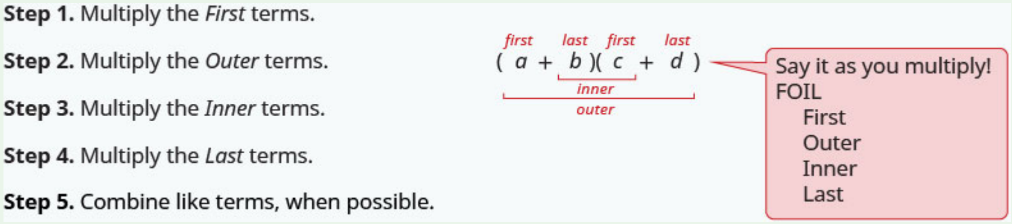
Multiply using the FOIL method: $(y+17)(y+3)$.

Show answer

$y^2+20y+51$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

HOW TO: Multiply two binomials using the FOIL method



Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

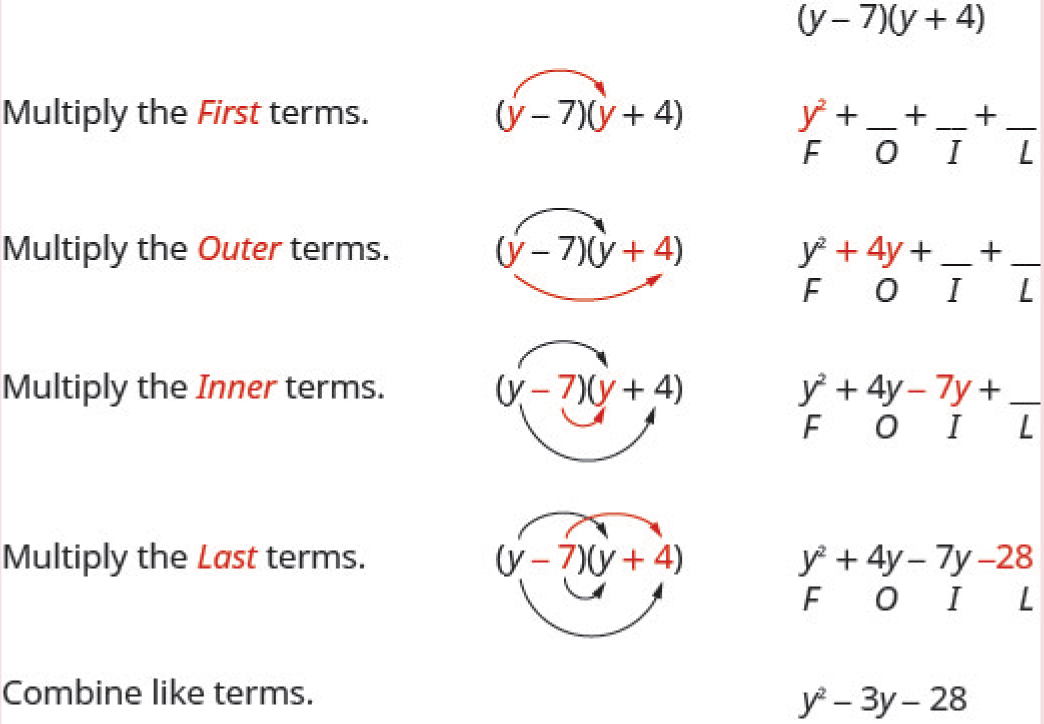
Step 5. Combine like terms, when possible.

When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

EXAMPLE 12

Multiply: $(y-7)(y+4)$.

Solution



		$(y-7)(y+4)$
Multiply the *First* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} \underset{I}{- 7y} + \underset{L}{\_\_}$
Multiply the *Last* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} \underset{I}{- 7y} \underset{L}{- 28}$
Combine like terms.		$y^2 - 3y - 28$

TRY IT 12.1

Multiply: $(x-7)(x+5)$.

Show answer

$x^2 - 2x - 35$

TRY IT 12.2

Multiply: $(b-3)(b+6)$.

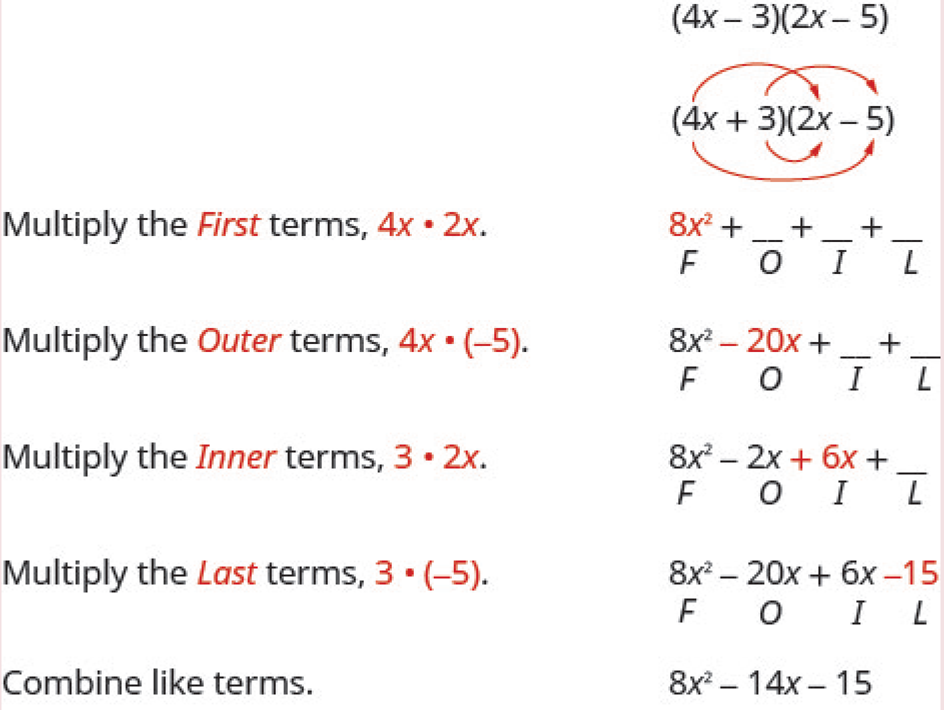
Show answer

$b^2 + 3b - 18$

EXAMPLE 13

Multiply: $(4x+3)(2x-5)$.

Solution



	$(4x - 3)(2x - 5)$
	$(4x + 3)(2x - 5)$
Multiply the *First* terms, $4x \cdot 2x$.	$\underset{F}{8x^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer* terms, $4x \cdot (-5)$.	$\underset{F}{8x^2} \underset{O}{- 20x} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner* terms, $3 \cdot 2x$.	$\underset{F}{8x^2} \underset{O}{- 2x} \underset{I}{+ 6x} + \underset{L}{\_\_}$
Multiply the *Last* terms, $3 \cdot (-5)$.	$\underset{F}{8x^2} \underset{O}{- 20x} \underset{I}{+ 6x} \underset{L}{-15}$
Combine like terms.	$8x^2 - 14x - 15$

TRY IT 13.1

Multiply: $(3x + 7)(5x - 2)$.

Show answer

$15x^2 + 29x - 14$

TRY IT 13.2

Multiply: $(4y + 5)(4y - 10)$.

Show answer

$16y^2 - 20y - 50$

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

EXAMPLE 14

Multiply: $(3x - y)(2x - 5)$.

Solution

	$(3x - y)(2x - 5)$
	 $(3x - y)(2x - 5)$
Multiply the *First*.	$\underset{F}{6x^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{6x^2} \underset{O}{- 15x} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{6x^2} \underset{O}{- 15x} \underset{I}{- 2xy} + \underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{6x^2} \underset{O}{- 15x} \underset{I}{- 2xy} \underset{L}{+ 5y}$
Combine like terms—there are none.	$6x^2 - 15x - 2xy + 5y$

TRY IT 14.1

Multiply: $(10c - d)(c - 6)$.

Show answer

$10c^2 - 60c - cd + 6d$

TRY IT 14.2

Multiply: $(7x - y)(2x - 5)$.

Show answer

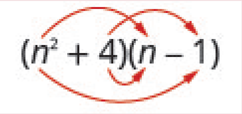
$14x^2 - 35x - 2xy + 10y$

Be careful of the exponents in the next example.

EXAMPLE 15

Multiply: $\left(n^2+4\right)(n-1)$.

Solution

	$(n^2+4)(n-1)$
	
Multiply the *First*.	$\underset{F}{n^3}+\underset{O}{\_\_}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{n^3}\underset{O}{-n^2}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{n^3}\underset{O}{-n^2}\underset{I}{+4n}+\underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{n^3}\underset{O}{-n^2}\underset{I}{+4n}\underset{L}{-4}$
Combine like terms—there are none.	n^3-n^2+4n-4

TRY IT 15.1

Multiply: $\left(x^2+6\right)(x-8)$.

Show answer

$x^3-8x^2+6x-48$

TRY IT 15.2

Multiply: $\left(y^2+7\right)(y-9)$.

Show answer

$y^3-9y^2+7y-63$

EXAMPLE 16

Multiply: $(3pq+5)(6pq-11)$.

Solution

	$(3pq+5)(6pq-11)$	
Multiply the *First*.	$\underset{F}{18p^2q^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$	
Multiply the *Outer*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} + \underset{I}{\_\_} + \underset{L}{\_\_}$	
Multiply the *Inner*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} + \underset{L}{\_\_}$	
Multiply the *Last*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} \underset{L}{- 55}$	
Combine like terms—there are none.	$18p^2q^2 - 3pq - 55$	

TRY IT 16.1

Multiply: $(2ab+5)(4ab-4)$.

Show answer

$8a^2b^2+12ab-20$

TRY IT 16.2

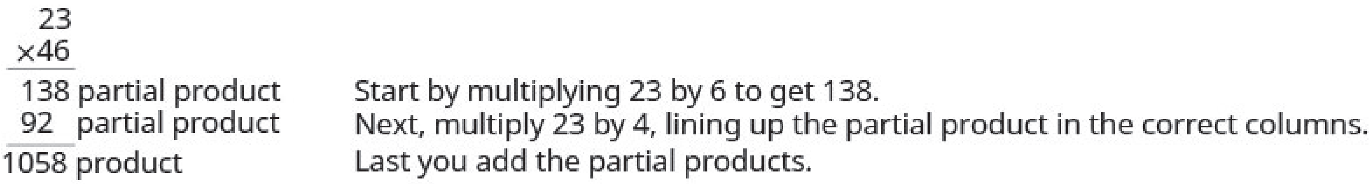
Multiply: $(2xy+3)(4xy-5)$.

Show answer

$8x^2y^2+2xy-15$

Multiply a Binomial by a Binomial Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.



$\begin{array}{r} 23 \\ \times 46 \\ \hline 138 \end{array}$ partial product	Start by multiplying 23 by 6 to get 138.
92 partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058 product	Last you add the partial products.

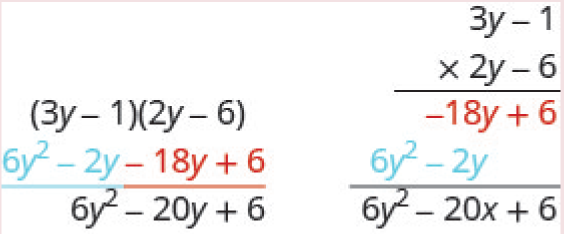
Now we'll apply this same method to multiply two binomials.

EXAMPLE 17

Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Solution

It does not matter which binomial goes on the top.

Multiply $3y-1$ by -6	Partial Product $-18y+6$	 $(3y-1)(2y-6)$ $6y^2-2y-18y+6$ $6y^2-20y+6$ $3y-1$ $\times\ 2y-6$ $-18y+6$ $6y^2-2y$ $6y^2-20x+6$
Multiple $3y-1$ by $2y$	Partial Product $6y^2-2y$	
Add like terms.	Product $6y^2-20y+6$	Notice the partial products are the same as the terms in the FOIL method.

TRY IT 17.1

Multiply using the Vertical Method: $(5m-7)(3m-6)$.

Show answer

$15m^2 - 51m + 42$

TRY IT 17.2

Multiply using the Vertical Method: $(6b - 5)(7b - 3)$.

Show answer

$42b^2 - 53b + 15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

HOW TO: Multiplying Two Binomials

To multiply binomials, use the: To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

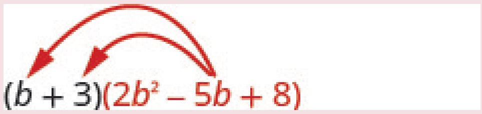
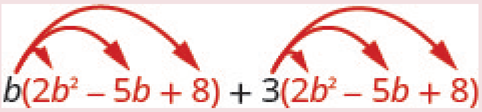
Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

EXAMPLE 18

Multiply using the Distributive Property: $(b + 3)\left(2b^2 - 5b + 8\right)$.

Solution

	 $(b+3)(2b^2-5b+8)$
Distribute.	 $b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

TRY IT 18.1

Multiply using the Distributive Property: $(y-3)\left(y^2-5y+2\right)$.

Show answer

$y^3-8y^2+17y-6$

TRY IT 18.2

Multiply using the Distributive Property: $(x+4)\left(2x^2-3x+5\right)$.

Show answer

$2x^3+5x^2-7x+20$

Now let's do this same multiplication using the Vertical Method.

EXAMPLE 19

Multiply using the Vertical Method: $(b+3)\left(2b^2-5b+8\right)$.

Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2-5b+8)$ by 3.	$2b^2-5b+8$ $\times \quad b+3$ $6b^2-15b+24$
	$2b^3-5b^2+8b$
Multiply $(2b^2-5b+8)$ by b.	$2b^3+b^2-7b+24$
Add like terms.	

TRY IT 19.1

Multiply using the Vertical Method: $(y-3)\left(y^2-5y+2\right)$.

Show answer

$y^3-8y^2+17y-6$

TRY IT 19.2

Multiply using the Vertical Method: $(x+4)\left(2x^2-3x+5\right)$.

Show answer

$2x^3+5x^2-7x+20$

We have now seen two methods you can use to multiply a trinomial by a binomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

HOW TO: Multiply a Trinomial by a Binomial

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1
- Multiplying Exponents 2
- Multiplying Exponents 3

Key Concepts

- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:
    1. Multiply the **First** terms.
    2. Multiply the **Outer** terms.
    3. Multiply the **Inner** terms.
    4. Multiply the **Last** terms.
- **Multiplying Two Binomials**—To multiply binomials, use the:
    - Distributive Property ((Figure))
    - FOIL Method ((Figure))
- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
    - Distributive Property ((Figure))

Practice Makes Perfect

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

1. $4\left(w+10\right)$	2. $6\left(b+8\right)$
3. $-3\left(a+7\right)$	4. $-5\left(p+9\right)$
5. $2\left(x-7\right)$	6. $7\left(y-4\right)$
7. $-3\left(k-4\right)$	8. $-8\left(j-5\right)$
9. $q\left(q+5\right)$	10. $k\left(k+7\right)$
11. $-b\left(b+9\right)$	12. $-y\left(y+3\right)$
13. $-x\left(x-10\right)$	14. $-p\left(p-15\right)$
15. $6r\left(4r+s\right)$	16. $5c\left(9c+d\right)$
17. $12x\left(x-10\right)$	18. $9m\left(m-11\right)$
19. $-9a\left(3a+5\right)$	20. $-4p\left(2p+7\right)$
21. $3\left(p^2+10p+25\right)$	22. $6\left(y^2+8y+16\right)$
23. $-8x\left(x^2+2x-15\right)$	24. $-5t\left(t^2+3t-18\right)$
25. $5q^3\left(q^3-2q+6\right)$	26. $4x^3\left(x^4-3x+7\right)$
27. $-8y\left(y^2+2y-15\right)$	28. $-5m\left(m^2+3m-18\right)$
29. $5q^3\left(q^2-2q+6\right)$	30. $9r^3\left(r^2-3r+5\right)$
31. $-4z^2\left(3z^2+12z-1\right)$	32. $-3x^2\left(7x^2+10x-1\right)$
33. $\left(2m-9\right)m$	34. $\left(8j-1\right)j$
35. $\left(w-6\right)\cdot 8$	36. $\left(k-4\right)\cdot 5$
37. $4\left(x+10\right)$	38. $6\left(y+8\right)$
39. $15\left(r-24\right)$	40. $12\left(v-30\right)$
41. $-3\left(m+11\right)$	42. $-4\left(p+15\right)$
43. $-8\left(z-5\right)$	44. $-3\left(x-9\right)$
45. $u\left(u+5\right)$	46. $q\left(q+7\right)$
47. $n\left(n^2-3n\right)$	48. $s\left(s^2-6s\right)$
49. $6x\left(4x+y\right)$	50. $5a\left(9a+b\right)$
51. $5p\left(11p-5q\right)$	52. $12u\left(3u-4v\right)$
53. $3\left(v^2+10v+25\right)$	54. $6\left(x^2+8x+16\right)$

55. $2n\left(4n^2-4n+1\right)$	56. $3r\left(2r^2-6r+2\right)$
57. $-8y\left(y^2+2y-15\right)$	58. $-5m\left(m^2+3m-18\right)$
59. $5q^3\left(q^2-2q+6\right)$	60. $9r^3\left(r^2-3r+5\right)$
61. $-4z^2\left(3z^2+12z-1\right)$	62. $-3x^2\left(7x^2+10x-1\right)$
63. $(2y-9)\,y$	64. $(8b-1)\,b$

Multiply a Binomial by a Binomial

In the following exercises, multiply the following binomials using: a) the Distributive Property b) the FOIL method c) the Vertical Method.

65. $(w+5)\,(w+7)$	66. $(y+9)\,(y+3)$
67. $(p+11)\,(p-4)$	68. $(q+4)\,(q-8)$

In the following exercises, multiply the binomials. Use any method.

69. $(x+8)\,(x+3)$	70. $(y+7)\,(y+4)$
71. $(y-6)\,(y-2)$	72. $(x-7)\,(x-2)$
73. $(w-4)\,(w+7)$	74. $(q-5)\,(q+8)$
75. $(p+12)\,(p-5)$	76. $(m+11)\,(m-4)$
77. $(6p+5)\,(p+1)$	78. $(7m+1)\,(m+3)$
79. $(2t-9)\,(10t+1)$	80. $(3r-8)\,(11r+1)$
81. $(5x-y)\,(3x-6)$	82. $(10a-b)\,(3a-4)$
83. $(a+b)\,(2a+3b)$	84. $(r+s)\,(3r+2s)$
85. $(4z-y)\,(z-6)$	86. $(5x-y)\,(x-4)$
87. $\left(x^2+3\right)(x+2)$	88. $\left(y^2-4\right)(y+3)$
89. $\left(x^2+8\right)\left(x^2-5\right)$	90. $\left(y^2-7\right)\left(y^2-4\right)$
91. $(5ab-1)\,(2ab+3)$	92. $(2xy+3)\,(3xy+2)$
93. $(6pq-3)\,(4pq-5)$	94. $(3rs-7)\,(3rs-4)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using a) the Distributive Property b) the Vertical Method.

95. $(x+5)\left(x^2+4x+3\right)$	96. $(u+4)\left(u^2+3u+2\right)$
97. $(y+8)\left(4y^2+y-7\right)$	98. $(a+10)\left(3a^2+a-5\right)$

In the following exercises, multiply. Use either method.

99. $(w-7)\left(w^2-9w+10\right)$	100. $(p-4)\left(p^2-6p+9\right)$
101. $(3q+1)\left(q^2-4q-5\right)$	102. $(6r+1)\left(r^2-7r-9\right)$

Mixed Practice

103. $(10y-6)+(4y-7)$	104. $(15p-4)+(3p-5)$
105. $\left(x^2-4x-34\right)-\left(x^2+7x-6\right)$	106. $\left(j^2-8j-27\right)-\left(j^2+2j-12\right)$
107. $5q\left(3q^2-6q+11\right)$	108. $8t\left(2t^2-5t+6\right)$
109. $(s-7)(s+9)$	110. $(x-5)(x+13)$
111. $\left(y^2-2y\right)(y+1)$	112. $\left(a^2-3a\right)(4a+5)$
113. $(3n-4)\left(n^2+n-7\right)$	114. $(6k-1)\left(k^2+2k-4\right)$
115. $(7p+10)(7p-10)$	116. $(3y+8)(3y-8)$
117. $\left(4m^2-3m-7\right)m^2$	118. $\left(15c^2-4c+5\right)c^4$
119. $(5a+7b)(5a+7b)$	120. $(3x-11y)(3x-11y)$
121. $(4y+12z)(4y-12z)$	

Everyday Math

122. **Mental math** You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 13 times 15. Think of 13 as $10+3$ and 15 as $10+5$.

a. Multiply $(10+3)(10+5)$ by the FOIL method.
b. Multiply $13 \cdot 15$ without using a calculator.
c. Which way is easier for you? Why?

123. **Mental math** You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 18 times 17. Think of 18 as $20-2$ and 17 as $20-3$.

a. Multiply $(20-2)(20-3)$ by the FOIL method.
b. Multiply $18 \cdot 17$ without using a calculator.
c. Which way is easier for you? Why?

Writing Exercises

124. Which method do you prefer to use when multiplying two binomials: the Distributive Property, the FOIL method, or the Vertical Method? Why?

125. Which method do you prefer to use when multiplying a trinomial by a binomial: the Distributive Property or the Vertical Method? Why?

126. Multiply the following:

$(x+2)(x-2)$
$(y+7)(y-7)$
$(w+5)(w-5)$

Explain the pattern that you see in your answers.

127. Multiply the following:

$(m-3)(m+3)$
$(n-10)(n+10)$
$(p-8)(p+8)$

Explain the pattern that you see in your answers.

128. Multiply the following:

$(p+3)(p+3)$
$(q+6)(q+6)$
$(r+1)(r+1)$

Explain the pattern that you see in your answers.

129. Multiply the following:

$(x-4)(x-4)$
$(y-1)(y-1)$
$(z-7)(z-7)$

Explain the pattern that you see in your answers.

Answers

1. $4w + 40$	3. $-3a - 21$
5. $2x - 14$	7. $-3k + 12$
9. $q^2 + 5q$	11. -$b^2 - 9b$
13. -$x^2 + 10x$	15. $24r^2 + 6rs$
17. $12x^2 - 120x$	19. $-27a^2 - 45a$
21. $3p^2 + 30p + 75$	23. $-8x^3 - 16x^2 + 120x$
25. $5q^6 - 10q^4 + 30q^3$	27. $-8y^3 - 16y^2 + 120y$
29. $5q^5 - 10q^4 + 30q^3$	31. $-12z^4 - 48z^3 + 4z^2$
33. $2m^2 - 9m$	35. $8w - 48$
37. $4x + 40$	39. $15r - 360$
41. $-3m - 33$	43. $-8z + 40$
45. $u^2 + 5u$	47. $n^3 - 3n^2$
49. $24x^2 + 6xy$	51. $55p^2 - 25pq$
53. $3v^2 + 30v + 75$	55. $8n^3 - 8n^2 + 2n$
57. $-8y^3 - 16y^2 + 120y$	59. $5q^5 - 10q^4 + 30q^3$
61. $-12z^4 - 48z^3 + 4z^2$	63. $2y^2 - 9y$
65. $w^2 + 12w + 35$	67. $p^2 + 7p - 44$
69. $x^2 + 11x + 24$	71. $y^2 - 8y + 12$
73. $w^2 + 3w - 28$	75. $p^2 + 7p - 60$
77. $6p^2 + 11p + 5$	79. $20t^2 - 88t - 9$
81. $15x^2 - 3xy - 30x + 6y$	83. $2a^2 + 5ab + 3b^2$
85. $4z^2 - 24z - zy + 6y$	87. $x^3 + 2x^2 + 3x + 6$
89. $x^4 + 3x^2 - 40$	91. $10a^2b^2 + 13ab - 3$
93. $24p^2q^2 - 42pq + 15$	95. $x^3 + 9x^2 + 23x + 15$
97. $4y^3 + 33y^2 + y - 56$	99. $w^3 - 16w^2 + 73w - 70$
101. $3q^3 - 11q^2 - 19q - 5$	103. $14y - 13$
105. $-11x - 28$	107. $15q^3 - 30q^2 + 55q$
109. $s^2 + 2s - 63$	111. $y^3 - y^2 - 2y$

113. $3n^3 - n^2 - 25n + 28$	115. $49p^2 - 100$
117. $4m^4 - 3m^3 - 7m^2$	119. $25a^2 + 70ab + 49b^2$
121. $16y^2 - 144z^2$	123. a) 306 b) 306 c) Answers will vary.
125. Answers will vary.	127. Answers will vary.
129. Answers will vary.	

Attributions

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8.3 Special Products

Learning Objectives

By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$.	
What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$

Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$

And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x^2+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example we squared a binomial and the result was a trinomial.
$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_$

Now look at the ***first term*** in each result. Where did it come from?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!
$(a+b)^2 = a^2 + \_\_\_\_ + \_\_\_\_$

To get the ***first term*** of the product, ***square the first term***.

Where did the ***last term*** come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.
$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + b^2$

*To get the **last term** of the product, **square the last term**.*

Finally, look at the **middle term**. Notice it came from adding the “outer” and the “inner” terms—which are both the same! So the middle term is double the product of the two terms of the binomial.
$(a+b)^2 = \_\_\_\_ + 2ab + \_\_\_\_$
$(a-b)^2 = \_\_\_\_ - 2ab + \_\_\_\_$

*To get the **middle term** of the product, **multiply the terms and double their product**.*

Putting it all together:
Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$$

HOW TO:

To square a binomial:

- square the first term
- square the last term
- double their product

A number example helps verify the pattern.

	$(10+4)^2$
Square the first term.	$10^2 + \_\_\_\_ +$
Square the last term.	$10^2 + \_\_\_\_ + 4^2$
Double their product.	$10^2 + 2 \cdot 10 \cdot 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you’d follow the Order of Operations.

$$(10+4)^2$$
$$(14)^2$$
$$196$$

The pattern works!

EXAMPLE 1

Multiply: $(x+5)^2$.

Solution

	$\left(\overset{a+b}{x+5}\right)^2$
Square the first term.	$\overset{a^2}{x^2} \overset{+}{+} \overset{2ab}{\_\_\_\_} \overset{+}{+} \overset{b^2}{\_\_\_\_}$
Square the last term.	$\overset{a^2}{x^2} \overset{+}{+} \overset{2ab}{\_\_\_\_} \overset{+}{+} \overset{b^2}{5^2}$
Double the product.	$\overset{a^2}{x^2} \overset{+}{+} \overset{2}{2} \overset{\cdot}{\cdot} \overset{a}{x} \overset{\cdot}{\cdot} \overset{b}{5} \overset{+}{+} \overset{b^2}{5^2}$
Simplify.	$x^2 + 10x + 25$

TRY IT 1.1

Multiply: $(x + 9)^2$.

Show answer

$x^2 + 18x + 81$

TRY IT 1.2

Multiply: $(y + 11)^2$.

Show answer

$y^2 + 22y + 121$

EXAMPLE 2

Multiply: $(y - 3)^2$.

Solution

	$(a - b)^2$ $(y - 3)^2$
Square the first term.	$a^2 - 2ab + b^2$ $y^2 - \_\_\_\_ + \_\_\_\_$
Square the last term.	$a^2 - 2ab + b^2$ $y^2 - \_\_\_\_ + 3^2$
Double the product.	$a^2 - 2 \cdot a \cdot b + b^2$ $y^2 - 2 \cdot y \cdot 3 + 3^2$
Simplify.	$y^2 - 6y + 9$

TRY IT 2.1

Multiply: $(x-9)^2$.

Show answer

$x^2 - 18x + 81$

TRY IT 2.2

Multiply: $(p-13)^2$.

Show answer

$p^2 - 26p + 169$

EXAMPLE 3

Multiply: $(4x+6)^2$.

Solution

	$\left(\overset{a}{4x}\overset{+}{+}\overset{b}{6}\right)^2$
Use the pattern.	$\overset{a^2}{(4x)^2}\overset{+\ 2\ \cdot\ a\ \cdot\ b}{+\,2\cdot 4x\cdot 6}\overset{+\ b^2}{+\,6^2}$
Simplify.	$16x^2+48x+36$

TRY IT 3.1

Multiply: $(6x+3)^2$.

Show answer
$36x^2+36x+9$

TRY IT 3.2

Multiply: $(4x+9)^2$.

Show answer
$16x^2+72x+81$

EXAMPLE 4

Multiply: $(2x-3y)^2$.

Solution

	$\left(\overset{a}{2x}\overset{-}{-}\overset{b}{3y}\right)^2$
Use the pattern.	$\overset{a^2}{(2x)^2}\overset{-\ 2\ \cdot\ a\ \cdot\ b}{-\,2\cdot 2x\cdot 3y}\overset{+\ b^2}{+\,(3y)^2}$
Simplify.	$4x^2-12xy+9y^2$

TRY IT 4.1

Multiply: $(2c-d)^2$.

Show answer

$4c^2-4cd+d^2$

TRY IT 4.2

Multiply: $(4x-5y)^2$.

Show answer

$16x^2-40xy+25y^2$

EXAMPLE 5

Multiply: $\left(4u^3+1\right)^2$.

Solution

	$\left(\overset{a}{4u^3}+\overset{b}{1}\right)^2$
Use the pattern.	$\overset{a^2}{(4u^3)^2}+\overset{2\cdot}{2\cdot}\overset{a}{4u^3}\overset{\cdot b}{\cdot 1}+\overset{b^2}{(1)^2}$
Simplify.	$16u^6+8u^3+1$

TRY IT 5.1

Multiply: $\left(2x^2+1\right)^2$.

Show answer

$4x^4+4x^2+1$

TRY IT 5.2

Multiply: $\left(3y^3+2\right)^2$.

Show answer
$9y^6+12y^3+4$

Multiply Conjugates Using the Product of Conjugates Pattern

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?
$(x-9)\,(x+9)$ $(y-8)\,(y+8)$ $(2x-5)\,(2x+5)$

Look at the first term of each binomial in each pair.

$(x-9)(x+9)$ $(y-8)(y+8)$ $(2x-5)(2x+5)$

Notice the first terms are the same in each pair.

Look at the last terms of each binomial in each pair.

$(x-9)(x+9)$ $(y-8)(y+8)$ $(2x-5)(2x+5)$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.

$$\left(\underset{\text{Difference}}{\underset{\uparrow}{x-9}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{x+9}}\right)\quad\left(\underset{\text{Difference}}{\underset{\uparrow}{y-8}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{y+8}}\right)\quad\left(\underset{\text{Difference}}{\underset{\uparrow}{2x-5}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{2x+5}}\right)$$

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a *conjugate pair* and is of the form $(a-b)\,,(a+b)$.

Conjugate Pair

A conjugate pair is two binomials of the form
$(a-b)\,,(a+b)$.

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let’s look for the pattern by using FOIL to multiply some conjugate pairs.

$$\begin{array}{ccc} (x-9)(x+9) & (y-8)(y+8) & (2x-5)(2x+5) \\ x^2+9x-9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

$$\begin{array}{ccc} (x+9)(x-9) & (y-8)(y+8) & (2x-5)(2x+5) \\ x^2-9x+9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

Each **first term** is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\_\_\_\_$$

To get the**first term, square the first term**.

The **last term** came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the**last term, square the last term**.

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

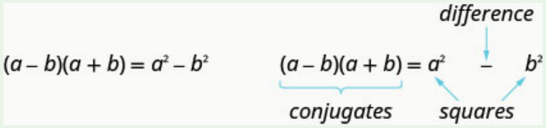
Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form a^2-b^2. This is called a difference of squares.

This leads to the pattern:

Product of Conjugates Pattern

If a and b are real numbers,



The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let’s test this pattern with a numerical example.

	$(10-2)(10+2)$
It is the product of conjudgates, so the result will be the difference of two squares.	____ – ____
Square the first term.	$10^2 -$ ____
Square the last term.	$10^2 - 2^2$
Simplify.	$100 - 4$
Simplify.	96
What do you get using the order of operations?	
	$(10-2)(10+2)$ $(8)(12)$ 96

Notice, the result is the same!

EXAMPLE 6

Multiply: $(x-8)(x+8)$.

Solution
First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.	$\left(\overset{a}{x} - \overset{b}{8}\right)\left(\overset{a}{x} + \overset{b}{8}\right)$
Square the first term, *x*.	$\overset{a^2}{x^2} - \overset{b^2}{\_\_}$
Square the last term, 8.	$\overset{a^2}{x^2} - \overset{b^2}{8^2}$
The product is a difference of squares.	$\overset{a^2}{x^2} - \overset{b^2}{64}$

TRY IT 6.1

Multiply: $(x - 5)(x + 5)$.

Show answer

$x^2 - 25$

TRY IT 6.2

Multiply: $(w - 3)(w + 3)$.

Show answer

$w^2 - 9$

EXAMPLE 7

Multiply: $(2x + 5)(2x - 5)$.

Solution

Are the binomials conjugates?

It is the product of conjugates.	$\left(\overset{a}{2x}+\overset{b}{5}\right)\left(\overset{a}{2x}-\overset{b}{5}\right)$
Square the first term, 2*x*.	$\overset{a^2}{(2x)^2}-\overset{b^2}{\_\_}$
Square the last term, 5.	$\overset{a^2}{(2x)^2}-\overset{b^2}{5^2}$
Simplify. The product is a difference of squares.	$\overset{a^2}{4x^2}-\overset{b^2}{25}$

TRY IT 7.1

Multiply: $\left(6x+5\right)\left(6x-5\right)$.

Show answer

$36x^2-25$

TRY IT 7.2

Multiply: $\left(2x+7\right)\left(2x-7\right)$.

Show answer

$4x^2-49$

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

EXAMPLE 8

Find the product: $\left(3+5x\right)\left(3-5x\right)$.

Solution

It is the product of conjugates.	$(\overset{a}{3} \overset{-}{+} \overset{b}{5x})(\overset{a}{3} \overset{+}{-} \overset{b}{5x})$
Use the pattern.	$\overset{a^2}{3^2} \overset{-}{-} \overset{b^2}{(5x)^2}$
Simplify.	$9 - 25x^2$

TRY IT 8.1

Multiply: $(7+4x)(7-4x)$.

Show answer

$49-16x^2$

TRY IT 8.2

Multiply: $(9-2y)(9+2y)$.

Show answer

$81-4y^2$

Now we'll multiply conjugates that have two variables.

EXAMPLE 9

Find the product: $(5m-9n)(5m+9n)$.

Solution

This fits the pattern.	$\left(\overset{a}{5m} \overset{-}{-} \overset{b}{9n}\right)\left(\overset{a}{5m} \overset{+}{+} \overset{b}{9n}\right)$
Use the pattern.	$\overset{a^2}{(5m)^2} \overset{-}{-} \overset{b^2}{(9n)^2}$
Simplify.	$25m^2 - 81n^2$

TRY IT 9.1

Find the product: $(4p-7q)\,(4p+7q)$.

Show answer
$16p^2 - 49q^2$

TRY IT 9.2

Find the product: $(3x-y)\,(3x+y)$.

Show answer
$9x^2 - y^2$

EXAMPLE 10

Find the product: $(cd-8)\,(cd+8)$.

Solution

This fits the pattern.	$\left(\overset{a}{cd} \overset{-}{-} \overset{b}{8}\right)\left(\overset{a}{cd} \overset{+}{+} \overset{b}{8}\right)$
Use the pattern.	$\overset{a^2}{(cd)^2} \overset{-}{-} \overset{b^2}{(8)^2}$
Simplify.	$c^2d^2 - 64$

TRY IT 10.1

Find the product: $(xy-6)(xy+6)$.

Show answer

x^2y^2-36

TRY IT 10.2

Find the product: $(ab-9)(ab+9)$.

Show answer

a^2b^2-81

EXAMPLE 11

Find the product: $\left(6u^2-11v^5\right)\left(6u^2+11v^5\right)$.

Solution

This fits the pattern.	$\left(\overset{a}{6u^2}\overset{-}{-}\overset{b}{11v^5}\right)\left(\overset{a}{6u^2}\overset{+}{+}\overset{b}{11v^5}\right)$
Use the pattern.	$\overset{a^2}{(6u^2)^2}\overset{-}{-}\overset{b^2}{(11v^5)^2}$
Simplify.	$36u^4-121v^{10}$

TRY IT 11.1

Find the product: $\left(3x^2-4y^3\right)\left(3x^2+4y^3\right)$.

Show answer

$9x^4-16y^6$

TRY IT 11.2

Find the product: $\left(2m^2-5n^3\right)\left(2m^2+5n^3\right)$.

Show answer

$4m^4-25n^6$

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.
Comparing the Special Product Patterns

Binomial Squares	**Product of Conjugates**
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	
– Squaring a binomial	– Multiplying conjugates
– Product is a **trinomial**	– Product is a **binomial**
– Inner and outer terms with FOIL are **the same.**	– Inner and outer terms with FOIL are **opposites.**
– Middle term is **double the product** of the terms.	– There is **no** middle term.

EXAMPLE 12

Choose the appropriate pattern and use it to find the product:

a) $(2x-3)(2x+3)$ b) $(5x-8)^2$ c) $(6m+7)^2$ d) $(5x-6)(6x+5)$

Solution

a. $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

This fits the pattern.	$\left(\overset{a}{2x}-\overset{b}{3}\right)\left(\overset{a}{2x}+\overset{b}{3}\right)$
Use the pattern.	$\overset{a^2}{(2x)^2}-\overset{b^2}{3^2}$
Simplify.	$4x^2-9$

b. $(8x-5)^2$ We are asked to square a binomial. It fits the **binomial squares** pattern.

	$\left(\overset{a}{8x}-\overset{b}{5}\right)^2$
Use the pattern.	$\overset{a^2}{(8x)^2}-\overset{2ab}{2\cdot 8x\cdot 5}+\overset{b^2}{5^2}$
Simplify.	$64x^2-80x+25$

c. $(6m+7)^2$ Again, we will square a binomial so we use the **binomial squares** pattern.

	$\left(\overset{a}{6m}+\overset{b}{7}\right)^2$
Use the pattern.	$\overset{a^2}{(6m)^2}+\overset{2ab}{2\cdot 6m\cdot 7}+\overset{b^2}{7^2}$
Simplify.	$36m^2+84m+49$

d. $(5x-6)(6x+5)$ This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

TRY IT 12.1

Choose the appropriate pattern and use it to find the product:

a) $(9b-2)(2b+9)$ b) $(9p-4)^2$ c) $(7y+1)^2$ d) $(4r-3)(4r+3)$

Show answer

a) FOIL; $18b^2 + 77b - 18$ b) Binomial Squares; $81p^2 - 72p + 16$ c) Binomial Squares; $49y^2 + 14y + 1$ d) Product of Conjugates; $16r^2 - 9$

TRY IT 12.2

Choose the appropriate pattern and use it to find the product:

a) $(6x + 7)^2$ b) $(3x - 4)(3x + 4)$ c) $(2x - 5)(5x - 2)$ d) $(6n - 1)^2$

Show answer

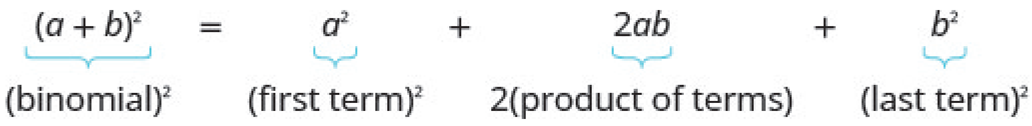
a) Binomial Squares; $36x^2 + 84x + 49$ b) Product of Conjugates; $9x^2 - 16$ c) FOIL; $10x^2 - 29x + 10$ d) Binomial Squares; $36n^2 - 12n + 1$

Access these online resources for additional instruction and practice with special products:

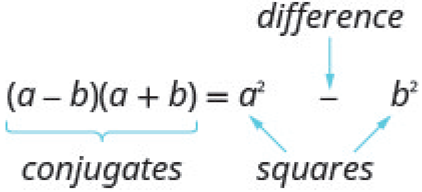
- Special Products

Key Concepts

- **Binomial Squares Pattern**
    - If a, b are real numbers,



    - $(a + b)^2 = a^2 + 2ab + b^2$
    - $(a - b)^2 = a^2 - 2ab + b^2$
    - To square a binomial: square the first term, square the last term, double their product.
- **Product of Conjugates Pattern**
    - If a, b are real numbers,



    - $(a - b)(a + b) = a^2 - b^2$
    - The product is called a difference of squares.
- **To multiply conjugates:**

- ◦ **square the first term square the last term** write it as a difference of squares

Glossary

conjugate pair
A conjugate pair is two binomials of the form $(a-b)$, $(a+b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

Practice Makes Perfect

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

1. $(q+12)^2$	2. $(w+4)^2$
3. $\left(x+\frac{2}{3}\right)^2$	4. $\left(y+\frac{1}{4}\right)^2$
5. $(y-6)^2$	6. $(b-7)^2$
7. $(p-13)^2$	8. $(m-15)^2$
9. $(4a+10)^2$	10. $(3d+1)^2$
11. $\left(3z+\frac{1}{5}\right)^2$	12. $\left(2q+\frac{1}{3}\right)^2$
13. $(2y-3z)^2$	14. $(3x-y)^2$
15. $\left(\frac{1}{8}x-\frac{1}{9}y\right)^2$	16. $\left(\frac{1}{5}x-\frac{1}{7}y\right)^2$
17. $\left(5u^2+9\right)^2$	18. $\left(3x^2+2\right)^2$
19. $\left(8p^3-3\right)^2$	20. $\left(4y^3-2\right)^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

Multiply Conjugates Using the Product of Conjugates Pattern

21. $(c-5)(c+5)$	22. $(m-7)(m+7)$
23. $\left(b+\frac{6}{7}\right)\left(b-\frac{6}{7}\right)$	24. $\left(x+\frac{3}{4}\right)\left(x-\frac{3}{4}\right)$
25. $(8j+4)(8j-4)$	26. $(5k+6)(5k-6)$
27. $(9c+5)(9c-5)$	28. $(11k+4)(11k-4)$
29. $(13-q)(13+q)$	30. $(11-b)(11+b)$
31. $(4-6y)(4+6y)$	32. $(5-3x)(5+3x)$
33. $(7w+10x)(7w-10x)$	34. $(9c-2d)(9c+2d)$
35. $\left(p+\frac{4}{5}q\right)\left(p-\frac{4}{5}q\right)$	36. $\left(m+\frac{2}{3}n\right)\left(m-\frac{2}{3}n\right)$
37. $(xy-9)(xy+9)$	38. $(ab-4)(ab+4)$
39. $\left(rs-\frac{2}{7}\right)\left(rs+\frac{2}{7}\right)$	40. $\left(uv-\frac{3}{5}\right)\left(uv+\frac{3}{5}\right)$
41. $\left(6m^3-4n^5\right)\left(6m^3+4n^5\right)$	42. $\left(2x^2-3y^4\right)\left(2x^2+3y^4\right)$
43. $\left(15m^2-8n^4\right)\left(15m^2+8n^4\right)$	44. $\left(12p^3-11q^2\right)\left(12p^3+11q^2\right)$

In the following exercises, find each product.

Recognize and Use the Appropriate Special Product Pattern

45.

a) $(2r+12)^2$

b) $(3p+8)(3p-8)$

c) $(7a+b)(a-7b)$

d) $(k-6)^2$

46.

a) $(p-3)(p+3)$

b) $(t-9)^2$

c) $(m+n)^2$

d) $(2x+y)(x-2y)$

47.

a) $\left(x^5+y^5\right)\left(x^5-y^5\right)$

b) $\left(m^3-8n\right)^2$

c) $(9p+8q)^2$

d) $\left(r^2-s^3\right)\left(r^3+s^2\right)$

48.

a) $\left(a^5-7b\right)^2$

b) $\left(x^2+8y\right)\left(8x-y^2\right)$

c) $\left(r^6+s^6\right)\left(r^6-s^6\right)$

d) $\left(y^4+2z\right)^2$

Everyday Math

49. **Mental math** You can use the binomial squares pattern to multiply numbers without a calculator. Say you need to square 65. Think of 65 as $60+5$.

a. Multiply $(60+5)^2$ by using the binomial squares pattern, $(a+b)^2=a^2+2ab+b^2$.

b. Square 65 without using a calculator.

c. Which way is easier for you? Why?

50. **Mental math** You can use the product of conjugates pattern to multiply numbers without a calculator. Say you need to multiply 47 times 53. Think of 47 as $50-3$ and 53 as $50+3$.

a. Multiply $(50-3)(50+3)$ by using the product of conjugates pattern, $(a-b)(a+b)=a^2-b^2$.

b. Multiply $47\cdot 53$ without using a calculator.

c. Which way is easier for you? Why?

Writing Exercises

52. Why does $(a+b)^2$ result in a trinomial, but $(a-b)\,(a+b)$ result in a binomial?	51. How do you decide which pattern to use?
54. Use the order of operations to show that $(3+5)^2$ is 64, and then use that numerical example to explain why $(a+b)^2 \neq a^2+b^2$.	53. Marta did the following work on her homework paper: $(3-y)^2$ 3^2-y^2 $9-y^2$ Explain what is wrong with Marta's work.

Answers

1. $q^2+24q+144$	3. $x^2+\frac{4}{3}x+\frac{4}{9}$
5. $y^2-12y+36$	7. $p^2-26p+169$
9. $16a^2+80a+100$	11. $9z^2+\frac{6}{5}z+\frac{1}{25}$
13. $4y^2-12yz+9z^2$	15. $\frac{1}{64}x^2-\frac{1}{36}xy+\frac{1}{81}y^2$
17. $25u^4+90u^2+81$	19. $64p^6-48p^3+9$
21. c^2-25	23. $b^2-\frac{36}{49}$
25. $64j^2-16$	27. $81c^2-25$
29. $169-q^2$	31. $16-36y^2$
33. $49w^2-100x^2$	35. $p^2-\frac{16}{25}q^2$
37. x^2y^2-81	39. $r^2s^2-\frac{4}{49}$
41. $36m^6-16n^{10}$	43. $225m^4-64n^8$
45. a) $4r^2+48r+144$ b) $9p^2-64$ c) $7a^2-48ab-7b^2$ d) $k^2-12k+36$	47. a) $x^{10}-y^{10}$ b) $m^6-16m^3n+64n^2$ c) $81p^2+144pq+64q^2$ d) $r^5+r^2s^2-r^3s^3-s^5$
49. a) 4,225 b) 4,225 c) Answers will vary.	51. Answers will vary.
53. Answers will vary.	

Attributions

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8.4 Greatest Common Factor and Factor by Grouping

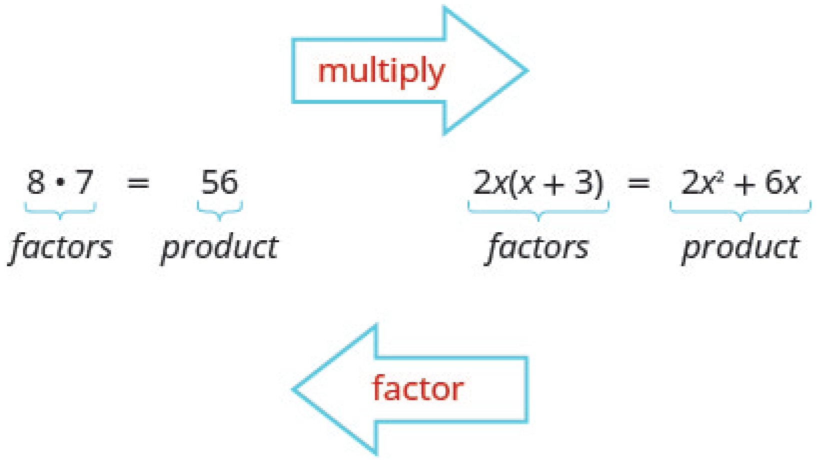
Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called factoring.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the greatest common factor of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

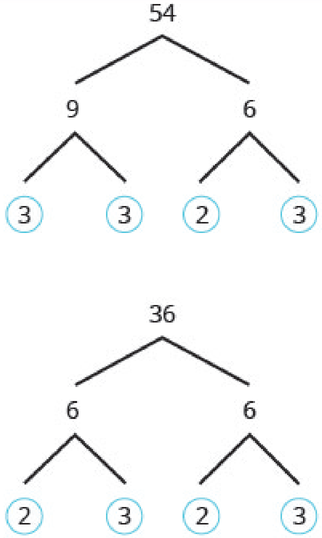
First we’ll find the GCF of two numbers.

EXAMPLE 1

How to Find the Greatest Common Factor of Two or More Expressions

Find the GCF of 54 and 36

Solution

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 54 and 36.	
Step 2. In each column, circle the common factors.	Circle the 2, 3, and 3 that are shared by both numbers.	$36 = 2 \cdot 2 \cdot 3 \cdot 3$ $18 = 2 \cdot 3 \cdot 3 \cdot 3$
Step 3. Bring down the common factors that all expressions share.	Bring down the 2, 3, and 3, and then multiply.	$\text{GCF} = 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		$\text{GCF} = 18$ The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18

$$54 = 18 \cdot 3$$

$$36 = 18 \cdot 2$$

TRY IT 1.1

Find the GCF of 48 and 80.

Show answer

16

TRY IT 1.2

Find the GCF of 18 and 40.

Show answer
2

We summarize the steps we use to find the GCF below.

HOW TO:

Find the Greatest Common Factor (GCF) of two expressions

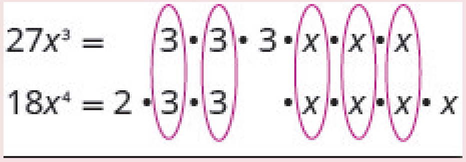
1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

EXAMPLE 2

Find the greatest common factor of $27x^3$ and $18x^4$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	
Bring down the common factors.	$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x$
Multiply the factors.	$\text{GCF} = 9x^3$
	The GCF of $27x^3$ and $18x^4$ is $9x^3$.

TRY IT 2.1

Find the GCF: $12x^2, 18x^3$.

Show answer

$3x^2$

TRY IT 2.2

Find the GCF: $16y^2, 24y^3$.

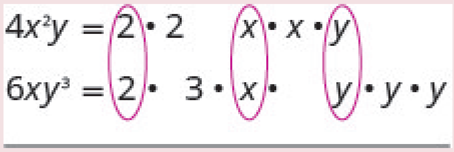
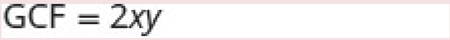
Show answer

$8y^2$

EXAMPLE 3

Find the GCF of $4x^2y, 6xy^3$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	 $4x^2y = 2 \cdot 2 \quad x \cdot x \cdot y$ $6xy^3 = 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$
Bring down the common factors.	 $\text{GCF} = 2 \cdot x \cdot y$
Multiply the factors.	 $\text{GCF} = 2xy$
	The GCF of $4x^2y$ and $6xy^3$ is 2xy.

TRY IT 3.1

Find the GCF: $6ab^4, 8a^2b$.

Show answer
$2ab$

TRY IT 3.2

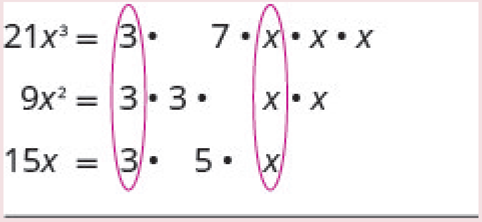
Find the GCF: $9m^5n^2, 12m^3n$.

Show answer
$3m^3n$

EXAMPLE 4

Find the GCF of: $21x^3, 9x^2, 15x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	 $21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

TRY IT 4.1

Find the greatest common factor: $25m^4, 35m^3, 20m^2$.

Show answer

$5m^2$

TRY IT 4.2

Find the greatest common factor: $14x^3, 70x^2, 105x$.

Show answer

$7x$

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2 \cdot 6$ or $3 \cdot 4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:

$$\begin{array}{ll} 2\left(x+7\right) & \text{factors} \\ 2 \cdot x + 2 \cdot 7 & \\ 2x + 14 & \text{product} \end{array}$$

Now we will start with a product, like $2x + 14$, and end with its factors, $2\left(x + 7\right)$. To do this we apply the Distributive Property “in reverse.”

We state the Distributive Property here just as you saw it in earlier chapters and “in reverse.”

Distributive Property

If a, b, c are real numbers, then

$a\left(b + c\right) = ab + ac$ and $ab + ac = a\left(b + c\right)$

The form on the left is used to multiply. The form on the right is used to factor.

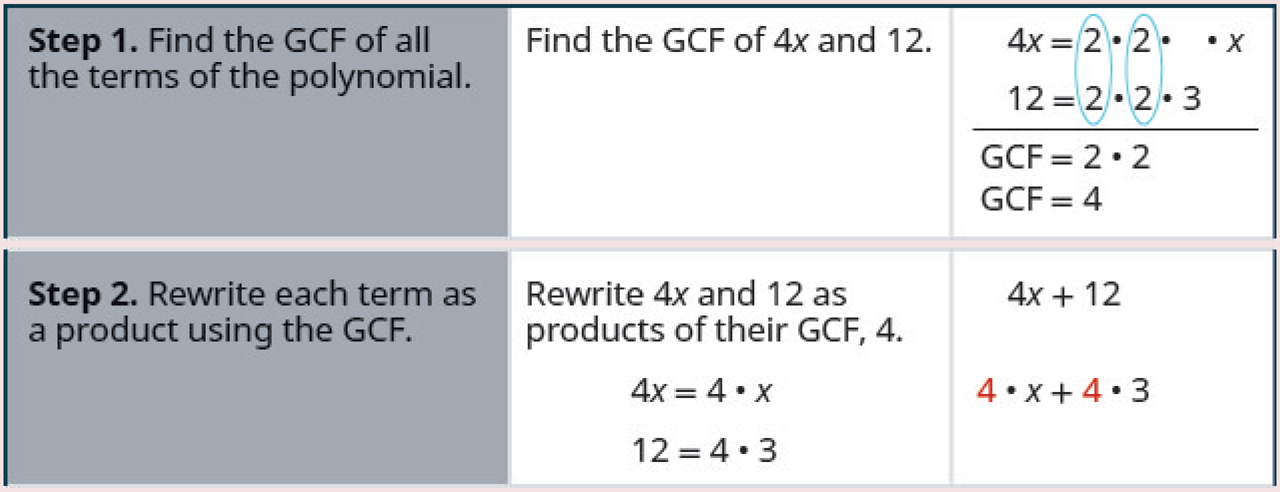
So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 5

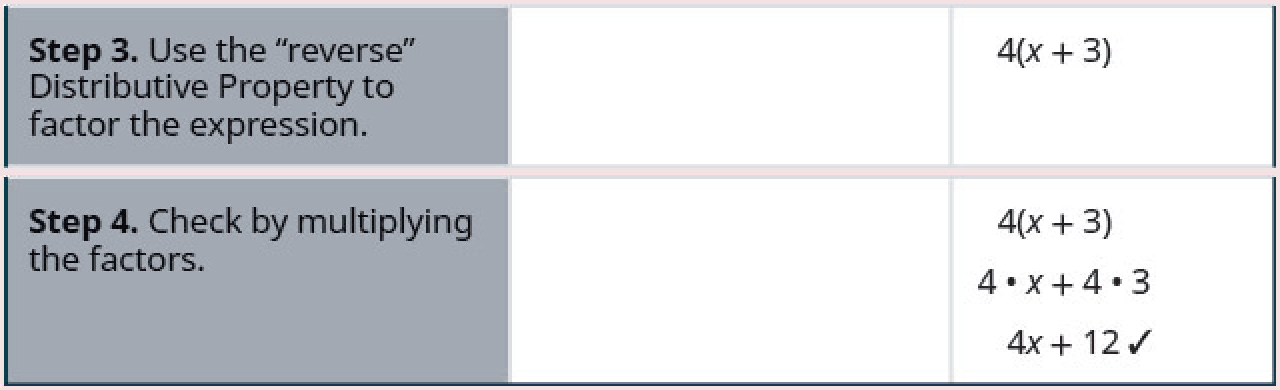
How to Factor the Greatest Common Factor from a Polynomial

Factor: $4x + 12$.

Solution



Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x = 2 \cdot 2 \cdot \quad \cdot x$ $12 = 2 \cdot 2 \cdot 3$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$



Step 3. Use the “reverse” Distributive Property to factor the expression.		$4(x+3)$
Step 4. Check by multiplying the factors.		$4(x+3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

TRY IT 5.1

Factor: $6a + 24$.

Show answer

$6\left(a+4\right)$

TRY IT 5.2

Factor: $2b + 14$.

Show answer

$2\left(b+7\right)$

HOW TO:

Factor the greatest common factor from a polynomial.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the “reverse” Distributive Property to factor the expression.
4. Check by multiplying the factors.

Factor as a Noun and a Verb

We use “factor” as both a noun and a verb.

Noun	7 is a factor of 14
Verb	factor 3 from $3a + 3$

EXAMPLE 6

Factor: $5a + 5$.

Solution

Find the GCF of $5a$ and 5.	$5a = 5 \cdot a$ $5 = 5$ $\text{GCF} = 5$
	$5a + 5$
Rewrite each term as a product using the GCF.	$5 \cdot a + 5 \cdot 1$
Use the Distributive Property “in reverse” to factor the GCF.	$5(a + 1)$
Check by mulitplying the factors to get the orginal polynomial.	
$5\left(a+1\right)$	
$5 \cdot a + 5 \cdot 1$	
$5a + 5\checkmark$	

TRY IT 6.1

Factor: $14x + 14$.

Show answer
$14\left(x+1\right)$

TRY IT 6.1

Factor: $12p + 12$.

Show answer
$12\left(p+1\right)$

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

EXAMPLE 7

Factor: $12x - 60$.

Solution

Find the GCF of 12*x* and 60.	$12x = 2 \cdot 2 \cdot 3 \cdot x$ $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $\text{GCF} = 2 \cdot 2 \cdot 3$ $\text{GCF} = 12$
	$12x - 60$
Rewrite each term as a product using the GCF.	$12 \cdot x - 12 \cdot 5$
Factor the GCF.	$12(x - 5)$
Check by mulitplying the factors.	
$12\,(x - 5)$	
$12 \cdot x - 12 \cdot 5$	
$12x - 60 ✓$	

TRY IT 7.1

Factor: $18u - 36$.

Show answer
$8\,(u - 2)$

TRY IT 7.2

Factor: $30y - 60$.

Show answer
$30\,(y - 2)$

Now we’ll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

EXAMPLE 8

Factor: $4y^2 + 24y + 28$.

Solution
We start by finding the GCF of all three terms.

Find the GCF of $4y^2, 24y$ and 28.	$4y^2 = 2 \cdot 2 \cdot \quad y \cdot y$ $24y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot y$ $28 = 2 \cdot 2 \cdot \quad 7$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
	$4y^2 + 24y + 28$
Rewrite each term as a product using the GCF.	$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$
Factor the GCF.	$4(y^2 + 6y + 7)$
Check by mulitplying.	
$4\left(y^2 + 6y + 7\right)$	
$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$	
$4y^2 + 24y + 28$✓	

TRY IT 8.1

Factor: $5x^2 - 25x + 15$.

Show answer
$5\left(x^2 - 5x + 3\right)$

TRY IT 8.2

Factor: $3y^2 - 12y + 27$.

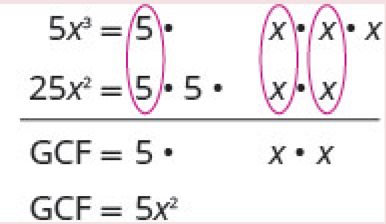
Show answer

$3\left(y^2 - 4y + 9\right)$

EXAMPLE 9

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.	 $5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 5 \cdot x \cdot x$ $\text{GCF} = 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x - 5)$
Check.	
$5x^2\left(x - 5\right)$	
$5x^2 \cdot x - 5x^2 \cdot 5$	
$5x^3 - 25x^2 \checkmark$	

TRY IT 9.1

Factor: $2x^3 + 12x^2$.

Show answer

$2x^2\left(x + 6\right)$

TRY IT 9.2

Factor: $6y^3 - 15y^2$.

Show answer
$3y^2\left(2y - 5\right)$

EXAMPLE 10

Factor: $21x^3 - 9x^2 + 15x$.

Solution
In a previous example we found the GCF of $21x^3, 9x^2, 15x$ to be $3x$.

	$21x^3 - 9x^2 + 15x$
Rewrite each term using the GCF, 3*x*.	$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$
Factor the GCF.	$3x(7x^2 - 3x + 5)$
Check.	
$3x\left(7x^2 - 3x + 5\right)$	
$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$	
$21x^3 - 9x^2 + 15x\checkmark$	

TRY IT 10.1

Factor: $20x^3 - 10x^2 + 14x$.

Show answer
$2x\left(10x^2 - 5x + 7\right)$

TRY IT 10.2

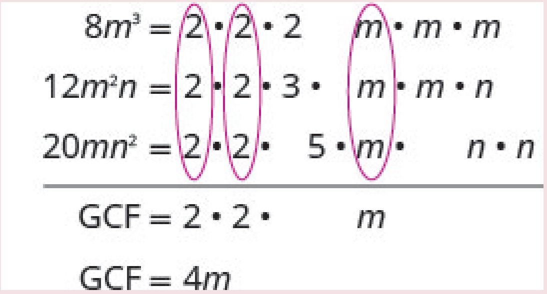
Factor: $24y^3 - 12y^2 - 20y$.

Show answer
$4y\left(6y^2-3y-5\right)$

EXAMPLE 11

Factor: $8m^3-12m^2n+20mn^2$.

Solution

Find the GCF of $8m^3, 12m^2n, 20mn^2$.	 $8m^3 = 2 \cdot 2 \cdot 2 \quad m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot \quad m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot \quad 5 \cdot m \cdot \quad n \cdot n$ $\text{GCF} = 2 \cdot 2 \cdot \quad m$ $\text{GCF} = 4m$
	$8m^3 - 12m^2n + 20mn^2$
Rewrite each term.	$4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Factor the GCF.	$4m(2m^2 - 3m\,n + 5n^2)$
Check.	
$4m\left(2m^2-3mn+5n^2\right)$	
$4m\cdot 2m^2-4m\cdot 3mn+4m\cdot 5n^2$	
$8m^3-12m^2n+20mn^2 \checkmark$	

TRY IT 11.1

Factor: $9xy^2+6x^2y^2+21y^3$.

Show answer
$3y^2\left(3x+2x^2+7y\right)$

TRY IT 11.2

Factor: $3p^3 - 6p^2q + 9pq^3$.

Show answer
$3p\left(p^2 - 2pq + 3q^2\right)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 12

Factor: $-8y - 24$.

Solution
When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the GCF of 8y and 24 is 8. Since the expression −8y − 24 has a negative leading coefficient, we use −8 as the GCF.	$8y = 2 \cdot 2 \cdot 2 \cdot y$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $\text{GCF} = 2 \cdot 2 \cdot 2$ $\text{GCF} = 8$
Rewrite each term using the GCF.	$-8y - 24$ $-8 \cdot y + (-8) \cdot 3$
Factor the GCF.	$-8(y + 3)$
Check.	
$-8\left(y + 3\right)$	
$-8 \cdot y + \left(-8\right) \cdot 3$	
$-8y - 24$✓	

TRY IT 12.1

Factor: $-16z - 64$.

Show answer
$-8\left(8z + 8\right)$

TRY IT 12.2

Factor: $-9y-27$.

Show answer
$-9\left(y+3\right)$

EXAMPLE 13

Factor: $-6a^2+36a$.

Solution
The leading coefficient is negative, so the GCF will be negative.?

Since the leading coefficient is negative, the GCF is negative, −6*a*.	$6a^2 = \enclose{circle}{2} \cdot \enclose{circle}{3} \cdot \enclose{circle}{a} \cdot a$ $36a = \enclose{circle}{2} \cdot 2 \cdot \enclose{circle}{3} \cdot 3 \cdot \enclose{circle}{a}$ $\text{GCF} = 2 \cdot 3 \cdot a$ $\text{GCF} = 6a$ $-6a^2 + 36a$
Rewrite each term using the GCF.	$-6a \cdot a - (-6a) \cdot 6$
Factor the GCF.	$-6a(a-6)$
Check.	
$-6a\left(a-6\right)$	
$-6a\cdot a+\left(-6a\right)\left(-6\right)$	
$-6a^2+36a✓$	

TRY IT 13.1

Factor: $-4b^2+16b$.

Show answer
$-4b\left(b-4\right)$

TRY IT 13.2

Factor: $-7a^2+21a$.

Show answer
$-7a\left(a-3\right)$

EXAMPLE 14

Factor: $5q\left(q+7\right)-6\left(q+7\right)$.

Solution
The GCF is the binomial $q+7$.

	$5q(q+7)-6(q+7)$
Factor the GCF, $(q+7)$.	$(q+7)(5q-6)$
Check on your own by multiplying.	

TRY IT 14.1

Factor: $4m\left(m+3\right)-7\left(m+3\right)$.

Show answer
$\left(m+3\right)\left(4m-7\right)$

TRY IT 14.2

Factor: $8n\left(n-4\right)+5\left(n-4\right)$.

Show answer
$\left(n-4\right)\left(8n+5\right)$

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

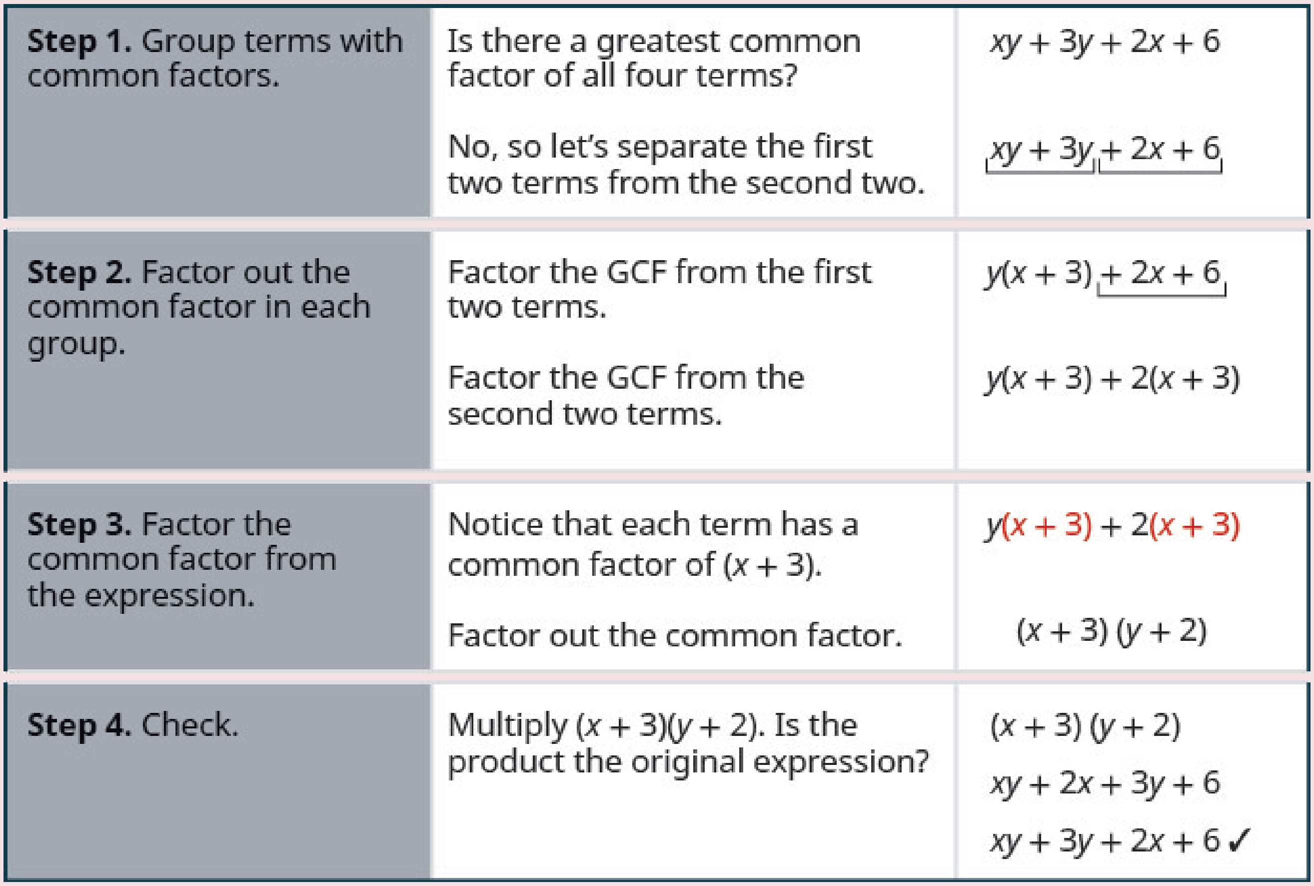
(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

EXAMPLE 15

How to Factor by Grouping

Factor: $xy+3y+2x+6$.

Solution



Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$ $\underbrace{xy+3y}\underbrace{+2x+6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x+3)\underbrace{+2x+6}$ $y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$. Factor out the common factor.	$y(x+3)+2(x+3)$ $(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

TRY IT 15.1

Factor: $xy + 8y + 3x + 24$.

Show answer
$\left(x + 8\right)\left(y + 3\right)$

TRY IT 15.2

Factor: $ab + 7b + 8a + 56$.

Show answer
$\left(a + 7\right)\left(b + 8\right)$

HOW TO:

Factor by grouping.

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

EXAMPLE 16

Factor: $x^2 + 3x - 2x - 6$.

Solution

There is no GCF in all four terms.	$x^2 + 3x - 2x - 6$
Separate into two parts.	$\underbrace{x^2 + 3x}\underbrace{-2x - 6}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x\left(x+3\right) - 2\left(x+3\right)$ $\left(x+3\right)\left(x-2\right)$
Check on your own by multiplying.	

TRY IT 16.1

Factor: $x^2 + 2x - 5x - 10$.

Show answer
$\left(x-5\right)\left(x+2\right)$

TRY IT 16.2

Factor: $y^2 + 4y - 7y - 28$.

Show answer
$\left(y+4\right)\left(y-7\right)$

Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- Greatest Common Factor (GCF)
- Factoring Out the GCF of a Binomial
- Greatest Common Factor (GCF) of Polynomials

Key Concepts

- **Finding the Greatest Common Factor (GCF):** To find the GCF of two expressions:
    1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

- **Factor the Greatest Common Factor from a Polynomial:** To factor a greatest common factor from a polynomial:
    1. Find the GCF of all the terms of the polynomial.
    2. Rewrite each term as a product using the GCF.
    3. Use the ‘reverse’ Distributive Property to factor the expression.
    4. Check by multiplying the factors.
- **Factor by Grouping:** To factor a polynomial with 4 four or more terms
    1. Group terms with common factors.
    2. Factor out the common factor in each group.
    3. Factor the common factor from the expression.
    4. Check by multiplying the factors.

Glossary

factoring
Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.
greatest common factor
The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).
Type your textbox content here.

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. 8, 18	2. 24, 40
3. 72, 162	4. 150, 275
5. 10*a*, 50	6. 5*b*, 30
7. $3x, 10x^2$	8. $21b^2, 14b$
9. $8w^2, 24w^3$	10. $30x^2, 18x^3$
11. $10p^3q, 12pq^2$	12. $8a^2b^3, 10ab^2$
13. $12m^2n^3, 30m^5n^3$	14. $28x^2y^4, 42x^4y^4$
15. $10a^3, 12a^2, 14a$	16. $20y^3, 28y^2, 40y$
17. $35x^3, 10x^4, 5x^5$	18. $27p^2, 45p^3, 9p^4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

19. $4x + 20$	20. $8y + 16$
21. $6m + 9$	22. $14p + 35$
23. $9q + 9$	24. $7r + 7$
25. $8m - 8$	26. $4n - 4$
27. $9n - 63$	28. $45b - 18$
29. $3x^2 + 6x - 9$	30. $4y^2 + 8y - 4$
31. $8p^2 + 4p + 2$	32. $10q^2 + 14q + 20$
33. $8y^3 + 16y^2$	34. $12x^3 - 10x$
35. $5x^3 - 15x^2 + 20x$	36. $8m^2 - 40m + 16$
37. $12xy^2 + 18x^2y^2 - 30y^3$	38. $21pq^2 + 35p^2q^2 - 28q^3$
39. $-2x - 4$	40 $-3b + 12$
41. $5x\left(x+1\right) + 3\left(x+1\right)$	42. $2x\left(x-1\right) + 9\left(x-1\right)$
43. $3b\left(b-2\right) - 13\left(b-2\right)$	44. $6m\left(m-5\right) - 7\left(m-5\right)$

Factor by Grouping

In the following exercises, factor by grouping.

45. $xy + 2y + 3x + 6$	46. $mn + 4n + 6m + 24$
47. $uv - 9u + 2v - 18$	48. $pq - 10p + 8q - 80$
49. $b^2 + 5b - 4b - 20$	50. $m^2 + 6m - 12m - 72$
51. $p^2 + 4p - 9p - 36$	52. $x^2 + 5x - 3x - 15$

Mixed Practice

In the following exercises, factor.

53. $-20x - 10$	54. $5x^3 - x^2 + x$
55. $3x^3 - 7x^2 + 6x - 14$	56. $x^3 + x^2 - x - 1$
57. $x^2 + xy + 5x + 5y$	58. $5x^3 - 3x^2 - 5x - 3$

Everyday Math

59. **Area of a rectangle** The area of a rectangle with length 6 less than the width is given by the expression $w^2 - 6w$, where $w =$ width. Factor the greatest common factor from the polynomial.	60. **Height of a baseball** The height of a baseball *t* seconds after it is hit is given by the expression $-16t^2 + 80t + 4$. Factor the greatest common factor from the polynomial.

Writing Exercises

61. The greatest common factor of 36 and 60 is 12. Explain what this means.	62. What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, and y^c.

Answers

1. 2	3. 18
5. 10	7. x
9. $8w^2$	11. $2pq$
13. $6m^2n^3$	15. $2a$
17. $5x^3$	19. $4\left(x+5\right)$
21. $3\left(2m+3\right)$	23. $9\left(q+1\right)$
25. $8\left(m-1\right)$	27. $9\left(n-7\right)$
29. $3\left(x^2+2x-3\right)$	31. $2\left(4p^2+2p+1\right)$
33. $8y^2\left(y+2\right)$	35. $5x\left(x^2-3x+4\right)$
37. $6y^2\left(2x+3x^2-5y\right)$	39. $-2\left(x+4\right)$
41. $\left(x+1\right)\left(5x+3\right)$	43. $\left(b-2\right)\left(3b-13\right)$
45. $\left(y+3\right)\left(x+2\right)$	47. $\left(u+2\right)\left(v-9\right)$
49. $\left(b-4\right)\left(b+5\right)$	51. $\left(p-9\right)\left(p+4\right)$
53. $-10\left(2x+1\right)$	55. $\left(x^2+2\right)\left(3x-7\right)$
57. $\left(x+y\right)\left(x+5\right)$	59. $w\left(w-6\right)$
61. Answers will vary.	

Attributions

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8.5 Factor Quadratic Trinomials with Leading Coefficient 1

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $x^2 + bxy + cy^2$

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication—to start with the product and end up with the factors. Let's look at an example of multiplying binomials to refresh your memory.

$$\begin{array}{rl} (x+2)(x+3) & \text{factors} \\ F \quad O \quad I \quad L & \\ x^2 + 3x + 2x + 6 & \\ x^2 + 5x + 6 & \text{product} \end{array}$$

To factor the trinomial means to start with the product, $x^2 + 5x + 6$, and end with the factors, $(x+2)(x+3)$. You need to think about where each of the terms in the trinomial came from.

The *first term* came from multiplying the first term in each binomial. So to get x^2 in the product, each binomial must start with an *x*.

$$\begin{array}{l} x^2 + 5x + 6 \\ \quad (x)(x) \end{array}$$

The *last term* in the trinomial came from multiplying the last term in each binomial. So the last terms must multiply to 6

What two numbers multiply to 6?

The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use?

Consider the *middle term*. It came from adding the outer and inner terms.

So the numbers that must have a product of 6 will need a sum of 5. We'll test both possibilities and summarize the results in the table below—the table will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	**Sum of factors**
$1, 6$	$1+6=7$
$2, 3$	$2+3=5$

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. So we have the factors of x^2+5x+6. They are $(x+2)(x+3)$.

$$\begin{array}{ll} x^2+5x+6 & \text{product} \\ (x+2)(x+3) & \text{factors} \end{array}$$

You should check this by multiplying.

Looking back, we started with x^2+5x+6, which is of the form x^2+bx+c, where $b=5$ and $c=6$. We factored it into two binomials of the form $(x+m)$ and $(x+n)$.

$$\begin{array}{ll} x^2+5x+6 & x^2+bx+c \\ (x+2)(x+3) & (x+m)(x+n) \end{array}$$

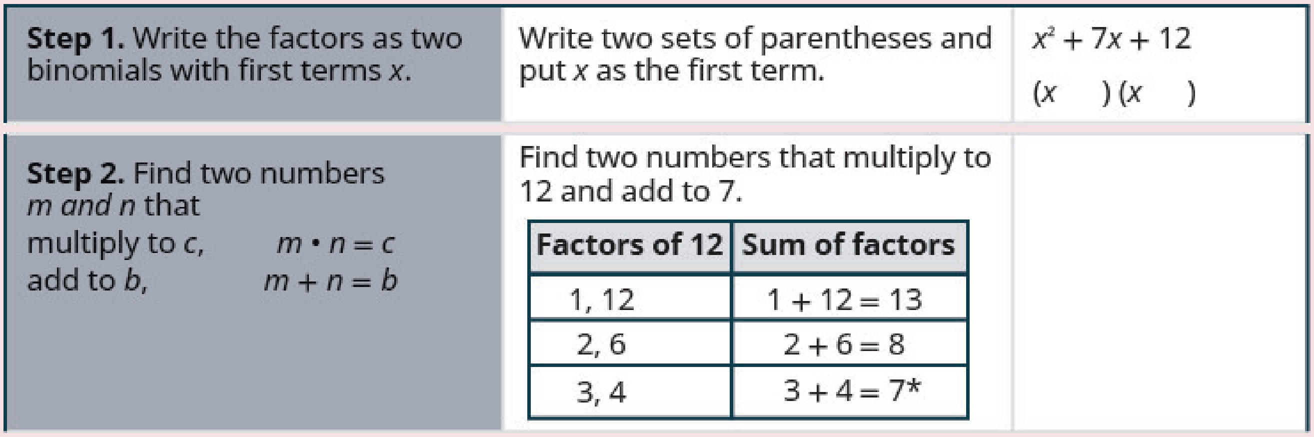
To get the correct factors, we found two numbers *m* and *n* whose product is *c* and sum is *b*.

EXAMPLE 1

How to Factor Trinomials of the Form x^2+bx+c

Factor: $x^2+7x+12$.

Solution



Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2+7x+12$ $(x\quad)(x\quad)$
Step 2. Find two numbers *m and n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 12 and add to 7.	

Factors of 12	**Sum of factors**
1, 12	$1+12=13$
2, 6	$2+6=8$
3, 4	$3+4=7$*

Step 3. Use *m and n* as the last terms of the factors.	Use 3 and 4 as the last terms of the binomials.	$(x+3)\,(x+4)$
Step 4. Check by multiplying the factors.		$(x+3)\,(x+4)$ $x^2+4x+3x+12$ $x^2+7x+12$ ✓

TRY IT 1.1

Factor: x^2+6x+8.

Show answer
$(x+2)\,(x+4)$

TRY IT 1.2

Factor: $y^2+8y+15$.

Show answer
$(y+3)\,(y+5)$

Let’s summarize the steps we used to find the factors.

HOW TO:

Factor trinomials of the form x^2+bx+c.

1. Write the factors as two binomials with first terms *x*: $(x)(x)$.
2. Find two numbers *m* and *n* that
   Multiply to *c*, $m \cdot n = c$
   Add to *b*, $m + n = b$
3. Use *m* and *n* as the last terms of the factors: $(x+m)\,(x+n)$.
4. Check by multiplying the factors.

EXAMPLE 2

Factor: $u^2 + 11u + 24$.

Solution

Notice that the variable is *u*, so the factors will have first terms *u*.

$$u^2 + 11u + 24$$

Write the factors as two binomials with first terms u. $(u)\,(u)$

Find two numbers that: multiply to 24 and add to 11

Factors of 24	**Sum of factors**
$1, 24$	$1 + 24 = 25$
$2, 12$	$2 + 12 = 14$
$3, 8$	$3 + 8 = 11^*$
$4, 6$	$4 + 6 = 10$

Use 3 and 8 as the last terms of the binomials. $(u + 3)\,(u + 8)$

Check.

$$(u + 3)\,(u + 8)$$
$$u^2 + 3u + 8u + 24$$
$$u^2 + 11u + 24\checkmark$$

TRY IT 2.1

Factor: $q^2 + 10q + 24$.

Show answer

$(q + 4)\,(q + 6)$

TRY IT 2.2

Factor: $t^2 + 14t + 24$.

Show answer

$(t + 2)\,(t + 12)$

EXAMPLE 3

Factor: $y^2 + 17y + 60$.

Solution

$y^2 + 17y + 60$

Write the factors as two binomials with first terms y. $(y)\,(y)$

Find two numbers that multiply to 60 and add to 17

Factors of 60	**Sum of factors**
$1, 60$	$1 + 60 = 61$
$2, 30$	$2 + 30 = 32$
$3, 20$	$3 + 20 = 23$
$4, 15$	$4 + 15 = 19$
$5, 12$	$5 + 12 = 17^*$
$6, 10$	$6 + 10 = 16$

Use 5 and 12 as the last terms. $(y + 5)\,(y + 12)$

Check.

$$(y + 5)(y + 12)$$
$$(y^2 + 12y + 5y + 60)$$
$$(y^2 + 17y + 60)\checkmark$$

TRY IT 3.1

Factor: $x^2 + 19x + 60$.

Show answer
$(x + 4)\,(x + 15)$

TRY IT 3.2

Factor: $v^2 + 23v + 60$.

Show answer

$(v+3)(v+20)$

Factor Trinomials of the Form $x^2 + bx + c$ with *b* Negative, *c* Positive

In the examples so far, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

Remember: To get a negative sum and a positive product, the numbers must both be negative.

Again, think about FOIL and where each term in the trinomial came from. Just as before,

- the first term, x^2, comes from the product of the two first terms in each binomial factor, *x* and *y*;
- the positive last term is the product of the two last terms
- the negative middle term is the sum of the outer and inner terms.

How do you get a *positive product* and a *negative sum*? With two negative numbers.

EXAMPLE 4

Factor: $t^2 - 11t + 28$.

Solution

Again, with the positive last term, 28, and the negative middle term, $-11t$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

$$t^2 - 11t + 28$$

Write the factors as two binomials with first terms t. $(t)(t)$

Find two numbers that: multiply to 28 and add to -11.

Factors of 28	**Sum of factors**
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11^*$

Use -4, -7 as the last terms of the binomials. $(t-4)(t-7)$

Check.

$$(t-4)(t-7)$$
$$t^2-7t-4t+28$$
$$t^2-11t+28\checkmark$$

TRY IT 4.1

Factor: $u^2-9u+18$.

Show answer
$(u-3)(u-6)$

TRY IT 4.2

Factor: $y^2-16y+63$.

Show answer
$(y-7)(y-9)$

Factor Trinomials of the Form $x^2 + bx + c$ with c Negative

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

Remember: To get a negative product, the numbers must have different signs.

EXAMPLE 5

Factor: z^2+4z-5.

Solution
To get a negative last term, multiply one positive and one negative. We need factors of -5 that add to positive 4

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4$
$-1, 5$	$-1 + 5 = 4^{*}$

Notice: We listed both $1, -5$ and $-1, 5$ to make sure we got the sign of the middle term correct.

$$z^2 + 4z - 5$$

Factors will be two binomials with first terms z. $(z)(z)$

Use -1, 5 as the last terms of the binomials. $(z-1)(z+5)$

Check.

$$\begin{gathered}(z-1)(z+5)\\ z^2 + 5z - 1z - 5\\ z^2 + 4z - 5\checkmark\end{gathered}$$

TRY IT 5.1

Factor: $h^2 + 4h - 12$.

Show answer
$(h-2)(h+6)$

TRY IT 5.2

Factor: $k^2 + k - 20$.

Show answer
$(k-4)(k+5)$

Let’s make a minor change to the last trinomial and see what effect it has on the factors.

EXAMPLE 6

Factor: $z^2 - 4z - 5$.

Solution
This time, we need factors of -5 that add to -4.

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4^{*}$
$-1, 5$	$-1 + 5 = 4$

$$z^2 - 4z - 5$$

Factors will be two binomials with first terms z. $(z)\,(z)$

Use 1, -5 as the last terms of the binomials. $(z+1)\,(z-5)$

Check.

$$(z+1)\,(z-5)$$
$$z^2 - 5z + 1z - 5$$
$$z^2 - 4z - 5\checkmark$$

Notice that the factors of $z^2 - 4z - 5$ are very similar to the factors of $z^2 + 4z - 5$. It is very important to make sure you choose the factor pair that results in the correct sign of the middle term.

TRY IT 6.1

Factor: $x^2 - 4x - 12$.

Show answer

$(x+2)\,(x-6)$

TRY IT 6.2

Factor: $y^2 - y - 20$.

Show answer

$(y+4)\,(y-5)$

EXAMPLE 7

Factor: $q^2 - 2q - 15$.

Solution

Factors will be two binomials with first terms q. $$q^2 - 2q - 15$$ $$(q)(q)$$

You can use 3, -5 as the last terms of the binomials. $$(q+3)(q-5)$$

Factors of -15	**Sum of factors**
$1, -15$	$1 + (-15) = -14$
$-1, 15$	$-1 + 15 = 14$
$3, -5$	$3 + (-5) = -2^*$
$-3, 5$	$-3 + 5 = 2$

Check.

$(q+3)(q-5)$

$q^2 - 5q + 3q - 15$

$q^2 - 2q - 15$✓

TRY IT 7.1

Factor: $r^2 - 3r - 40$.

Show answer

$(r+5)(r-8)$

TRY IT 7.1

Factor: $s^2 - 3s - 10$.

Show answer

$(s+2)(s-5)$

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

EXAMPLE 8

Factor: $y^2 - 6y + 15$.

Solution

	$y^2 - 6y + 15$
Factors will be two binomials with first terms y.	$(y)\,(y)$

Factors of 15	**Sum of factors**
$-1, -15$	$-1 + (-15) = -16$
$-3, -5$	$-3 + (-5) = -8$

As shown in the table, none of the factors add to -6; therefore, the expression is prime.

TRY IT 8.1

Factor: $m^2 + 4m + 18$.

Show answer
prime

TRY IT 8.2

Factor: $n^2 - 10n + 12$.

Show answer
prime

EXAMPLE 9

Factor: $2x + x^2 - 48$.

Solution

$2x + x^2 - 48$

First we put the terms in decreasing degree order. $x^2 + 2x - 48$

Factors will be two binomials with first terms x. $(x)\,(x)$

As shown in the table, you can use $-6, 8$ as the last terms of the binomials.

$(x - 6)\,(x + 8)$

Factors of -48	**Sum of factors**
$-1, 48$	$-1 + 48 = 47$
$-2, 24$ $-3, 16$ $-4, 12$ $-6, 8$	$-2 + 24 = 22$ $-3 + 16 = 13$ $-4 + 12 = 8$ $6 + 8 = 2$

Check.

$(x - 6)\,(x + 8)$

$x^2 - 6q + 8q - 48$

$x^2 + 2x - 48$✓

TRY IT 9.1

Factor: $9m + m^2 + 18$.

Show answer

$(m + 3)\,(m + 6)$

TRY IT 9.2

Factor: $-7n + 12 + n^2$.

Show answer

$(n - 3)\,(n - 4)$

Let's summarize the method we just developed to factor trinomials of the form $x^2 + bx + c$.

HOW TO:

Factor trinomials of the form $x^2 + bx + c$.

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When *c* is positive, *m* and *n* have the same sign.

b positive	b negative
m, n positive	m, n negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

When *c* is negative, *m* and *n* have opposite signs.

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the Form $x^2 + bxy + cy^2$

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in the previous objective.

EXAMPLE 10

Factor: $x^2 + 12xy + 36y^2$.

Solution

	$x^2 + 12xy + 36y^2$
Note that the first terms are x. last terms contain y.	$(x_y)(x_y)$
Find the numbers that multiply to 36 and add to 12	

Factors of 36	**Sum of factors**
1, 36	$1 + 36 = 37$
2, 18	$2 + 18 = 20$
3, 12	$3 + 12 = 15$
4, 9	$4 + 9 = 13$
6, 6	$6 + 6 = 12^*$

Use 6 and 6 as the coefficients of the last terms. $(x + 6y)(x + 6y)$

Check your answer.

$$(x + 6y)(x + 6y)$$
$$x^2 + 6xy + 6xy + 36y^2$$
$$x^2 + 12xy + 36y^2 \checkmark$$

TRY IT 10.1

Factor: $u^2 + 11uv + 28v^2$.

Show answer
$(u + 4v)(u + 7v)$

TRY IT 10.2

Factor: $x^2 + 13xy + 42y^2$.

Show answer
$(x + 6y)(x + 7y)$

EXAMPLE 11

Factor: $r^2 - 8rx - 9s^2$.

Solution
We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$r^2 - 8rx - 9s^2$$

Note that the first terms are r, last terms contain s. $$(r_s)(r_s)$$

Find the numbers that multiply to -9 and add to -8.

Factors of -9	**Sum of factors**
$1, -9$	$1 + (-9) = -8$*
$-1, 9$	$-1 + 9 = 8$
$3, -3$	$3 + (-3) = 0$

Check your answer.Use 1, -9 as coefficients of the last terms. $(r + s)(r - 9s)$

$$(r - 9s)(r + s)$$
$$r^2 + rs - 9rs - 9s^2$$
$$r^2 - 8rs - 9s^2 \checkmark$$

TRY IT 11.1

Factor: $a^2 - 11ab + 10b^2$.

Show answer
$(a - b)(a - 10b)$

TRY IT 11.2

Factor: $m^2 - 13mn + 12n^2$.

Show answer
$(m - n)(m - 12n)$

EXAMPLE 12

Factor: $u^2 - 9uv - 12v^2$.

Solution
We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2 - 9uv - 12v^2$$

Note that the first terms are u, last terms contain v. $$(u_v)\,(u_v)$$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	**Sum of factors**
$1, -12$	$1 + (-12) = -11$
$1, 12$	$-1 + 12 = 11$
$2, -6$	$2 + (-6) = -4$
$-2, 6$	$-2 + 6 = 4$
$3, -4$	$3 + (-4) = -1$
$-3, 4$	$-3 + 4 = 1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

TRY IT 12.1

Factor: $x^2 - 7xy - 10y^2$.

Show answer
prime

TRY IT 12.2

Factor: $p^2 + 15pq + 20q^2$.

Show answer
prime

Key Concepts

- **Factor trinomials of the form** $x^2 + bx + c$
  1. Write the factors as two binomials with first terms *x*: $(x)\,(x)$.
  2. Find two numbers *m* and *n* that
     Multiply to *c*, $m \cdot n = c$

   Add to *b*, $m + n = b$

3. Use *m* and *n* as the last terms of the factors: $(x + m)\,(x + n)$.
4. Check by multiplying the factors.

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

1. $x^2 + 4x + 3$	2. $y^2 + 8y + 7$
3. $m^2 + 12m + 11$	4. $b^2 + 14b + 13$
5. $a^2 + 9a + 20$	6. $m^2 + 7m + 12$
7. $p^2 + 11p + 30$	8. $w^2 + 10x + 21$
9. $n^2 + 19n + 48$	10. $b^2 + 14b + 48$
11. $a^2 + 25a + 100$	12. $u^2 + 101u + 100$
13. $x^2 - 8x + 12$	14. $q^2 - 13q + 36$
15. $y^2 - 18x + 45$	16. $m^2 - 13m + 30$
17. $x^2 - 8x + 7$	18. $y^2 - 5y + 6$
19. $p^2 + 5p - 6$	20. $n^2 + 6n - 7$
21. $y^2 - 6y - 7$	22. $v^2 - 2v - 3$
23. $x^2 - x - 12$	24. $r^2 - 2r - 8$
25. $a^2 - 3a - 28$	26. $b^2 - 13b - 30$
27. $w^2 - 5w - 36$	28. $t^2 - 3t - 54$
29. $x^2 + x + 5$	30. $x^2 - 3x - 9$
31. $8 - 6x + x^2$	32. $7x + x^2 + 6$
33. $x^2 - 12 - 11x$	34. $-11 - 10x + x^2$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

35. $p^2 + 3pq + 2q^2$	36. $m^2 + 6mn + 5n^2$
37. $r^2 + 15rs + 36s^2$	38. $u^2 + 10uv + 24v^2$
39. $m^2 - 12mn + 20n^2$	40. $p^2 - 16pq + 63q^2$
41. $x^2 - 2xy - 80y^2$	42. $p^2 - 8pq - 65q^2$
43. $m^2 - 64mn - 65n^2$	44. $p^2 - 2pq - 35q^2$
45. $a^2 + 5ab - 24b^2$	46. $r^2 + 3rs - 28s^2$
47. $x^2 - 3xy - 14y^2$	48. $u^2 - 8uv - 24v^2$
49. $m^2 - 5mn + 30n^2$	50. $c^2 - 7cd + 18d^2$

Mixed Practice

In the following exercises, factor each expression.

51. $u^2 - 12u + 36$	52. $w^2 + 4w - 32$
53. $x^2 - 14x - 32$	54. $y^2 + 41y + 40$
55. $r^2 - 20rs + 64s^2$	56. $x^2 - 16xy + 64y^2$
57. $k^2 + 34k + 120$	58. $m^2 + 29m + 120$
59. $y^2 + 10y + 15$	60. $z^2 - 3z + 28$
61. $m^2 + mn - 56n^2$	62. $q^2 - 29qr - 96r^2$
63. $u^2 - 17uv + 30v^2$	64. $m^2 - 31mn + 30n^2$
65. $c^2 - 8cd + 26d^2$	66. $r^2 + 11rs + 36s^2$

Everyday Math

67. **Consecutive integers** Deirdre is thinking of two consecutive integers whose product is 56. The trinomial $x^2 + x - 56$ describes how these numbers are related. Factor the trinomial.	68. **Consecutive integers** Deshawn is thinking of two consecutive integers whose product is 182. The trinomial $x^2 + x - 182$ describes how these numbers are related. Factor the trinomial.

Writing Exercises

69. Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of *m* and *n*.

70. How do you determine whether to use plus or minus signs in the binomial factors of a trinomial of the form $x^2 + bx + c$ where b and c may be positive or negative numbers?

71. Will factored $x^2 - x - 20$ as $(x + 5)(x - 4)$. Bill factored it as $(x + 4)(x - 5)$. Phil factored it as $(x - 5)(x - 4)$. Who is correct? Explain why the other two are wrong.

72. Look at (Figure), where we factored $y^2 + 17y + 60$. We made a table listing all pairs of factors of 60 and their sums. Do you find this kind of table helpful? Why or why not?

Answers

1. $(x + 1)(x + 3)$
3. $(m + 1)(m + 11)$
5. $(a + 4)(a + 5)$
7. $(p + 5)(p + 6)$
9. $(n + 3)(n + 16)$
11. $(a + 5)(a + 20)$
13. $(x - 2)(x - 6)$
15. $(y - 3)(y - 15)$
17. $(x - 1)(x - 7)$
19. $(p - 1)(p + 6)$
21. $(y + 1)(y - 7)$
23. $(x - 4)(x + 3)$
25. $(a - 7)(a + 4)$
27. $(w - 9)(w + 4)$
29. prime
31. $(x - 4)(x - 2)$
33. $(x - 12)(x + 1)$
35. $(p + q)(p + 2q)$
37. $(r + 3s)(r + 12s)$
39. $(m - 2n)(m - 10n)$
41. $(x + 8y)(x - 10y)$
43. $(m + n)(m - 65n)$
45. $(a + 8b)(a - 3b)$
47. prime
49. prime
51. $(u - 6)(u - 6)$
53. $(x + 2)(x - 16)$
55. $(r - 4s)(r - 16s)$
57. $(k + 4)(k + 30)$
59. prime
61. $(m + 8n)(m - 7n)$
63. $(u - 15v)(u - 2v)$
65. prime
67. $(x + 8)(x - 7)$
69. Answers may vary
71. Answers may vary

Attributions

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8.6 Divide Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Divide a polynomial by a monomial

Divide a Polynomial by a Monomial

In the last chapter, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

The sum,	$\frac{y}{5}+\frac{2}{5}$,
simplifies to	$\frac{y+2}{5}$.

Now we will do this in reverse to split a single fraction into separate fractions.

We'll state the fraction addition property here just as you learned it and in reverse.

Fraction Addition

If a, b, and c are numbers where $c \neq 0$, then

$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ and $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

For example,	$\dfrac{y+2}{5}$
can be written	$\dfrac{y}{5}+\dfrac{2}{5}.$

We use this form of fraction addition to divide polynomials by monomials.

Division of a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 1

Find the quotient: $\dfrac{7y^2+21}{7}$.

Solution

	$\dfrac{7y^2+21}{7}$
Divide each term of the numerator by the denominator.	$\dfrac{7y^2}{7}+\dfrac{21}{7}$
Simplify each fraction.	y^2+3

TRY IT 1.1

Find the quotient: $\dfrac{8z^2+24}{4}$.

Show answer

$2z^2+6$

TRY IT 1.2

Find the quotient: $\frac{18z^2 - 27}{9}$.

Show answer
$2z^2 - 3$

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

EXAMPLE 2

Find the quotient: $\left(18x^3 - 36x^2\right) \div 6x$.

Solution

	$\left(18x^3 - 36x^2\right) \div 6x$
Rewrite as a fraction.	$\frac{18x^3 - 36x^2}{6x}$
Divide each term of the numerator by the denominator.	$\frac{18x^3}{6x} - \frac{36x^2}{6x}$
Simplify.	$3x^2 - 6x$

TRY IT 2.1

Find the quotient: $\left(27b^3 - 33b^2\right) \div 3b$.

Show answer
$9b^2 - 11b$

TRY IT 2.2

Find the quotient: $\left(25y^3 - 55y^2\right) \div 5y$.

Show answer

$5y^2 - 11y$

When we divide by a negative, we must be extra careful with the signs.

EXAMPLE 3

Find the quotient: $\dfrac{12d^2 - 16d}{-4}$.

Solution

	$\dfrac{12d^2 - 16d}{-4}$
Divide each term of the numerator by the denominator.	$\dfrac{12d^2}{-4} - \dfrac{16d}{-4}$
Simplify. Remember, subtracting a negative is like adding a positive!	$-3d^2 + 4d$

TRY IT 3.1

Find the quotient: $\dfrac{25y^2 - 15y}{-5}$.

Show answer

$-5y^2 + 3y$

TRY IT 3.2

Find the quotient: $\dfrac{42b^2 - 18b}{-6}$.

Show answer
$-7b^2+3b$

EXAMPLE 4

Find the quotient: $\dfrac{105y^5+75y^3}{5y^2}$.

Solution

	$\dfrac{105y^5+75y^3}{5y^2}$
Separate the terms.	$\dfrac{105y^5}{5y^2}+\dfrac{75y^3}{5y^2}$
Simplify.	$21y^3+15y$

TRY IT 4.1

Find the quotient: $\dfrac{60d^7+24d^5}{4d^3}$.

Show answer
$15d^4+6d^2$

TRY IT 4.2

Find the quotient: $\dfrac{216p^7-48p^5}{6p^3}$.

Show answer
$36p^4-8p^2$

EXAMPLE 5

Find the quotient: $\left(15x^3y - 35xy^2\right) \div (-5xy)$.

Solution

	$\left(15x^3y - 35xy^2\right) \div (-5xy)$
Rewrite as a fraction.	$\dfrac{15x^3y - 35xy^2}{-5xy}$
Separate the terms.	$\dfrac{15x^3y}{-5xy} - \dfrac{35xy^2}{-5xy}$
Simplify.	$-3x^2 + 7y$

TRY IT 5.1

Find the quotient: $\left(32a^2b - 16ab^2\right) \div (-8ab)$.

Show answer
$-4a + 2b$

TRY IT 5.2

Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$.

Show answer
$8a^5b + 6a^3b^2$

EXAMPLE 6

Find the quotient: $\dfrac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$.

Solution

	$\dfrac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$
Separate the terms.	$\dfrac{36x^3y^2}{9x^2y} + \dfrac{27x^2y^2}{9x^2y} - \dfrac{9x^2y^3}{9x^2y}$
Simplify.	$4xy + 3y - y^2$

TRY IT 6.1

Find the quotient: $\dfrac{40x^3y^2 + 24x^2y^2 - 16x^2y^3}{8x^2y}$.

Show answer

$5xy + 3y - 2y^2$

TRY IT 6.2

Find the quotient: $\dfrac{35a^4b^2 + 14a^4b^3 - 42a^2b^4}{7a^2b^2}$.

Show answer

$5a^2 + 2a^2b - 6b^2$

EXAMPLE 7

Find the quotient: $\dfrac{10x^2 + 5x - 20}{5x}$.

Solution

	$\dfrac{10x^2+5x-20}{5x}$
Separate the terms.	$\dfrac{10x^2}{5x}+\dfrac{5x}{5x}-\dfrac{20}{5x}$
Simplify.	$2x+1+\dfrac{4}{x}$

TRY IT 7.1

Find the quotient: $\dfrac{18c^2+6c-9}{6c}$.

Show answer

$3c+1-\dfrac{3}{2c}$

TRY IT 7.2

Find the quotient: $\dfrac{10d^2-5d-2}{5d}$.

Show answer

$2d-1-\dfrac{2}{5d}$

Access these online resources for additional instruction and practice with dividing polynomials:

- Divide a Polynomial by a Monomial
- Divide a Polynomial by a Monomial 2

Key Concepts

- **Fraction Addition**
  - If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a+b}{c}=\tfrac{a}{c}+\frac{b}{c}$$

- **Division of a Polynomial by a Monomial**
    - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Practice Makes Perfect

Dividing Polynomial by Monomial

In the following exercises, divide each polynomial by the monomial.

1. $\frac{30b+75}{5}$	2. $\frac{45y+36}{9}$
3. $\frac{42x^2-14x}{7}$	4. $\frac{8d^2-4d}{2}$
5. $\left(55w^2-10w\right)\div 5w$	6. $\left(16y^2-20y\right)\div 4y$
7. $\left(8x^3+6x^2\right)\div 2x$	8. $\left(9n^4+6n^3\right)\div 3n$
9. $\frac{20b^2-12b}{-4}$	10. $\frac{18y^2-12y}{-6}$
11. $\frac{51m^4+72m^3}{-3}$	12. $\frac{35a^4+65a^2}{-5}$
13. $\frac{412z^8-48z^5}{4z^3}$	14. $\frac{310y^4-200y^3}{5y^2}$
15. $\frac{51y^4+42y^2}{3y^2}$	16. $\frac{46x^3+38x^2}{2x^2}$
17. $\left(35x^4-21x\right)\div(-7x)$	18. $\left(24p^2-33p\right)\div(-3p)$
19. $\left(48y^4-24y^3\right)\div\left(-8y^2\right)$	20. $\left(63m^4-42m^3\right)\div\left(-7m^2\right)$
21. $\left(45x^3y^4+60xy^2\right)\div(5xy)$	22. $\left(63a^2b^3+72ab^4\right)\div(9ab)$
23. $\frac{49c^2d^2-70c^3d^3-35c^2d^4}{7cd^2}$	24. $\frac{52p^5q^4+36p^4q^3-64p^3q^2}{4p^2q}$
25. $\frac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$	26. $\frac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$
27. $\frac{12q^2+3q-1}{3q}$	28. $\frac{4w^2+2w-5}{2w}$
29. $\frac{20y^2+12y-1}{-4y}$	30. $\frac{10x^2+5x-4}{-5x}$
31. $\frac{63a^3-108a^2+99a}{9a^2}$	32. $\frac{36p^3+18p^2-12p}{6p^2}$

Everyday Math

33. Handshakes At a company meeting, every employee shakes hands with every other employee. The number of handshakes is given by the expression $\frac{n^2-n}{2}$, where n represents the number of employees. How many handshakes will there be if there are 10 employees at the meeting?

34. Average cost Pictures Plus produces digital albums. The company's average cost (in dollars) to make x albums is given by the expression $\frac{7x+500}{x}$.

  a. Find the quotient by dividing the numerator by the denominator.
  b. What will the average cost (in dollars) be to produce 20 albums?

Writing Exercises

35. Divide $\frac{10x^2+x-12}{2x}$ and explain with words how you get each term of the quotient.

36. James divides $48y+6$ by 6 this way: $\frac{48y+\overline{)6}}{\overline{)6}} = 48y$. What is wrong with his reasoning?

Answers

1. $6b+15$	3. $6x^2-2x$	5. $11w-2$
7. $4x^2+3x$	9. $-5b^2+3b$	11. $-17m^4-24m^3$
13. $103z^5-12z^2$	15. $17y^2+14$	17. $-5x^3+3$
19. $-6y^2+3y$	21. $9x^2y^3+12y$	23. $7c-10c^2d-5cd^2$
25. $6r^3+11r^2s-8rs^3$	27. $4q+1-\frac{1}{3q}$	29. $-5y-3+\frac{1}{4y}$
31. $7a-12+\frac{11}{a}$	33. 45	35. Answers will vary.

Attributions

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8.7 Chapter Review

Review Exercises

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

1. a) $a^2 - b^2$ b) $24d^3$ c) $x^2 + 8x - 10$ d) $m^2n^2 - 2mn + 6$ e) $7y^3 + y^2 - 2y - 4$	2. a) $11c^4 - 23c^2 + 1$ b) $9p^3 + 6p^2 - p - 5$ c) $\frac{3}{7}x + \frac{5}{14}$ d) 10 e) $2y - 12$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

3. a) $5p^3 - 8p^2 + 10p - 4$ b) $-20q^4$ c) $x^2 + 6x + 12$ d) $23r^2s^2 - 4rs + 5$ e) 100	4. a) $3x^2 + 9x + 10$ b) $14a^2bc$ c) $6y + 1$ d) $n^3 - 4n^2 + 2n - 8$ e) -19

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

5. $-14k+19k$	6. $5y^3+8y^3$
7. $-9c-18c$	8. $12q-(-6q)$
9. $3m^2+7n^2-3m^2$	10. $12x-4y-9x$
11. $13a+b$	12. $6x^2y-4x+8xy^2$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

13. $\left(9p^2-5p+3\right)+\left(4p^2-4\right)$	14. $\left(5x^2+12x+1\right)+\left(6x^2-8x+3\right)$
15. $\left(7y^2-8y\right)-(y-4)$	16. $\left(10m^2-8m-1\right)-\left(5m^2+m-2\right)$
17. Find the sum of $\left(a^2+6a+9\right)$ and $\left(5a^3-7\right)$	18. Subtract $\left(3s^2+10\right)$ from $\left(15s^2-2s+8\right)$

Evaluate a Polynomial for a Given Value of the Variable

In the following exercises, evaluate each polynomial for the given value.

19. Evaluate $10-12x$ when: a) $x=3$ b) $x=0$ c) $x=-1$	20. Evaluate $3y^2-y+1$ when: a) $y=5$ b) $y=-1$ c) $y=0$
21. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial $-4p^2+460p$. Find the revenue received when $p=75$ dollars.	22. Randee drops a stone off the 200 foot high cliff into the ocean. The polynomial $-16t^2+200$ gives the height of a stone t seconds after it is dropped from the cliff. Find the height after $t=3$ seconds.

Multiply Monomials

In the following exercises, multiply the monomials.

23. $\left(-9n^7\right)(-16n)$	24. $\left(-15x^2\right)\left(6x^4\right)$
25. $\left(\frac{5}{9}ab^2\right)\left(27ab^3\right)$	26. $\left(7p^5q^3\right)\left(8pq^9\right)$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

27. $-4\left(y+13\right)$	28. $7\left(a+9\right)$
29. $p\left(p+3\right)$	30. $-5\left(r-2\right)$
31. $-6u\left(2u+7\right)$	32. $-m\left(m+15\right)$
33. $3q^{2}\left(q^{2}-7q+6\right)$ 3	34. $9\left(b^{2}+6b+8\right)$
35. $\left(b-4\right)\cdot 11$	36. $\left(5z-1\right)z$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using: a) the Distributive Property, b) the FOIL method, c) the Vertical Method.

37. $\left(6y-7\right)\left(2y-5\right)$	38. $\left(x-4\right)\left(x+10\right)$

In the following exercises, multiply the binomials. Use any method.

39. $\left(y-4\right)\left(y-8\right)$	40. $\left(x+3\right)\left(x+9\right)$
41. $\left(q+16\right)\left(q-3\right)$	42. $\left(p-7\right)\left(p+4\right)$
43. $\left(u^{2}+6\right)\left(u^{2}-5\right)$	44. $\left(5m-8\right)\left(12m+1\right)$
45. $\left(8mn+3\right)\left(2mn-1\right)$	46. $\left(9x-y\right)\left(6x-5\right)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using a) the Distributive Property, b) the Vertical Method.

47. $\left(3x-4\right)\left(6x^{2}+x-10\right)$	48. $\left(n+1\right)\left(n^{2}+5n-2\right)$

In the following exercises, multiply. Use either method.

49. $\left(7m+1\right)\left(m^{2}-10m-3\right)$	50. $\left(y-2\right)\left(y^{2}-8y+9\right)$

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

51. $(q-15)^2$	52. $(c+11)^2$
53. $(8u+1)^2$	54. $\left(x+\frac{1}{3}\right)^2$
55. $(4a-3b)^2$	56. $\left(3n^3-2\right)^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

57. $\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)$	58. $(s-7)(s+7)$
59. $(6-r)(6+r)$	60. $(12c+13)(12c-13)$
61. $\left(5p^4-4q^3\right)\left(5p^4+4q^3\right)$	62. $\left(u+\frac{3}{4}v\right)\left(u-\frac{3}{4}v\right)$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

63. $(6a+11)(6a-11)$	64. $(3m+10)^2$
65. $\left(c^4+9d\right)^2$	66. $(5x+y)(x-5y)$
67. $\left(a^2+4b\right)\left(4a-b^2\right)$	68. $\left(p^5+q^5\right)\left(p^5-q^5\right)$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

69. $\left(35x^2-75x\right)\div 5x$	70. $\frac{42z^2-18z}{6}$
71. $\frac{550p^6-300p^4}{10p^3}$	72. $\frac{81n^4+105n^2}{-3}$
73. $\frac{96a^5b^2-48a^4b^3-56a^2b^4}{8ab^2}$	74. $\left(63xy^3+56x^2y^4\right)\div(7xy)$
75. $\frac{105y^5+50y^3-5y}{5y^3}$	76. $\frac{57m^2-12m+1}{-3m}$

Review Exercise Answers

1. a) binomial b) monomial c) trinomial d) trinomial e) other polynomial	3. a) 3 b) 4 c) 2 d) 4 e) 0	5. $5k$
7. $-27c$	9. $7n^2$	11. $13\text{a}+b$
13. $13p^2-5p-1$	15. $7y^2-9y+4$	17. $5a^3+a^2+6a+2$
19. a) -26 b) 10 c) 22	21. 12,000	23. $144n^8$
25. $15a^2b^5$	27. $-4y-52$	29. p^2+3p
31. $-12u^2-42u$	33. $3q^4-21q^3+18q^2$	35. $11b-44$
37. a) $12y^2-44y+35$ b) $12y^2-44y+35$ c) $12y^2-44y+35$	39. $y^2-12y+32$	41. $q^2+13q-48$
43. u^4+u^2-30	45. $16m^2n^2-2mn-3$	47. a) $18x^3-21x^2-34x+40$ b) $18x^3-21x^2-34x+40$
49. $7m^3-69m^2-31m-3$	51. $q^2-30q+225$	53. $64u^2+16u+1$
55. $16a^2-24ab+9b^2$	57. $y^2-\frac{4}{25}$	59. $36-r^2$
61. $25p^8-16q^6$	63. $36a^2-121$	65. $c^8+18c^4d+81d^2$
67. $4a^3+3a^2b-4b^3$	69. $7x-15$	71. $55p^3-30p$
73. $12a^4-6a^3b-7ab^2$	75. $21y^2+10-\frac{1}{y^2}$	

Chapter Practice Test

In the following exercises, simplify each expression. 1. $\left(12a^2-7a+4\right)+\left(3a^2+8a-10\right)$	2. For the polynomial $10x^4+9y^2-1$ a) Is it a monomial, binomial, or trinomial? b) What is its degree?
3. $\left(9p^2-5p+1\right)-\left(2p^2-6\right)$	4. $\left(-9r^4s^5\right)\left(4rs^7\right)$
5. $(v-9)(9v-5)$	6. $(m+6)(m+12)$
7. $(n-6)\left(n^2-5n+4\right)$	8. $(4c-11)(3c-8)$
9. $(7p-5)(7p+5)$	10. $(2x-15y)(5x+7y)$
11. $(9v-2)^2$	12. $\frac{12x^3+42x^2-6x}{2x}$
13. $\frac{64x^3-x}{4x}$	14. $\frac{70xy^4+95x^3y}{5xy}$
15. $\frac{y^2-5y-18}{y}$	16. A helicopter flying at an altitude of 1000 feet drops a rescue package. The polynomial $-16t^2+1000$ gives the height of the package t seconds after it was dropped. Find the height when $t=6$ seconds.

Practice Test Answers

1. $15a^2+a-6$	2. a) Trinomial, b) 4	3. $7p^2-5p+7$
4. $-36r^5s^{12}$	5. $9v^2-86v+45$	6. $m^2+18m+72$
7. $n^3-11n^2+34n-24$	8. $12c^2-65c+88$	9. $49p^2-25$
10. $10x^2-61xy-105y^2$	11. $81v^2-36v+4$	12. $6x^2+21x-3$
13. $16x^2-\frac{1}{4}$	14. $14y^3+19x^2$	15. $y-5-\frac{18}{y}$
16. 424 feet		

CHAPTER 9 Trigonometry

Trigonometry is a part of geometry that takes its origin in the ancient study of the relationship of the sides and angles of a right triangle. “Trigon” from Greek means triangle and “metron” means measure.

Applications of trigonometry are essential to many disciplines like carpentry, engineering, surveying, and astronomy, just to name a few.

How tall is the Riverpole? Do we have to climb the pole to find out? Fortunately, with the knowledge of trigonometry, we can find out the measurements of tall objects without too much hassle.

In this chapter we will explore the basic properties of angles and triangles, and the applications of the Pythagorean Theorem and trigonometric ratios.

Riverpole by Vaughn Warren – Kamloops, BC.

9.1 Use Properties of Angles, Triangles, and the Pythagorean Theorem

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of angles
- Use the properties of triangles
- Use the Pythagorean Theorem

Use the Properties of Angles

Are you familiar with the phrase 'do a 180'? It means to make a full turn so that you face the opposite direction. It comes from the fact that the measure of an angle that makes a straight line is 180 degrees. See (Figure 1).

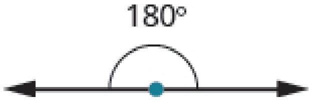


Figure 1

An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle and the common endpoint is called the vertex. An angle is named by its vertex. In (Figure 2), $\angle A$ is the angle with vertex at point A. The measure of $\angle A$ is written $m\angle A$.
$\angle A$ is the angle with vertex at $\text{point } A$.

Figure 2

We measure angles in degrees, and use the symbol ° to represent degrees. We use the abbreviation m to for the *measure* of an angle. So if $\angle A$ is 27°, we would write $m\angle A = 27$.

If the sum of the measures of two angles is 180°, then they are called supplementary angles. In (Figure 3), each pair of angles is supplementary because their measures add to 180°. Each angle is the *supplement* of the other.
The sum of the measures of supplementary angles is 180°.

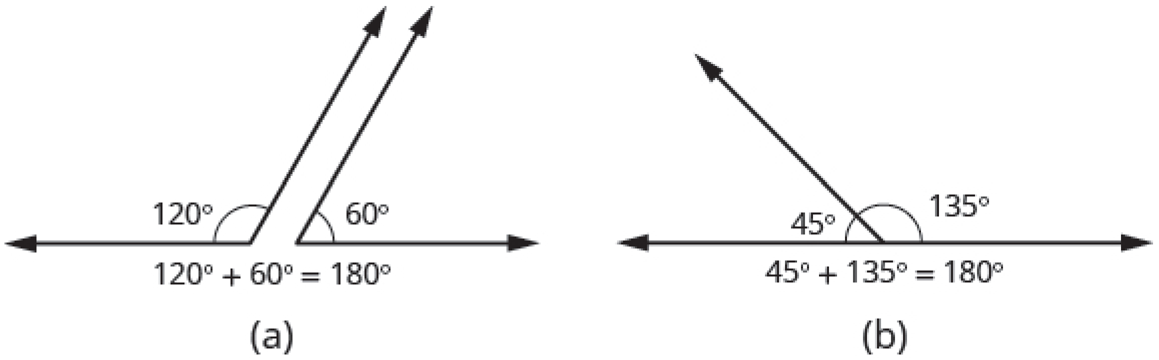


Figure 3

If the sum of the measures of two angles is $90°$, then the angles are complementary angles. In (Figure 4), each pair of angles is complementary, because their measures add to $90°$. Each angle is the *complement* of the other.

The sum of the measures of complementary angles is $90°$.

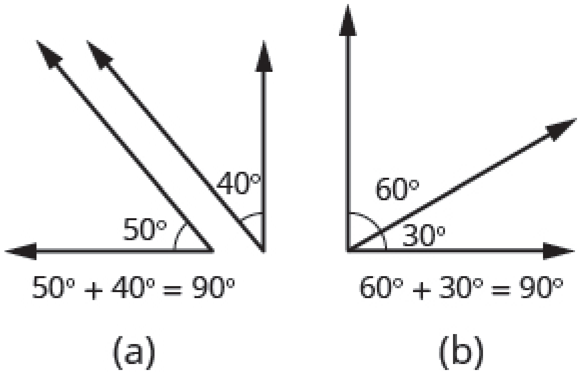


Figure 4

Supplementary and Complementary Angles

If the sum of the measures of two angles is $180°$, then the angles are supplementary.

If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180°$.

If the sum of the measures of two angles is $90°$, then the angles are complementary.

If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90°$.

In this section and the next, you will be introduced to some common geometry formulas. We will adapt our Problem Solving Strategy for Geometry Applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve geometric shapes, it will be helpful to draw a figure and then label it with the information from the problem. We will include this step in the Problem Solving Strategy for Geometry Applications.

HOW TO: Use a Problem Solving Strategy for Geometry Applications

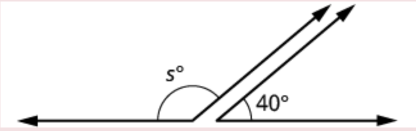
1. **Read** the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for and choose a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

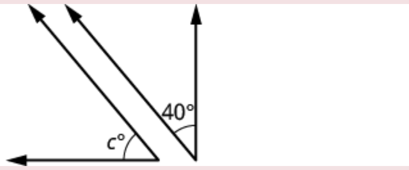
The next example will show how you can use the Problem Solving Strategy for Geometry Applications to answer questions about supplementary and complementary angles.

EXAMPLE 1

An angle measures $40°$. Find a) its supplement, and b) its complement.

Solution

a)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the supplement of a 40° angle.
Step 3. **Name.** Choose a variable to represent it.	let s = the measure of the supplement
Step 4. **Translate.** Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 180$ $s + 40 = 180$
Step 5. **Solve** the equation.	$s = 140$
Step 6. **Check:** $140 + 40 \overset{?}{=} 180$ $180 = 180$ ✓	
Step 7. **Answer** the question.	The supplement of the 40° angle is 140°.

b)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the complement of a 40° angle.
Step 3. **Name.** Choose a variable to represent it.	let c = the measure of the complement
Step 4. **Translate.** Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 90$
Step 5. **Solve** the equation.	$c + 40 = 90$ $c = 50$
Step 6. **Check:** $50 + 40 \overset{?}{=} 90$ $90 = 90$ ✓	
Step 7. **Answer** the question.	The complement of the 40° angle is 50°.

TRY IT 1.1

An angle measures $25°$. Find its: a) supplement b) complement.

Show answer

a. 155°
b. 65°

TRY IT 1.2

An angle measures $77°$. Find its: a) supplement b) complement.

Show answer

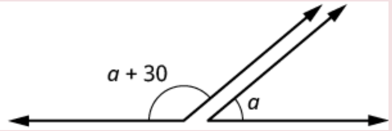
a. 103°
b. 13°

Did you notice that the words complementary and supplementary are in alphabetical order just like 90 and 180 are in numerical order?

EXAMPLE 2

Two angles are supplementary. The larger angle is $30°$ more than the smaller angle. Find the measure of both angles.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measures of both angles
Step 3. **Name.** Choose a variable to represent it. The larger angle is 30° more than the smaller angle.	let a = measure of smaller angle $a + 30$ = measure of larger angle
Step 4. **Translate.** Write the appropriate formula and substitute.	$m\angle A + m\angle B = 180$
Step 5. **Solve** the equation.	$(a + 30) + a = 180$ $2a + 30 = 180$ $2a = 150$ $a = 75$ measure of smaller angle $a + 30$ measure of larger angle $75 + 30$ 105
Step 6. **Check:** $m\angle A + m\angle B = 180$ $75 + 105 \overset{?}{=} 180$ $180 = 180 ✓$	
Step 7. **Answer** the question.	The measures of the angles are 75° and 105°.

TRY IT 2.1

Two angles are supplementary. The larger angle is $100°$ more than the smaller angle. Find the measures of both angles.

Show answer

40°, 140°

TRY IT 2.2

Two angles are complementary. The larger angle is $40°$ more than the smaller angle. Find the measures of both angles.

Show answer
25°, 65°

Use the Properties of Triangles

What do you already know about triangles? Triangle have three sides and three angles. Triangles are named by their vertices. The triangle in (Figure 5) is called ΔABC, read 'triangle ABC'. We label each side with a lower case letter to match the upper case letter of the opposite vertex.
ΔABC has vertices $A, B, \text{ and } C$ and sides $a, b, \text{and } c$.

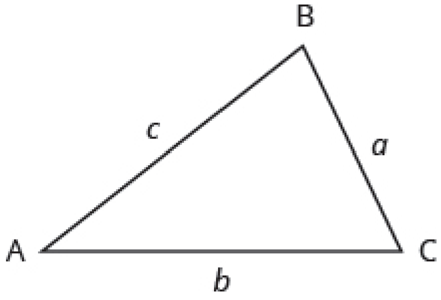


Figure 5

The three angles of a triangle are related in a special way. The sum of their measures is $180°$.
$m\angle A + m\angle B + m\angle C = 180°$

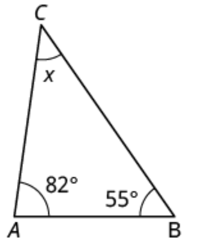
Sum of the Measures of the Angles of a Triangle

For any ΔABC, the sum of the measures of the angles is $180°$.
$m\angle A + m\angle B + m\angle C = 180°$

EXAMPLE 3

The measures of two angles of a triangle are $55°$ and $82°$. Find the measure of the third angle.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measure of the third angle in a triangle
Step 3. **Name.** Choose a variable to represent it.	let *x* = the measure of the angle
Step 4. **Translate.** Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. **Solve** the equation.	$55 + 82 + x = 180$ $137 + x = 180$ $x = 43$
Step 6. **Check:** $55 + 82 + 43 \overset{?}{=} 180$ $180 = 180\checkmark$	
Step 7. **Answer** the question.	The measure of the third angle is 43 degrees.

TRY IT 3.1

The measures of two angles of a triangle are $31°$ and $128°$. Find the measure of the third angle.

Show answer
21°

TRY IT 3.2

A triangle has angles of $49°$ and $75°$. Find the measure of the third angle.

Show answer
56°

Right Triangles

Some triangles have special names. We will look first at the right triangle. A right triangle has one 90° angle, which is often marked with the symbol shown in (Figure 6).

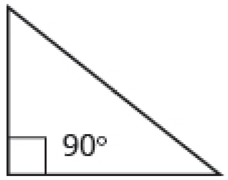


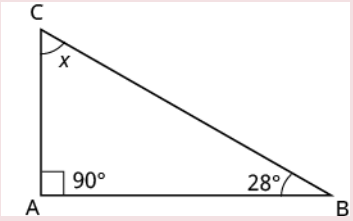
Figure 6

If we know that a triangle is a right triangle, we know that one angle measures 90° so we only need the measure of one of the other angles in order to determine the measure of the third angle.

EXAMPLE 4

One angle of a right triangle measures 28°. What is the measure of the third angle?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measure of an angle
Step 3. **Name.** Choose a variable to represent it.	let x = the measure of the angle
Step 4. **Translate.** Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. **Solve** the equation.	$x + 90 + 28 = 180$ $x + 118 = 180$ $x = 62$
Step 6. **Check:** $180 \overset{?}{=} 90 + 28 + 62$ $180 = 180$ ✓	
Step 7. **Answer** the question.	The measure of the third angle is 62°.

TRY IT 4.1

One angle of a right triangle measures $56°$. What is the measure of the other angle?

Show answer
34°

TRY IT 4.2

One angle of a right triangle measures $45°$. What is the measure of the other angle?

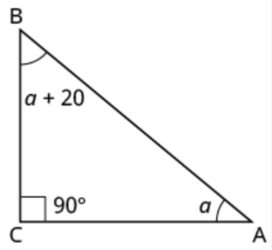
Show answer
45°

In the examples so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. So we will wait to draw the figure until we write expressions for all the angles we are looking for.

EXAMPLE 5

The measure of one angle of a right triangle is $20°$ more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the measures of all three angles
Step 3. **Name.** Choose a variable to represent it. Now draw the figure and label it with the given information.	Let $a = 1^{st}$ angle $a + 20 = 2^{nd}$ angle $90 = 3^{rd}$ angle (the right angle) 
Step 4. **Translate.** Write the appropriate formula and substitute into the formula.	$m\angle A + m\angle B + m\angle C = 180$ $a + (a + 20) + 90 = 180$
Step 5. **Solve** the equation.	$2a + 110 = 180$ $2a = 70$ $a = 35$ first angle $a + 20$ second angle $35 + 20$ 55 90 third angle
Step 6. **Check:** $35 + 55 + 90 \overset{?}{=} 180$ $180 = 180$ ✓	
Step 7. **Answer** the question.	The three angles measure 35°, 55°, and 90°.

TRY IT 5.1

The measure of one angle of a right triangle is $50°$ more than the measure of the smallest angle. Find the measures of all three angles.

Show answer
20°, 70°, 90°

TRY IT 5.2

The measure of one angle of a right triangle is $30°$ more than the measure of the smallest angle. Find the measures of all three angles.

Show answer
30°, 60°, 90°

Similar Triangles

When we use a map to plan a trip, a sketch to build a bookcase, or a pattern to sew a dress, we are working with similar figures. In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures. One is a scale model of the other. The corresponding sides of the two figures have the same ratio, and all their corresponding angles are have the same measures.

The two triangles in (Figure 7) are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ and their corresponding angles have equal measures.

ΔABC and ΔXYZ are similar triangles. Their corresponding sides have the same ratio and the corresponding angles have the same measure.

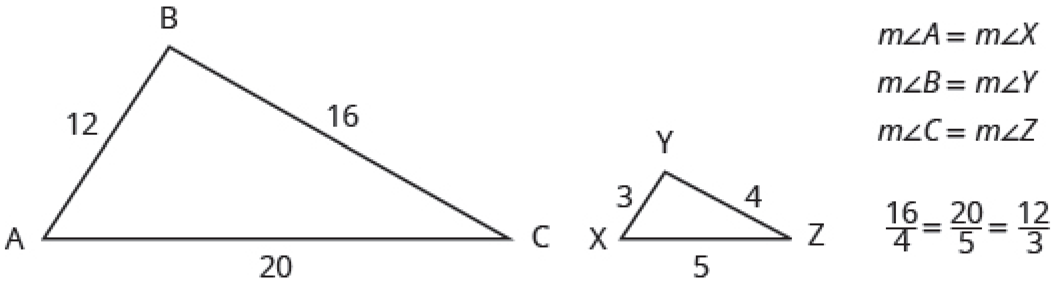
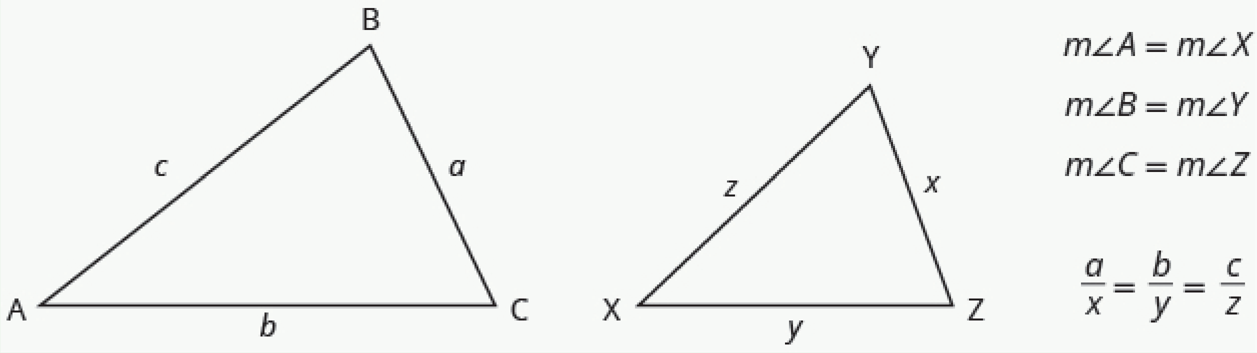


Figure 7

Properties of Similar Triangles

If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.



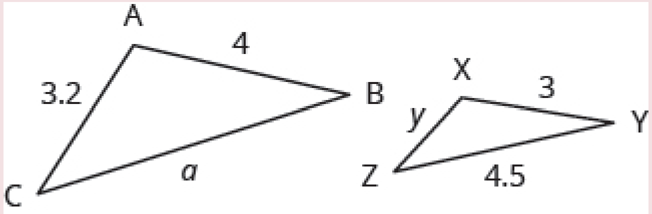
The length of a side of a triangle may be referred to by its endpoints, two vertices of the triangle. For example, in ΔABC:

the length a can also be written BC
the length b can also be written AC
the length c can also be written AB

We will often use this notation when we solve similar triangles because it will help us match up the corresponding side lengths.

EXAMPLE 6

ΔABC and ΔXYZ are similar triangles. The lengths of two sides of each triangle are shown. Find the lengths of the third side of each triangle.

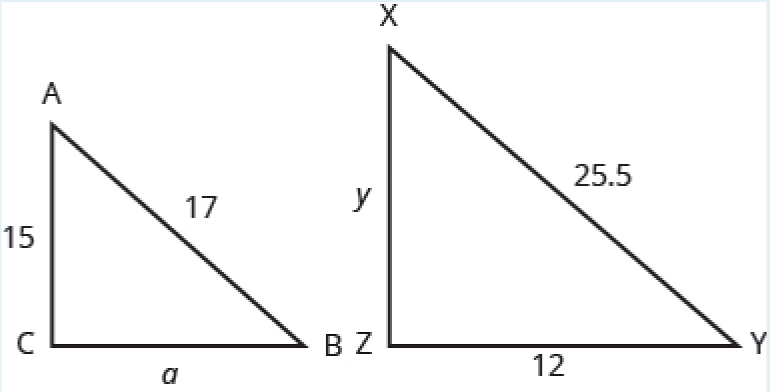


Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	The figure is provided.
Step 2. **Identify** what you are looking for.	The length of the sides of similar triangles
Step 3. **Name.** Choose a variable to represent it.	Let *a* = length of the third side of ΔABC *y* = length of the third side ΔXYZ
Step 4. **Translate.**	The triangles are similar, so the corresponding sides are in the same ratio. So $\frac{AB}{XY}=\frac{BC}{YZ}=\frac{AC}{XZ}$ Since the side $AB=4$ corresponds to the side $XY=3$, we will use the ratio $\frac{\text{AB}}{\text{XY}}=\frac{4}{3}$ to find the other sides. Be careful to match up corresponding sides correctly. To find *a*: To find *y*: sides of large triangle → $\frac{AB}{XY}=\frac{BC}{YZ}$ $\frac{AB}{XY}=\frac{AC}{XZ}$ sides of small triangle → $\frac{4}{3}=\frac{a}{4.5}$ $\frac{4}{3}=\frac{3.2}{y}$
Step 5. **Solve** the equation.	$3a=4(4.5)$ $4y=3(3.2)$ $3a=18$ $4y=9.6$ $a=6$ $y=2.4$
Step 6. **Check.**	$\frac{4}{3}\overset{?}{=}\frac{6}{4.5}$ $\frac{4}{3}\overset{?}{=}\frac{3.2}{2.4}$ $4(4.5)\overset{?}{=}6(3)$ $4(2.4)\overset{?}{=}3.2(3)$ $18=18\checkmark$ $9.6=9.6\checkmark$
Step 7. **Answer** the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

TRY IT 6.1

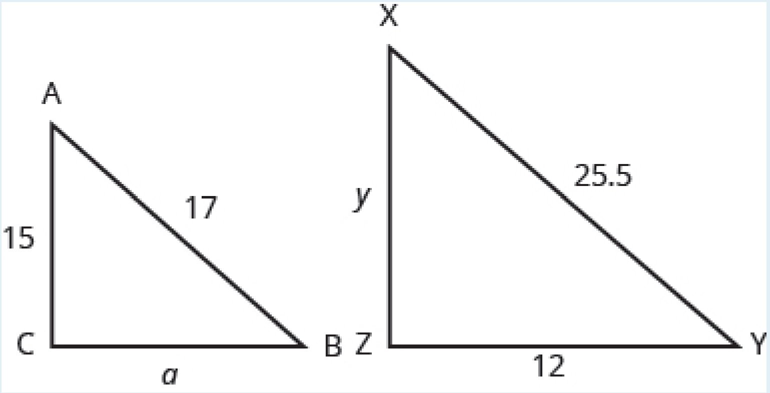
ΔABC is similar to ΔXYZ. Find a.



Show answer
8

TRY IT 6.2

ΔABC is similar to ΔXYZ. Find y.



Show answer
22.5

Use the Pythagorean Theorem

The Pythagorean Theorem is a special property of right triangles that has been used since ancient times. It is named after the Greek philosopher and mathematician Pythagoras who lived around 500 BCE.

Remember that a right triangle has a $90°$ angle, which we usually mark with a small square in the corner. The side of the triangle opposite the $90°$ angle is called the hypotenuse, and the other two sides are called the legs. See (Figure 8).

In a right triangle, the side opposite the $90°$ angle is called the hypotenuse and each of the other sides is called a leg.

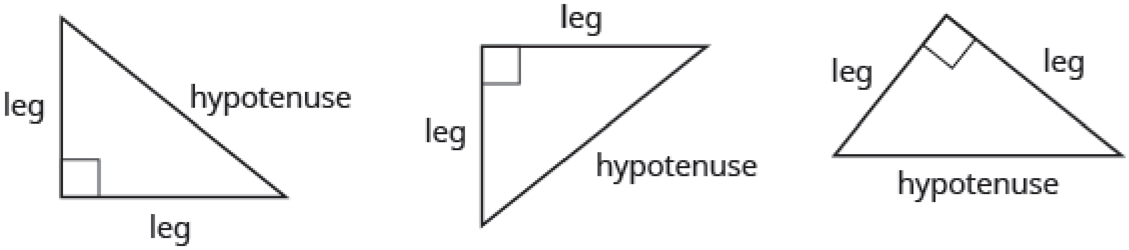


Figure 8

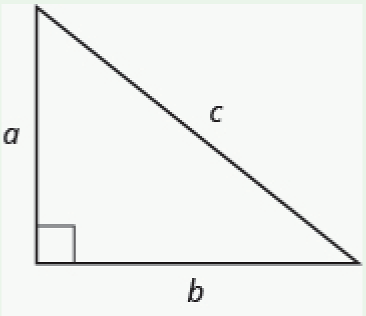
The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

The Pythagorean Theorem

In any right triangle $\Delta ABC,$

$$a^2 + b^2 = c^2$$

where c is the length of the hypotenuse a and b are the lengths of the legs.



To solve problems that use the Pythagorean Theorem, we will need to find square roots. We defined the notation $\sqrt{m}$ in this way:

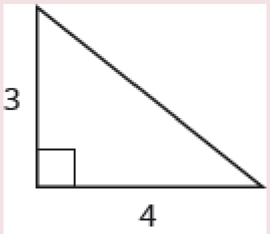
$$\text{If } m = n^2, \text{ then } \sqrt{m} = n \text{ for } n \geq 0$$

For example, we found that $\sqrt{25}$ is 5 because $5^2 = 25$.

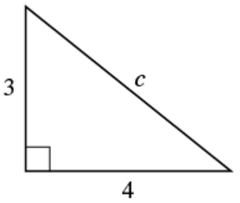
We will use this definition of square roots to solve for the length of a side in a right triangle.

EXAMPLE 7

Use the Pythagorean Theorem to find the length of the hypotenuse.

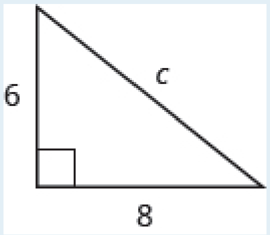


Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length of the hypotenuse of the triangle
Step 3. **Name.** Choose a variable to represent it.	Let $c = \text{the length of the hypotenuse}$ 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $3^2 + 4^2 = c^2$
Step 5. **Solve** the equation.	$9 + 16 = c^2$ $25 = c^2$ $\sqrt{25} = c^2$ $5 = c$
Step 6. **Check:** $3^2 + 4^2 = 5^2$ $9 + 16 \overset{?}{=} 25$ $25 = 25 \checkmark$	
Step 7. **Answer** the question.	The length of the hypotenuse is 5.

TRY IT 7.1

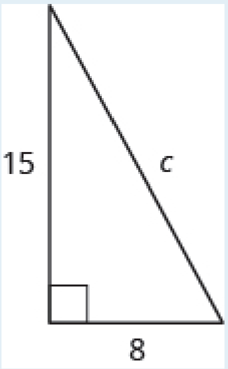
Use the Pythagorean Theorem to find the length of the hypotenuse.



Show answer
10

TRY IT 7.2

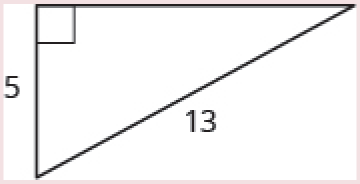
Use the Pythagorean Theorem to find the length of the hypotenuse.



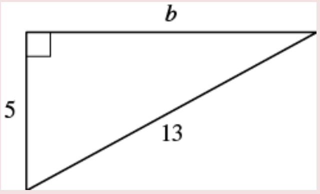
Show answer
17

EXAMPLE 8

Use the Pythagorean Theorem to find the length of the longer leg.

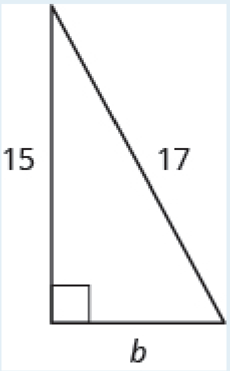


Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	The length of the leg of the triangle
Step 3. **Name.** Choose a variable to represent it.	Let $b =$ the leg of the triangle Label side b 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $5^2 + b^2 = 13^2$
Step 5. **Solve** the equation. Isolate the variable term. Use the definition of the square root. Simplify.	$25 + b^2 = 169$ $b^2 = 144$ $b^2 = \sqrt{144}$ $b = 12$
Step 6. **Check:**	$5^2 + 12^2 \overset{?}{=} 13^2$ $25 + 144 \overset{?}{=} 169$ $169 = 169$ ✓
Step 7. **Answer** the question.	The length of the leg is 12.

TRY IT 8.1

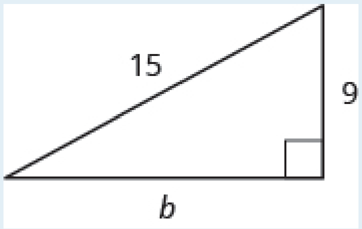
Use the Pythagorean Theorem to find the length of the leg.



Show answer
8

TRY IT 8.2

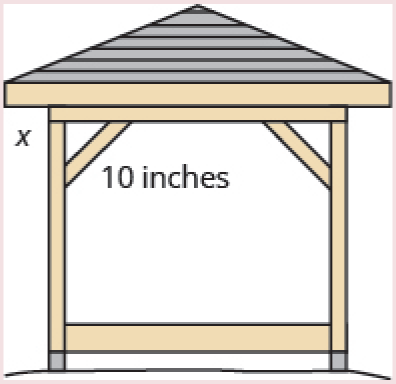
Use the Pythagorean Theorem to find the length of the leg.



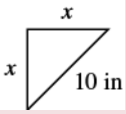
Show answer
12

EXAMPLE 9

Kelvin is building a gazebo and wants to brace each corner by placing a 10-inch wooden bracket diagonally as shown. How far below the corner should he fasten the bracket if he wants the distances from the corner to each end of the bracket to be equal? Approximate to the nearest tenth of an inch.



Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the distance from the corner that the bracket should be attached
Step 3. **Name.** Choose a variable to represent it.	Let x = the distance from the corner 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2+b^2=c^2$ $x^2+x^2=10^2$
Step 5. **Solve** the equation. Isolate the variable. Use the definition of the square root. Simplify. Approximate to the nearest tenth.	$2x^2=100$ $x^2=50$ $x=\sqrt{50}$ $b\approx 7.1$
Step 6. **Check:** $a^2+b^2=c^2$ $(7.1)^2+(7.1)^2 \overset{?}{\approx} 10^2$ Yes.	
Step 7. **Answer** the question.	Kelvin should fasten each piece of wood approximately 7.1″ from the corner.

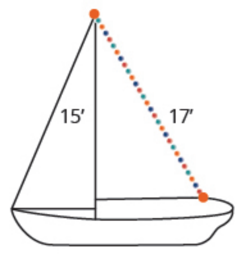
TRY IT 9.1

John puts the base of a 13-ft ladder 5 feet from the wall of his house. How far up the wall does the ladder reach?

Show answer
12 feet

TRY IT 9.2

Randy wants to attach a 17-ft string of lights to the top of the 15-ft mast of his sailboat. How far from the base of the mast should he attach the end of the light string?

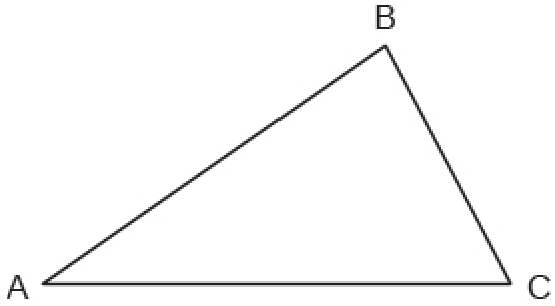


Show answer
8 feet

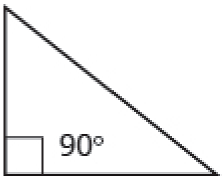
Key Concepts

- **Supplementary and Complementary Angles**
    - If the sum of the measures of two angles is 180°, then the angles are supplementary.
    - If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180$.
    - If the sum of the measures of two angles is 90°, then the angles are complementary.
    - If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90$.
- **Solve Geometry Applications**
    1. Read the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
    2. Identify what you are looking for.
    3. Name what you are looking for and choose a variable to represent it.
    4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
    5. Solve the equation using good algebra techniques.
    6. Check the answer in the problem and make sure it makes sense.
    7. Answer the question with a complete sentence.

- **Sum of the Measures of the Angles of a Triangle**
    - For any ΔABC, the sum of the measures is 180°
    - $m\angle A + m\angle B = 180$



- **Right Triangle**
    - A right triangle is a triangle that has one 90° angle, which is often marked with a ? symbol.



- **Properties of Similar Triangles**
    - If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths have the same ratio.

Glossary

angle
An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle.

complementary angles
If the sum of the measures of two angles is $90°$, then they are called complementary angles.

hypotenuse
The side of the triangle opposite the 90° angle is called the hypotenuse.

legs of a right triangle
The sides of a right triangle adjacent to the right angle are called the legs.

right triangle
A right triangle is a triangle that has one $90°$ angle.

similar figures
In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures.

supplementary angles
If the sum of the measures of two angles is $180°$, then they are called supplementary angles.

triangle
A triangle is a geometric figure with three sides and three angles.

vertex of an angle
When two rays meet to form an angle, the common endpoint is called the vertex of the angle.

Practice Makes Perfect

Use the Properties of Angles

In the following exercises, find a) the supplement and b) the complement of the given angle.

1. 53°

2. 16°

3. 29°

4. 72°

In the following exercises, use the properties of angles to solve.

5. Find the supplement of a 135° angle.

6. Find the complement of a 38° angle.

7. Find the complement of a 27.5° angle.

8. Find the supplement of a 109.5° angle.

9. Two angles are supplementary. The larger angle is 56° more than the smaller angle. Find the measures of both angles.

10. Two angles are supplementary. The smaller angle is 36° less than the larger angle. Find the measures of both angles.

11. Two angles are complementary. The smaller angle is 34° less than the larger angle. Find the measures of both angles.

12. Two angles are complementary. The larger angle is 52° more than the smaller angle. Find the measures of both angles.

Use the Properties of Triangles

In the following exercises, solve using properties of triangles.

13. The measures of two angles of a triangle are 26° and 98°. Find the measure of the third angle.

14. The measures of two angles of a triangle are 61° and 84°. Find the measure of the third angle.

15. The measures of two angles of a triangle are 105° and 31°. Find the measure of the third angle.

16. The measures of two angles of a triangle are 47° and 72°. Find the measure of the third angle.

17. One angle of a right triangle measures 33°. What is the measure of the other angle?

18. One angle of a right triangle measures 51°. What is the measure of the other angle?

19. One angle of a right triangle measures 22.5°. What is the measure of the other angle?

20. One angle of a right triangle measures 36.5°. What is the measure of the other angle?

21. The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

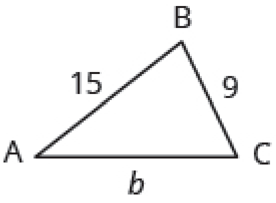
22. The measure of the smallest angle of a right triangle is 20° less than the measure of the other small angle. Find the measures of all three angles.

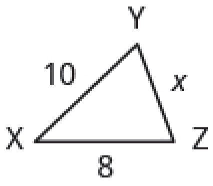
23. The angles in a triangle are such that the measure of one angle is twice the measure of the smallest angle, while the measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

24. The angles in a triangle are such that the measure of one angle is 20° more than the measure of the smallest angle, while the measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

Find the Length of the Missing Side

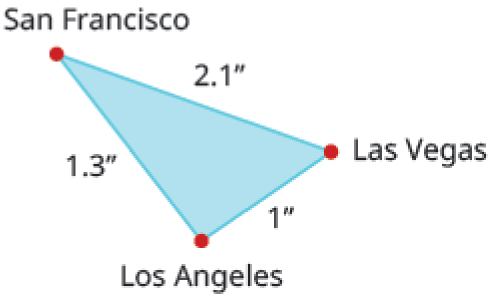
In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.





25. side b	26. side x

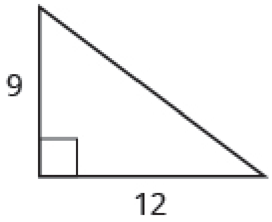
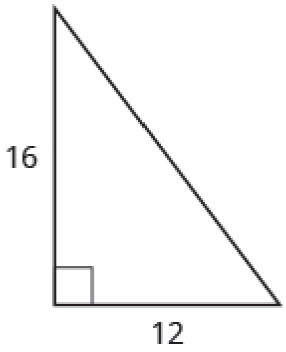
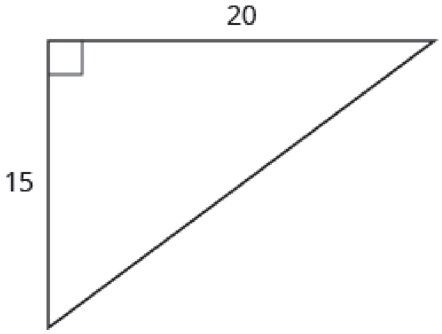
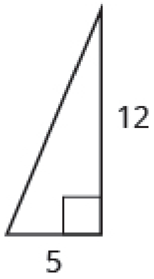
On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. The actual distance from Los Angeles to Las Vegas is 270 miles.



27. Find the distance from Los Angeles to San Francisco.	28. Find the distance from San Francisco to Las Vegas.

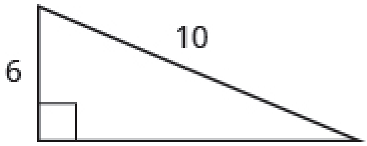
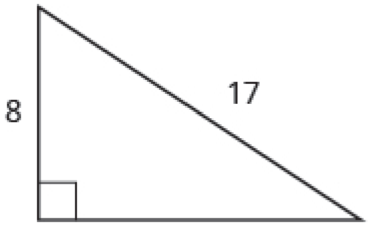
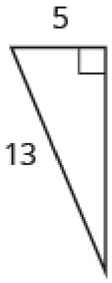
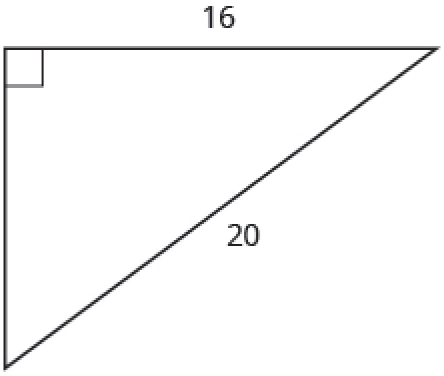
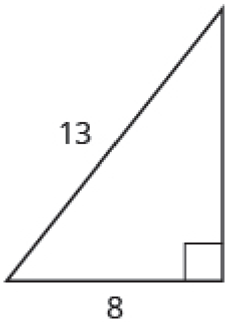
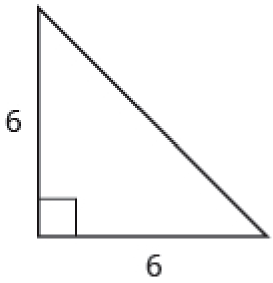
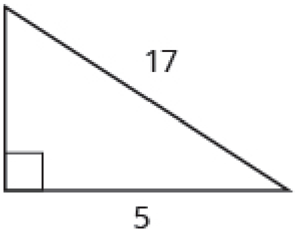
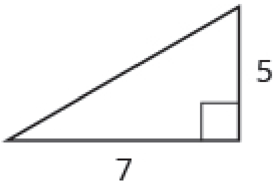
Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

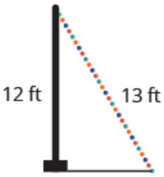
29. 	30. 
31. 	32. 

Find the Length of the Missing Side

In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

33. 	34. 
35. 	36. 
37. 	38. 
39. 	40. 

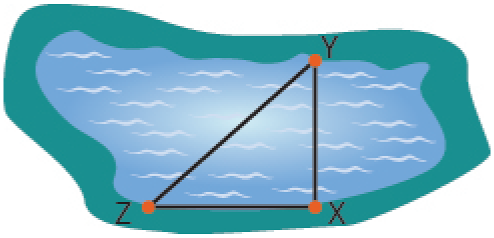
In the following exercises, solve. Approximate to the nearest tenth, if necessary.

41. A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display. How far from the base of the pole should the end of the string of lights be anchored?  	42. Pam wants to put a banner across her garage door to congratulate her son on his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door? 
43. Chi is planning to put a path of paving stones through her flower garden. The flower garden is a square with sides of 10 feet. What will the length of the path be?  	44. Brian borrowed a 20-foot extension ladder to paint his house. If he sets the base of the ladder 6 feet from the house, how far up will the top of the ladder reach? 

Everyday Math

45. **Building a scale model** Joe wants to build a doll house for his daughter. He wants the doll house to look just like his house. His house is 30 feet wide and 35 feet tall at the highest point of the roof. If the dollhouse will be 2.5 feet wide, how tall will its highest point be?

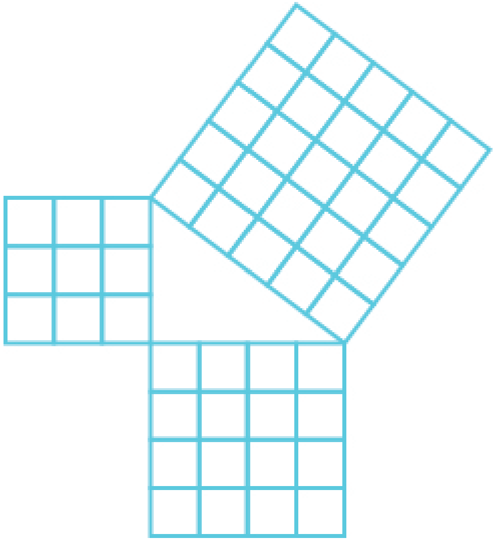
46. **Measurement** A city engineer plans to build a footbridge across a lake from point X to point Y, as shown in the picture below. To find the length of the footbridge, she draws a right triangle XYZ, with right angle at X. She measures the distance from X to Z, 800 feet, and from Y to Z, $1,000$ feet. How long will the bridge be?



Writing Exercises

47. Write three of the properties of triangles from this section and then explain each in your own words.

48. Explain how the figure below illustrates the Pythagorean Theorem for a triangle with legs of length 3 and 4.



Answers

1. a) 127° b) 37°	3. a) 151° b) 61°	5. 45°
7. 62.5°	9. 62°, 118°	11. 62°, 28°
13. 56°	15. 44°	17. 57°
19. 67.5°	21. 45°, 45°, 90°	23. 30°, 60°, 90°
25. 12	27. 351 miles	29. 15
31. 25	33. 8	35. 12
37. 10.2	39. 8	41. 5 feet
43. 14.1 feet	45. 2.9 feet	47. Answers will vary.

Attributions

This chapter has been adapted from “Use Properties of Angles, Triangles, and the Pythagorean Theorem” in *Prealgebra* (OpenStax) by Lynn Marecek, MaryAnne Anthony-Smith, and Andrea Honeycutt Mathis, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Copyright page for more information.

9.2 Solve Applications: Sine, Cosine and Tangent Ratios.

Learning Objectives

By the end of this section, you will be able to:

- Find missing side of a right triangle using sine, cosine, or tangent ratios
- Find missing angle of a right triangle using sine, cosine, or tangent ratios
- Solve applications using right angle trigonometry

Now, that we know the fundamentals of algebra and geometry associated with a right triangle, we can start exploring trigonometry. Many real life problems can be represented and solved using right angle trigonometry.

Sine, Cosine, and Tangent Ratios

We know that any right triangle has three sides and a right angle. The side opposite to the right angle is called the hypotenuse. The other two angles in a right triangle are acute angles (with a measure less than 90 degrees). One of those angles we call reference angle and we use θ (theta) to represent it.

The hypotenuse is always the longest side of a right triangle. The other two sides are called opposite side and adjacent side. The names of those sides depends on which of the two acute angles is being used as a reference angle.

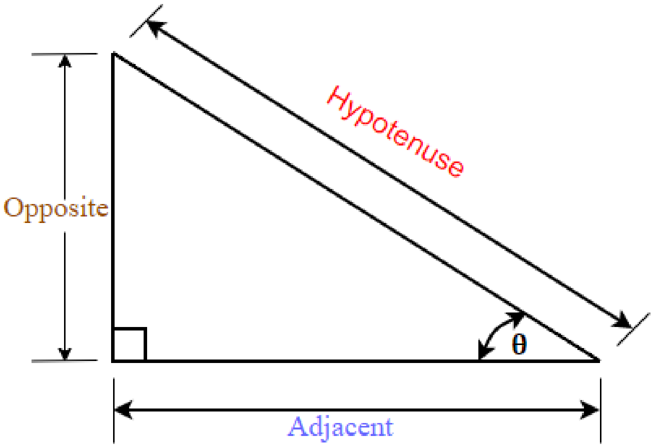
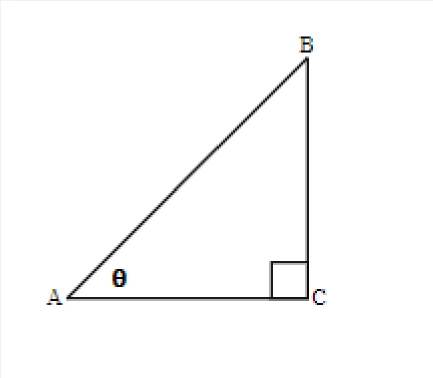


Figure 1.

In the right triangle each side is labeled with a lowercase letter to match the uppercase letter of the opposite vertex.

EXAMPLE 1

Label the sides of the triangle and find the hypotenuse, opposite, and adjacent.



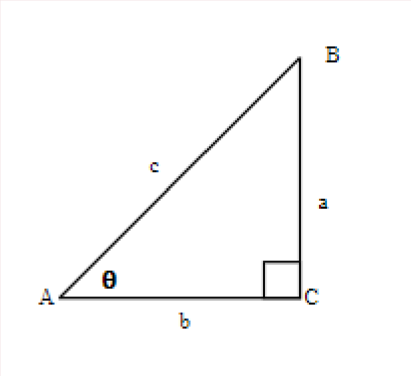
Solution

We labeled the sides with a lowercase letter to match the uppercase letter of the opposite vertex.

c is hypotenuse

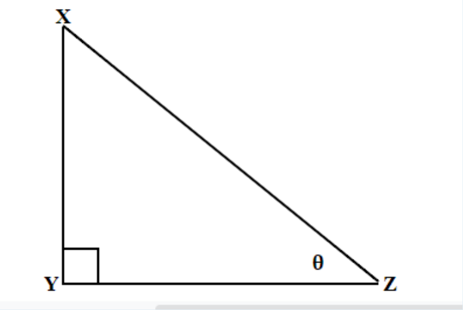
a is opposite

b is adjacent



TRY IT 1.1

Label the sides of the triangle and find the hypotenuse, opposite and adjacent.



Show answer

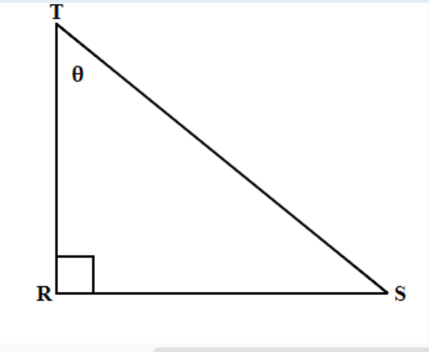
y is hypotenuse

z is opposite

x is adjacent

TRY IT 1.2

Label the sides of the triangle and find the hypotenuse, opposite and adjacent.



Show answer

r is hypotenuse

t is opposite

s is adjacent

Trigonometric Ratios

Trigonometric ratios are the ratios of the sides in the right triangle. For any right triangle we can define three basic trigonometric ratios: sine, cosine, and tangent.

Let us refer to Figure 1 and define the three basic trigonometric ratios as:

Three Basic Trigonometric Ratios

- sine θ = $\frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse side}}$
- cosine θ = $\frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse side}}$
- tangent θ = $\frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}}$

Where θ is the measure of a reference angle measured in degrees.

Very often we use the abbreviations for sine, cosine, and tangent ratios.

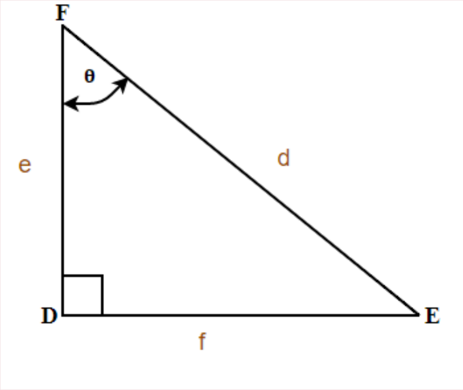
- sin θ = $\frac{\text{opp}}{\text{hyp}}$
- cos θ = $\frac{\text{adj}}{\text{hyp}}$
- tan θ = $\frac{\text{opp}}{\text{adj}}$

Some people remember the definition of the trigonometric ratios as SOH CAH TOA.

Let's use the ΔDEF from Example 1 to find the three ratios.

EXAMPLE 2

For the given triangle find the sine, cosine and tangent ratio.



Solution

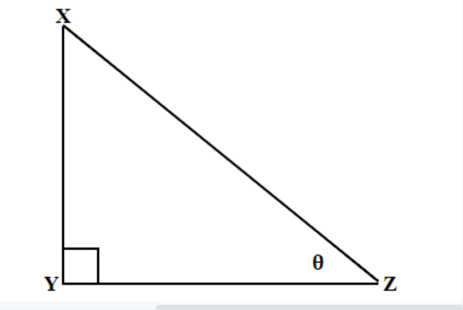
sin θ = $\frac{\text{f}}{\text{d}}$

cos θ = $\frac{\text{e}}{\text{d}}$

tan θ = $\frac{\text{f}}{\text{e}}$

TRY IT 2.1

For the given triangle find the sine cosine and tangent ratio.



Show answer

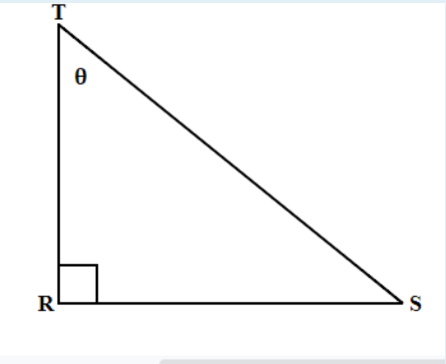
$\sin\theta = \frac{z}{y}$

$\cos\theta = \frac{x}{y}$

$\tan\theta = \frac{z}{x}$

TRY IT 2.2

For the given triangle find the sine, cosine and tangent ratio.



Show answer

$\sin\theta = \frac{t}{r}$

$\cos\theta = \frac{s}{r}$

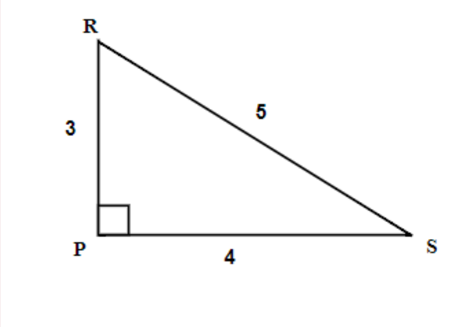
$\tan\theta = \frac{t}{s}$

In Example 2, our reference angles can be $\angle E$ or $\angle F$. Using the definition of trigonometric ratios, we can write sinE=$\frac{e}{d}$, cosE=$\frac{f}{d}$, and tanE= $\frac{e}{f}$.

When calculating we will usually round the ratios to four decimal places and at the end our final answer to one decimal place unless stated otherwise.

EXAMPLE 3

For the given triangle find the sine, cosine and tangent ratios. If necessary round to four decimal places.



Solution

We have two possible reference angles: R an S.

Using the definitions, the trigonometric ratios for angle R are:

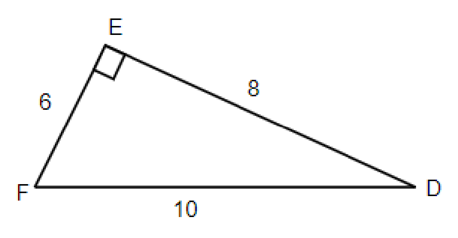
- sin R= $\frac{4}{5}$ = 0.8
- cos R= $\frac{3}{5}$ = 0.6
- tan R=$\frac{4}{3}$ = 1.3333…

Using the definitions, the trigonometric ratios for angle S:

- sin S = $\frac{3}{5}$ = 0.6
- cos S = $\frac{4}{5}$ = 0.8
- tan S = $\frac{3}{4}$ = 0.75

TRY IT 3.1

For the given triangle find the sine, cosine, and tangent ratios. If necessary round to four decimal places.

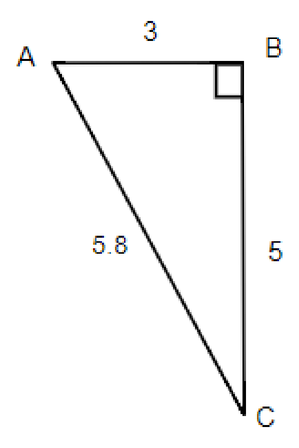


Show answer

- sin F = $\frac{8}{10}$ = 0.8
- cos F = $\frac{6}{10}$ =0.6
- tan F = $\frac{8}{6}$ = 1.3333…
- sin D =$\frac{6}{10}$ = 0.6
- cos D =$\frac{8}{10}$ = 0.8
- tan D =$\frac{6}{8}$ = 0.75

TRY IT 3.2

For given triangle find the sine, cosine and tangent ratios. If necessary round to four decimal places.



Show answer

- sin A = $\frac{5}{5.8}$ = 0.8621
- cos A = $\frac{3}{5.8}$ =0.5172
- tan A = $\frac{5}{3}$ = 1.6667

- sin C = $\frac{3}{5.8}$ = 0.5172
- cos C = $\frac{5}{5.8}$ = 0.8621
- tan C = $\frac{3}{5}$ = 0.6

Now, let us use a scientific calculator to find the trigonometric ratios. Can you find the sin, cos, and tan buttons on your calculator? To find the trigonometric ratios make sure your calculator is in Degree Mode.

EXAMPLE 4

Using a calculator find the trigonometric ratios. If necessary, round to 4 decimal places.

a) sin 30°

b) cos 45°

c) tan 60°

Solution

Make sure your calculator is in Degree Mode.

a) Using a calculator find that sin 30° = 0.5

b) Using a calculator find that cos 45° = 0.7071 Rounded to 4 decimal places.

c) Using a calculator find that tan 60° = 1.7321 Rounded to 4 decimal places.

TRY IT 4.1

Find the trigonometric ratios. If necessary, round to 4 decimal places.

a) sin 60°

b) cos 30°

c) tan 45°

Show answer
a) sin 60° = 0.8660

b) cos 30° = 0.8660

c) tan 45° = 1

TRY IT 4.2

Find the trigonometric ratios. If necessary, round to 4 decimal places.

a) sin 35°

b) cos 67°

c) tan 83°

Show answer

a) sin 35° = 0.5736

b) cos 67 ° = 0.3907

c) tan 83° = 8.1443

Finding Missing Sides of a Right Triangle

In this section you will be using trigonometric ratios to solve right triangle problems. We will adapt our problem solving strategy for trigonometry applications. In addition, since those problems will involve the right triangle, it is helpful to draw it (if the drawing is not given) and label it with the given information.We will include this in the first step of the problem solving strategy for trigonometry applications.

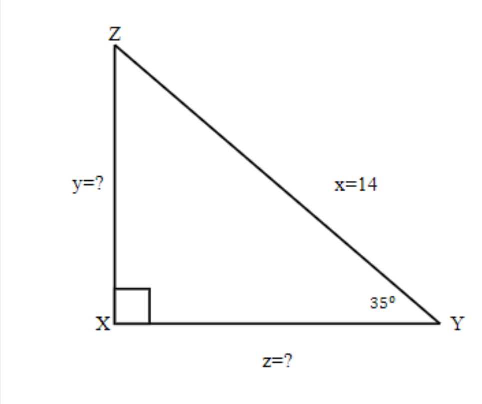
HOW TO: Solve Trigonometry Applications

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.
2. **Identify** what we are looking for.
3. **Label** what we are looking for by choosing a variable to represent it.
4. **Find** the required trigonometric ratio.
5. **Solve** the ratio using good algebra techniques.
6. **Check** the answer by substituting it back into the ratio in step 4 and by making sure it makes sense in the context of the problem.
7. **Answer** the question with a complete sentence.

In the next few examples, having given the measure of one acute angle and the length of one side of the right triangle, we will solve the right triangle for the missing sides.

EXAMPLE 5

Find the missing sides. Round your final answer to two decimal places

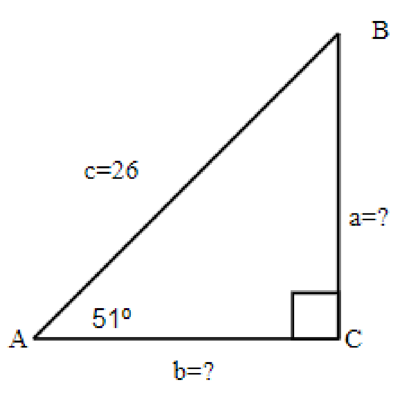


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle Y is our reference angle, y is opposite side, z is adjacent side, and x=14 is the hypotenuse.	
2. **Identify** what we are looking for.	a) the opposite side	b) adjacent side
3.**Label** what we are looking for by choosing a variable to represent it.	y=?	z=?
4. **Find** the required trigonometric ratio.	$\sin 35° = \frac{y}{14}$	$\cos 35° = \frac{z}{14}$
5. **Solve** the ratio using good algebra techniques.	14 sin 35° = y 8.03 = y	14 cos 35° = z 11.47 = z
6. **Check** the answer in the problem and by making sure it makes sense.	$0.57 \overset{?}{=} 8.03 \div 14$ 0.57 = 0.57 ✓	$0.82 \overset{?}{=} 11.47 \div 14$ 0.82 = 0.82 ✓
7. **Answer** the question with a complete sentence.	The opposite side is 8.03	The adjacent side is 11.47

TRY IT 5.1

Find the missing sides. Round your final answer to one decimal place.

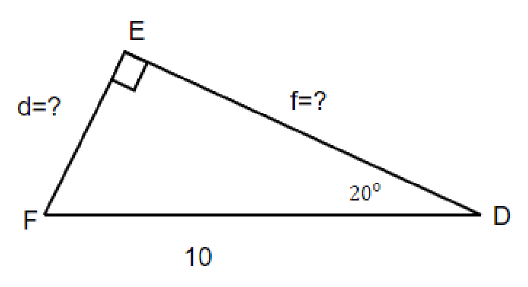


Show answer
a = 20.2

b = 16.4

TRY IT 5.2

Find the missing sides. Round your final answer to one decimal place.

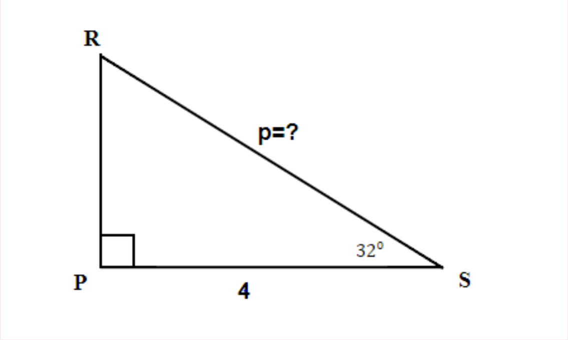


Show answer
d = 3.4

f = 9.4

EXAMPLE 6

Find the hypotenuse. Round your final answer to one decimal place.

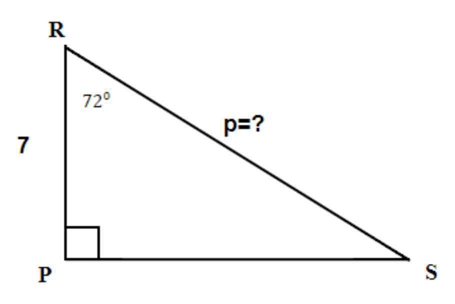


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle S is our reference angle, s is opposite side, r = 4 is the adjacent side, and p is the hypotenuse
2. **Identify** what we are looking for.	the hypotenuse
3.**Label** what we are looking for by choosing a variable to represent it.	p=?
4. **Find** the required trigonometric ratio.	$\cos 32° = \frac{4}{p}$
5. **Solve** the ratio using good algebra techniques.	$0.8480 = \frac{4}{p}$ p = 4.7170 Rounding the ratios to 4 decimal places
6. **Check** the answer in the problem and by making sure it makes sense.	$0.8480 \overset{?}{=} \frac{4}{4.7170}$ 0.8480 = 0.8480 ✓
7. **Answer** the question with a complete sentence.	The hypotenuse is 4.7 Round my final answer to one decimal place.

TRY IT 6.1..

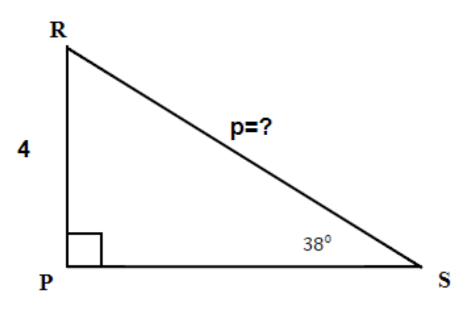
Find the hypotenuse. Round your final answer to one decimal place.



Show answer
p = 22.7

TRY IT 6.2

Find the hypotenuse. Round your final answer to one decimal place.



Show answer
p = 6.5

Finding Missing Angles of a Right Triangle

Sometimes we have a right triangle with only the sides given. How can we find the missing angles? To find the missing angles, we use the inverse of the trigonometric ratios. The inverse buttons $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are on your scientific calculator.

EXAMPLE 7

Find the angles. Round your final answer to one decimal place.

a) sin A = 0.5

b) cos B = 0.9735

c) tan C = 2.89358

Solution

Use your calculator and press the 2nd FUNCTION key and then press the SIN, COS, or TAN key

a) $A = \sin^{-1} 0.5$

$\angle A$ = 30°

b) $B = \cos^{-1} 0.9735$

$\angle B$ = 13.2° Rounded to one decimal place

c) $C = \tan^{-1} 2.89358$

$\angle C$ = 70.9° Rounded to one decimal place

TRY IT 7.1

Find the angles. Round your final answer to one decimal place.

a) sin X = 1

b) cos Y = 0.375

c) tan Z = 1.676767

Show answer

a) $\angle X$ = 90°

b) $\angle Y$ = 68°

c) $\angle Z$ = 59.2°

TRY IT 7.2

Find the angles. Round your final answer to one decimal place.

a) sin C = 0

b) cos D = 0.95

c) tan F = 6.3333

Show answer

a) $\angle C$ = 0°

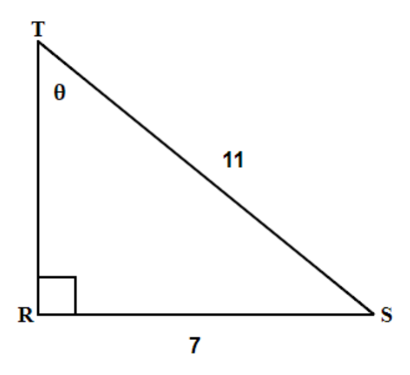
b) $\angle D$ = 18.2°

c) $\angle F$ = 81°

In the example below we have a right triangle with two sides given. Our acute angles are missing. Let us see what the steps are to find the missing angles.

EXAMPLE 8

Find the missing $\angle T$. Round your final answer to one decimal place.

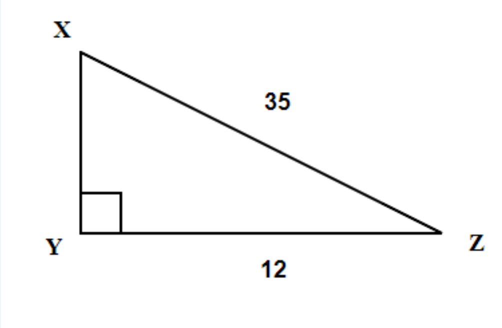


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle T is our reference angle, t = 7 is the opposite side, s is adjacent side, and r =11 is the hypotenuse
2. **Identify** what we are looking for.	angle T
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle T$ =?
4. **Find** the required trigonometric ratio.	sin T = $\frac{7}{11}$
5. **Solve** the ratio using good algebra techniques.	sin T = 0.6364 T = $\sin^{-1}0.6364$ $\angle T$ = 39.5239°
6. **Check** the answer in the problem and by making sure it makes sense.	sin 39.5239° $\overset{?}{=}$ 0.6364 0.6364 = 0.6364 ✓
7. **Answer** the question with a complete sentence.	The missing angle T is 39.5°.

TRY IT 8.1

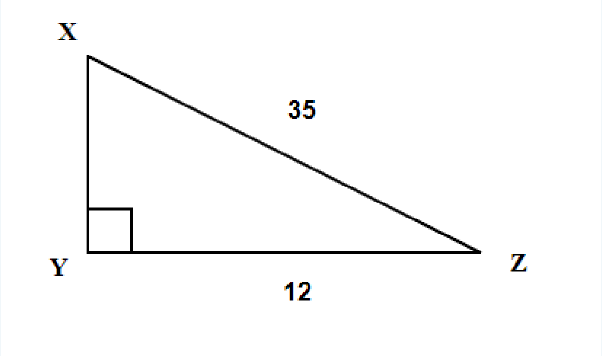
Find the missing angle X. Round your final answer to one decimal place.



Show answer
20.1°

TRY IT 8.2

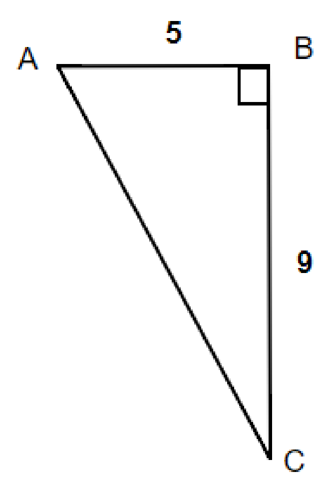
Find the missing angle Z. Round your final answer to one decimal place.



Show answer
69.9°

EXAMPLE 9

Find the missing angle A. Round your final answer to one decimal place.

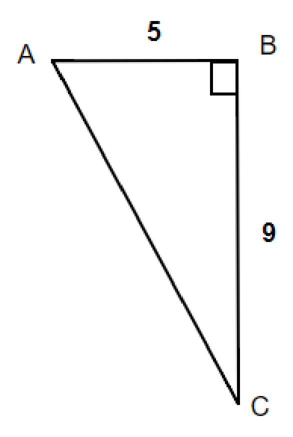


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle A is our reference angle, $a = 9$ is the opposite side, $c = 5$ is the adjacent side, and b is the hypotenuse
2. **Identify** what we are looking for.	angle A
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle A = ?$
4. **Find** the required trigonometric ratio.	$\tan A = \frac{9}{5}$
5. **Solve** the ratio using good algebra techniques.	$\tan A = 1.8$ $A = \tan^{-1} 1.8$ $\angle A = 60.9°$
6. **Check** the answer in the problem and by making sure it makes sense.	$\tan 60.9° \overset{?}{=} 1.8$ $1.8 = 1.8$ ✓
7. **Answer** the question with a complete sentence.	The missing angle A is 60.9°.

TRY IT 9.1

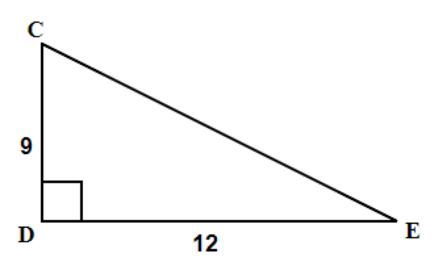
Find the missing angle C. Round your final answer to one decimal place.



Show answer
29.1°

TRY IT 9.2

Find the missing angle E. Round your final answer to one decimal place.



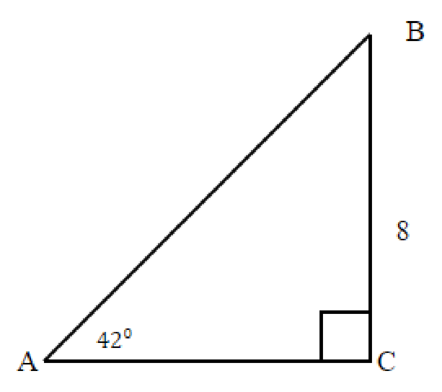
Show answer
36.9°

Solving a Right Triangle

From the section before we know that any triangle has three sides and three interior angles. In a right triangle, when all six parts of the triangle are known, we say that the right triangle is solved.

EXAMPLE 10

Solve the right triangle. Round your final answer to one decimal place.



Solution

Since the sum of angles in any triangle is 180°, the measure of angle B can be easy calculated.

$\angle B$ = 180° − 90° − 42°

$\angle B$ = 48°

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle A is our reference angle, a = 8 is the opposite side, b is the adjacent side, and c is the hypotenuse.	
2. **Identify** what we are looking for.	a) adjacent side	b) hypotenuse
3.**Label** what we are looking for by choosing a variable to represent it.	b = ?	c = ?
4. **Find** the required trigonometric ratio.	$\tan 42° = \frac{8}{b}$	$\sin 42° = \frac{8}{c}$
5. **Solve** the ratio using good algebra techniques.	$0.9004 = \frac{8}{b}$ 0.9004 b = 8 b = 8.8849	$0.6691 = \frac{8}{c}$ 0.6691 c = 8 c = 11.9563
6. **Check** the answer in the problem and by making sure it makes sense.	$\tan 42° \overset{?}{=} \frac{8}{8.8849}$ 0.9 = 0.9 ✓	$\sin 42° \overset{?}{=} \frac{8}{11.9563}$ 0.6691 = 0.6691 ✓
7. **Answer** the question with a complete sentence.	The adjacent side is 8.9. Rounded to one decimal place.	The hypotenuse is 12

We solved the right triangle

$\angle A$ = 42°

$\angle B$ = 48°

$\angle C$ = 90°

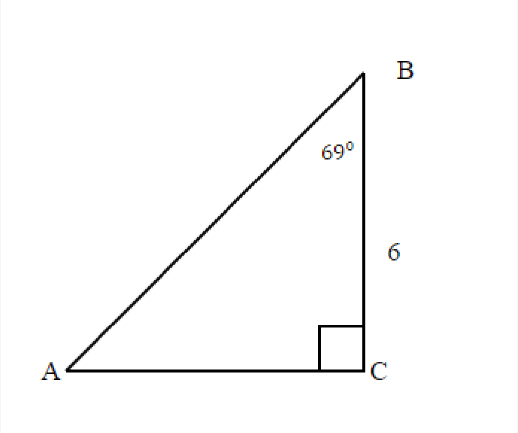
a = 8

b = 8.9

c = 12

TRY IT 10.1

Solve the right triangle. Round your final answer to one decimal place.



$\angle A$= 21°

$\angle B$ = 69°

$\angle C$ = 90°

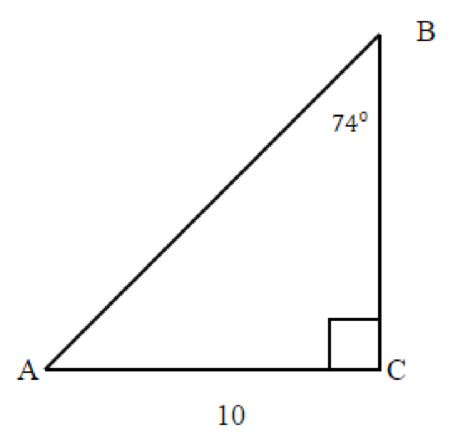
Show answer
a = 6

b = 15.6

c = 16.7

TRY IT 10.2

Solve the right triangle. Round your final answer to one decimal place.



$\angle A$=16°

$\angle B$ = 74°

$\angle C$ = 90°

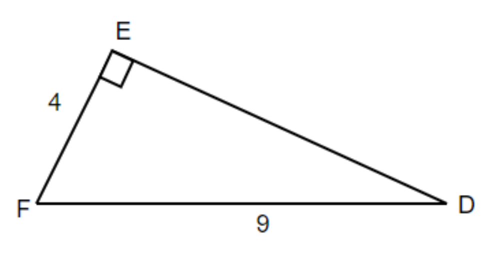
Show answer
a = 2.9

b = 10

c = 10.4

EXAMPLE 11

Solve the right triangle. Round to two decimal places.



Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Let angle D be our reference angle, d = 4 is the opposite side, f is the adjacent side, and e = 9 is the hypotenuse	
2. **Identify** what we are looking for.	a) angle D	b) adjacent
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle D$ =?	f = ?
4. **Find** the required trigonometric ratio.	sin D = $\frac{4}{9}$	$4^2 + f^2 = 9^2$
5. **Solve** the ratio using good algebra techniques.	sin D = 0.4444 D = $\sin^{-1}$0.4444 $\angle D$ = 26.3850°	$16 + f^2 = 81$ $f^2 = 81 - 16$ $f^2 = 65$ f = square root of 65 f = 8.06
6. **Check** the answer in the problem and by making sure it makes sense.	sin 26.3850° $\overset{?}{=} \frac{4}{9}$ 0.4444 =0.4444 ✓	$4^2 + 8.06^2 \overset{?}{=} 9^2$ 81 = 81 ✓
7. **Answer** the question with a complete sentence.	The missing angle D is 26.39°.	The adjacent side is 8.06 Rounded to two decimal places

The missing angle F = 180° – 90° – 26.39° = 63.64°

We solved the right triangle

$\angle D$ = 26.39°

$\angle E$ = 90°

$\angle F$ = 63.61°

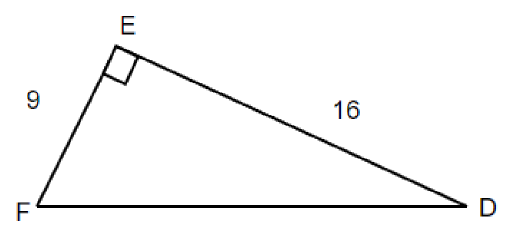
d = 4

e = 9

f = 8.06

TRY IT 11.1

Solve the right triangle. Round to one decimal place.



$\angle D$ = 29.3°

$\angle E$ = 90°

$\angle F$ = 60.7°

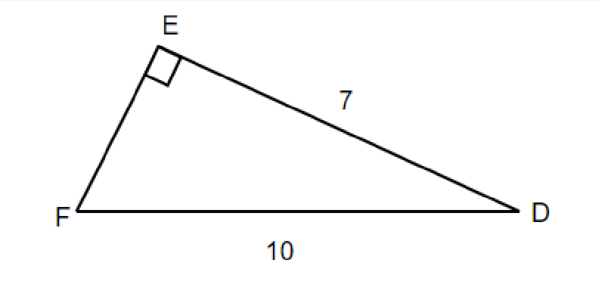
Show answer
d = 29.4

e = 18.4

f = 60.6

TRY IT 11.2

Solve the right triangle. Round to one decimal place.



$\angle D$ = 45.6°

$\angle E$ = 90°

$\angle F$ = 44.4°

Show answer
d = 7.1

e = 10

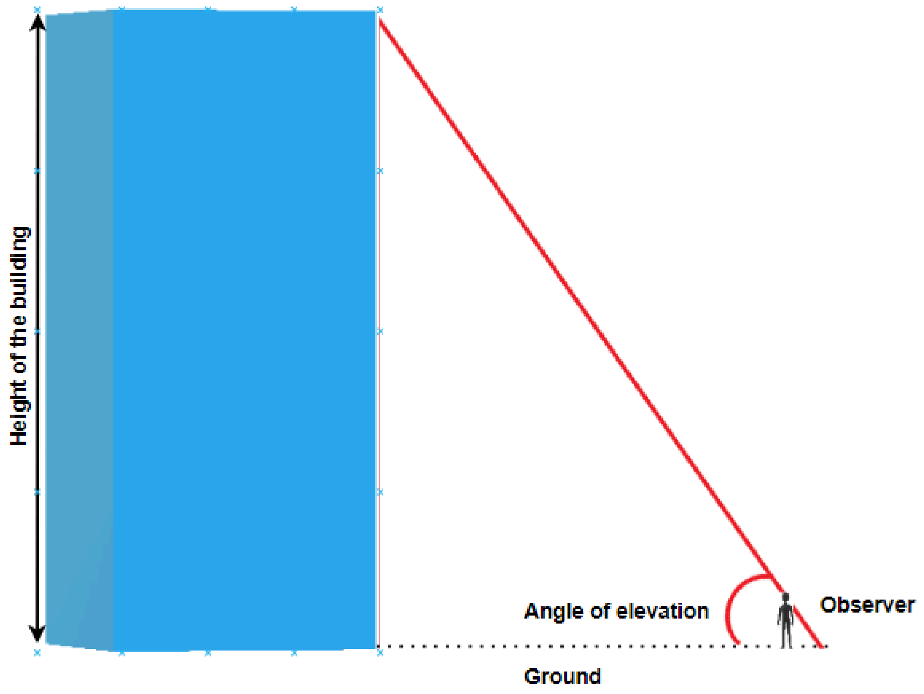
f = 7

Solve Applications Using Trigonometric Ratios

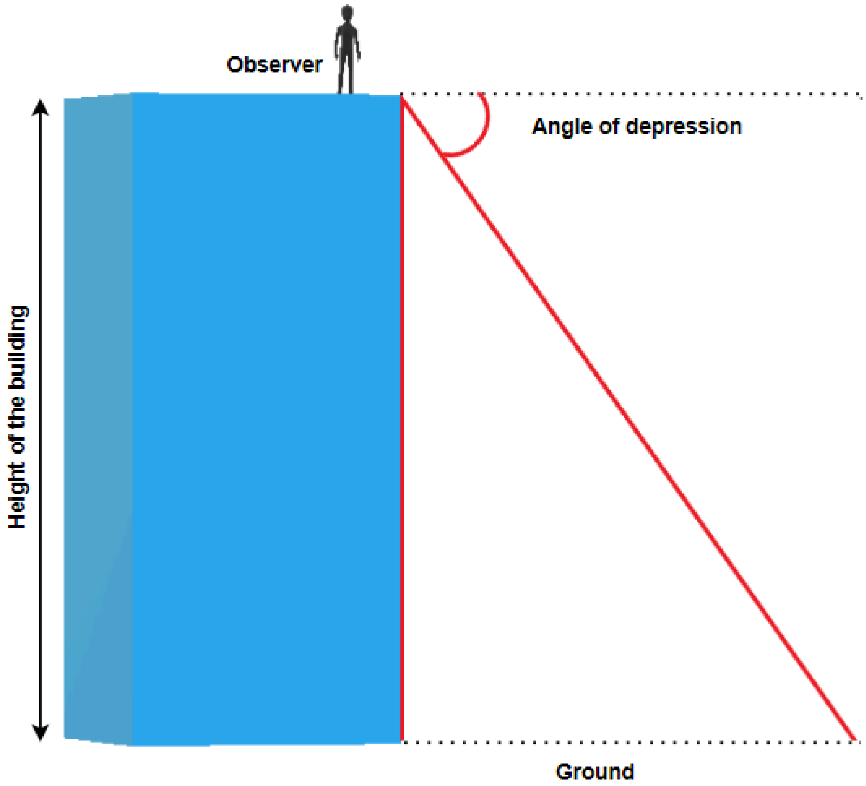
In the previous examples we were able to find missing sides and missing angles of a right triangle. Now, let's use the trigonometric ratios to solve real-life problems.

Many applications of trigonometric ratios involve understanding of an angle of elevation or angle of depression.

The angle of elevation is an angle between the horizontal line (ground) and the observer's line of sight.



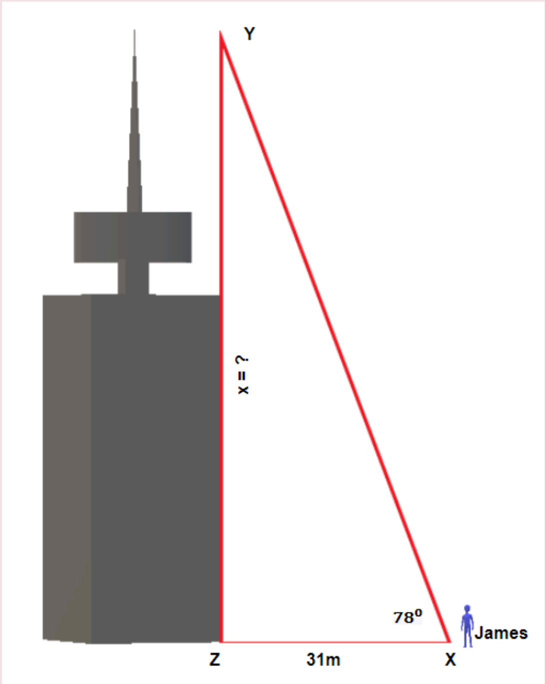
The angle of depression is the angle between horizontal line (that is parallel to the ground) and the observer's line of sight.



EXAMPLE 12

James is standing 31 metres away from the base of the Harbour Centre in Vancouver. He looks up to the top of the building at a 78° angle. How tall is the Harbour Centre?

Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	 Angle X is our reference angle, x is opposite side, y = 31 m is the adjacent side, and z is the hypotenuse.
2. **Identify** what we are looking for.	The opposite side
3.**Label** what we are looking for by choosing a variable to represent it.	x=?
4. **Find** the required trigonometric ratio.	$\tan 78° = \frac{x}{31}$
5. **Solve** the ratio using good algebra techniques.	$4.7046 = \frac{x}{31}$ $x = 145.8426$
6. **Check** the answer in the problem and by making sure it makes sense.	$4.7046 \overset{?}{=} \frac{145.8426}{31}$ $4.7046 = 4.7046$ ✓
7. **Answer** the question with a complete sentence.	The Harbour Centre is 145.8426 metres or rounded to 146 metres.

TRY IT 12.1

Nicole is standing 75 feet away from the base of the Living Shangri-La, the tallest building in British Columbia. She looks up to the top of the building at a 83.5° angle. How tall is the Living Shangri-La?

Show answer
658.3 feet.

TRY IT 12.2

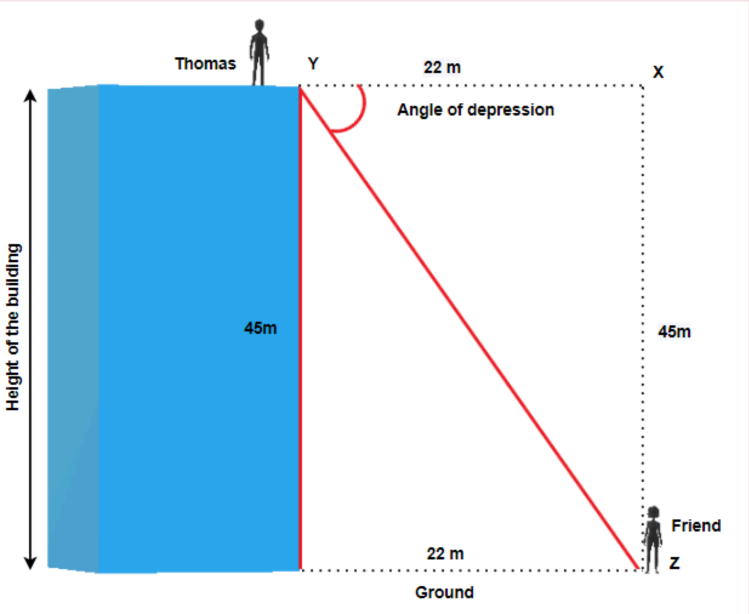
Kelly is standing 23 metres away from the base of the tallest apartment building in Prince George and looks at the top of the building at a 62° angle. How tall is the building?

Show answer
43.3 metres

EXAMPLE 13

Thomas is standing at the top of the building that is 45 metres high and looks at his friend that is standing on the ground, 22 metres from the base of the building. What is the angle of depression?

Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	 Angle Y is our reference angle, $y = 45$ m is the opposite side, $z = 22$ m is the adjacent side, and x is the hypotenuse
2. **Identify** what we are looking for.	angle Y
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle Y$ =?
4. **Find** the required trigonometric ratio.	$\tan Y = \frac{45}{22}$
5. **Solve** the ratio using good algebra techniques.	$\tan Y = 2.0455$ $Y = \tan^{-1} 2.0455$ $\angle Y = 63.9470°$
6. **Check** the answer in the problem and by making sure it makes sense.	$\tan 63.9470° \overset{?}{=} 2.0455$ $2.0455 = 2.0455$ ✓
7. **Answer** the question with a complete sentence.	The angle of depression is 63.9470° or 64° rounded to one decimal place.

TRY IT 13.1

Hemanth is standing on the top of a cliff 250 feet above the ground and looks at his friend that is standing on the ground, 40 feet from the base of the cliff. What is the angle of depression?

Show answer
80.9°

TRY IT 13.2

Klaudia is standing on the ground, 25 metres from the base of the cliff and looks up at her friend on the top of a cliff 100 metres above the ground. What is the angle of elevation?

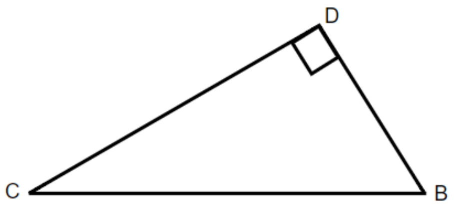
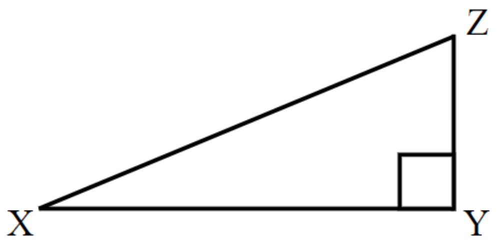
Show answer
76°

Key Concepts

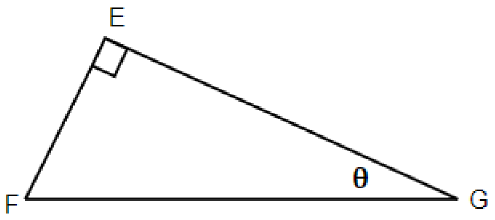
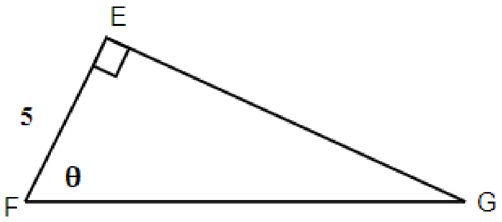
- Three Basic Trigonometric Ratios: (Where θ is the measure of a reference angle measured in degrees.)
    - sine θ = $\frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse side}}$
    - cosine θ = $\frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse side}}$
    - tangent θ = $\frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}}$
- **Problem-Solving Strategy for Trigonometry Applications**
    1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.
    2. **Identify** what we are looking for.
    3. **Label** what we are looking for by choosing a variable to represent it.
    4. **Find** the required trigonometric ratio.
    5. **Solve** the ratio using good algebra techniques.
    6. **Check** the answer by substituting it back into the ratio solved in step 5 and by making sure it makes sense in the context of the problem.
    7. **Answer** the question with a complete sentence.

Practice Makes Perfect

Label the sides of the triangle.

1 	2. 
3. If the reference angle in Question 1 is B, Find the adjacent ?	4. If the reference angle in Question 2 is Z, find the opposite ?

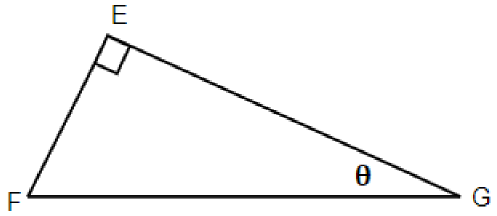
Label the sides of the triangle and find the hypotenuse, opposite and adjacent.

 5.	 6.

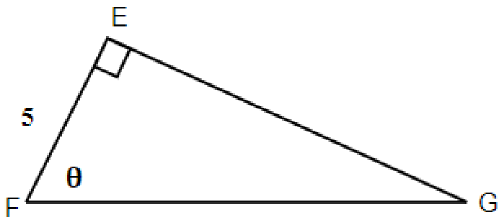
Use your calculator to find the given ratios. Round to four decimal places if necessary:

7. $\sin 47^\circ$	8. $\cos 82^\circ$
9. $\tan 12^\circ$	10. $\sin 30^\circ$

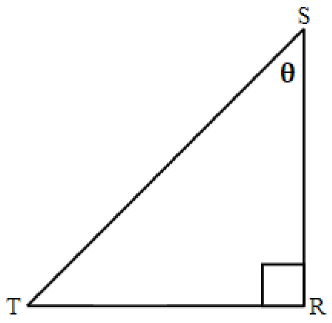
For the given triangles, find the sine, cosine and tangent of the θ.



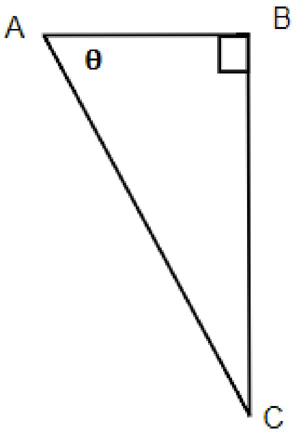
11.



12.



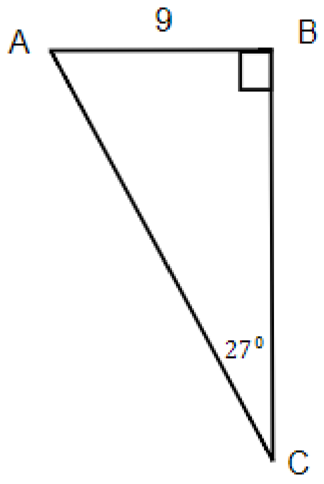
13.



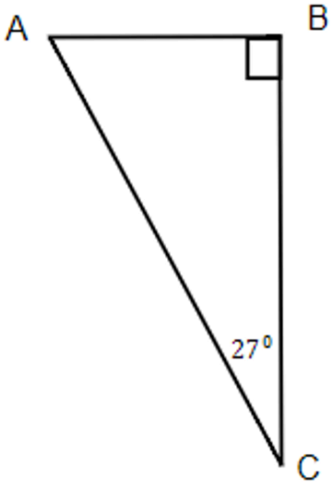
14.

For the given triangles, find the missing side. Round it to one decimal place.

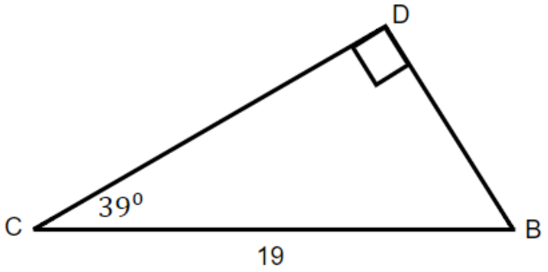
15. Find the hypotenuse.



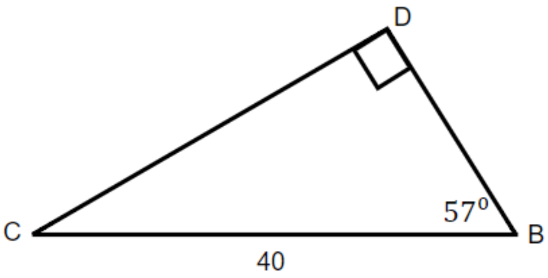
16. Find b if a = 6.



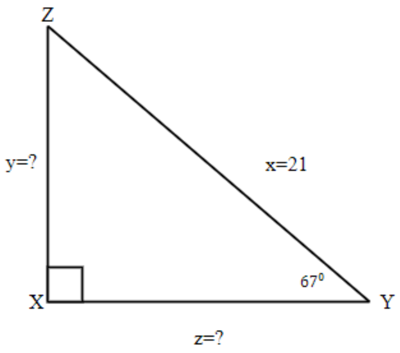
17. Find the opposite.



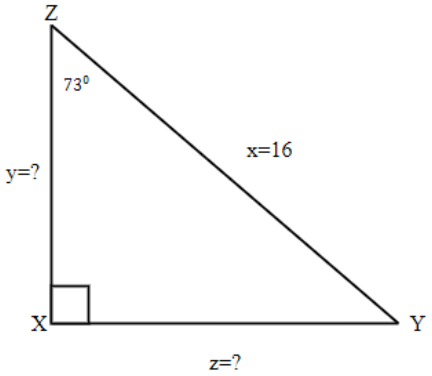
18. Find the adjacent.



For the given triangles, find the missing sides. Round it to one decimal place.

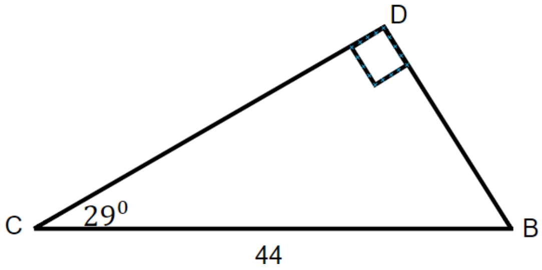
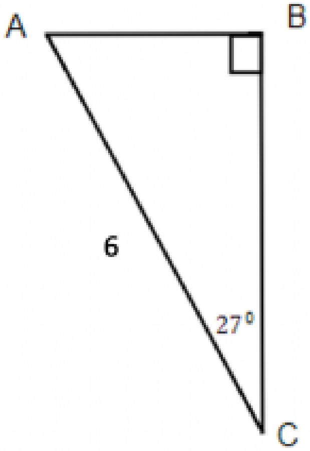
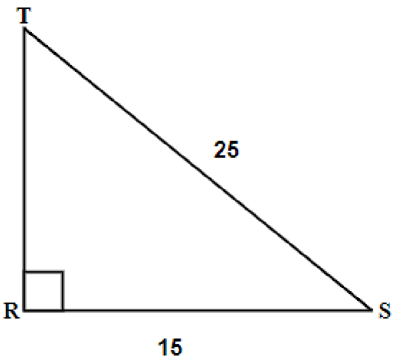
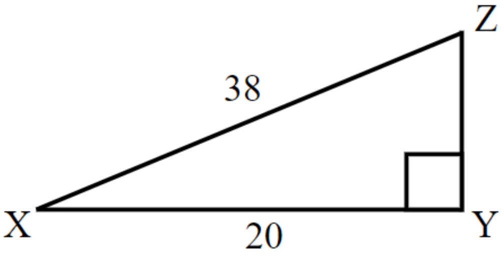


19.

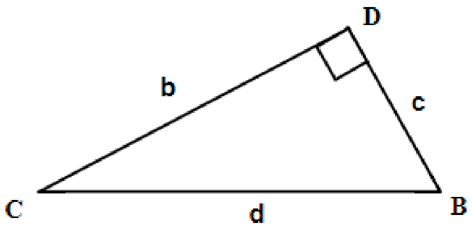
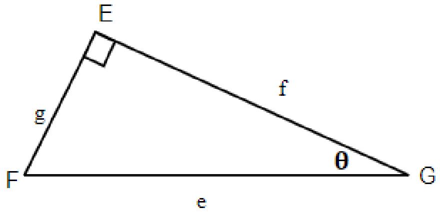
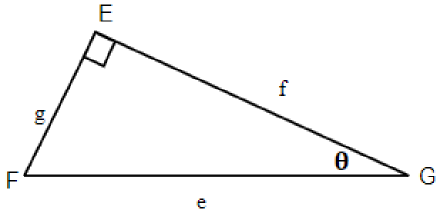


20.

Solve the triangles. Round to one decimal place.

 21.	 22.
 23.	 24.
25. A surveyor stands 75 metres from the bottom of a tree and looks up at the top of the tree at a 48° angle. How tall is the tree?	26. A tree makes a shadow that is 6 metres long when the angle of elevation to the sun is 52°. How tall is the tree?
27. A ladder that is 15 feet is leaning against a house and makes a 45° angle with the ground. How far is the base of the ladder from the house?	28. Matt is flying a kite and has let out 100 feet of string. The angle of elevation with the ground is 38°. How high is his kite above the ground?
29. Marta is flying a kite and has let out 28 metres of string. If the kite is 10 metres above the ground, what is the angle of elevation?	30. An airplane takes off from the ground at the angle of 25°. If the airplane traveled 200 kilometres, how high above the ground is it?

Answers

1.  	3. c	5.  g is opposite , f is adjacent, and e is hypotenuse
7. 0.7314	9. 0.2126	11.  $\sin\theta = \frac{g}{e}$, $\cos\theta = \frac{f}{e}$, $\tan\theta = \frac{g}{f}$
13. $\sin\theta = \frac{s}{r}$, $\cos\theta = \frac{t}{r}$, $\tan\theta = \frac{s}{t}$	15. b = 19.8	17. c = 12
19. y = 19.3, z = 8.2	21. $\angle B$ = 61° $\angle C$ = 29° $\angle D$ = 90° b = 38.5 c = 21.3 d = 44	23. $\angle T$ = 36.9° $\angle R$ = 90° $\angle S$ = 53.1° t = 15 r = 25 s = 20
25. 83.3 m	27. 10.6 ft	29. 20.9°

9.3 Chapter Review

Review Exercises

Use Properties of Angles

In the following exercises, solve using properties of angles.

1. What is the supplement of a 48° angle?

2. What is the complement of a 61° angle?

3. Two angles are complementary. The smaller angle is 24° less than the larger angle. Find the measures of both angles.

4. Two angles are supplementary. The larger angle is 45° more than the smaller angle. Find the measures of both angles.

Use Properties of Triangles

In the following exercises, solve using properties of triangles.

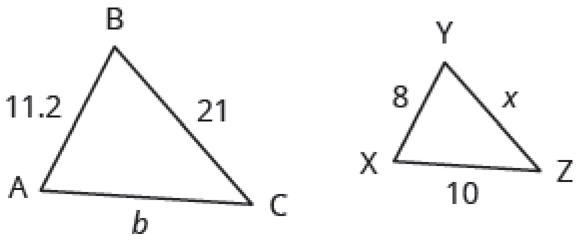
5. The measures of two angles of a triangle are 22 and 85 degrees. Find the measure of the third angle.

6. One angle of a right triangle measures 41.5 degrees. What is the measure of the other small angle?

7. One angle of a triangle is 30° more than the smallest angle. The largest angle is the sum of the other angles. Find the measures of all three angles.

8. One angle of a triangle is twice the measure of the smallest angle. The third angle is 60° more than the measure of the smallest angle. Find the measures of all three angles.

In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



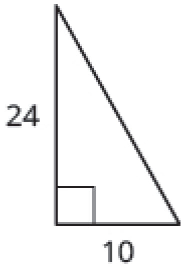
9. side x

10. side b

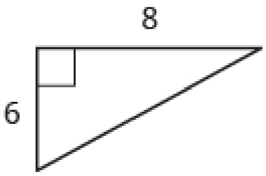
Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

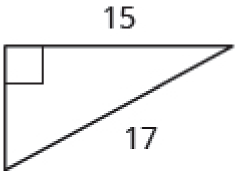
11.



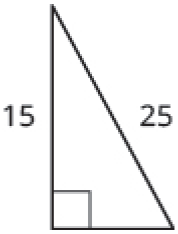
12.



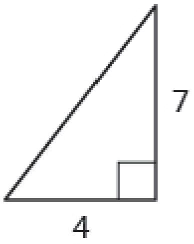
13.



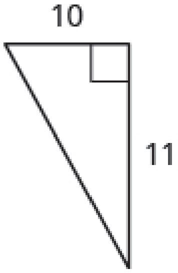
14.



15.

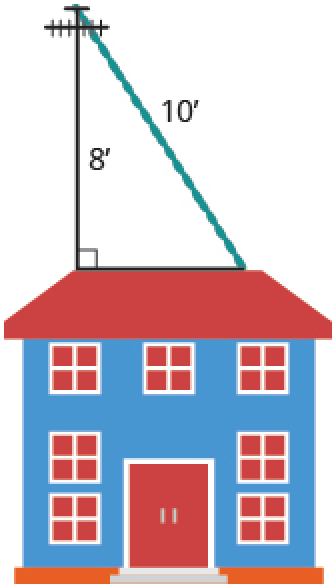


16.



In the following exercises, solve. Approximate to the nearest tenth, if necessary.

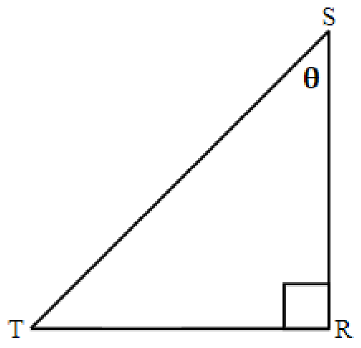
17. Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is 8 feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire?



18. Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?

Find missing side of a right triangle using sine, cosine, or tangent ratios.

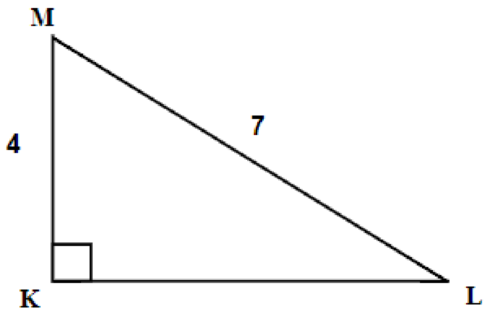
19. Label the triangle and find the sine cosine and tangent of θ.



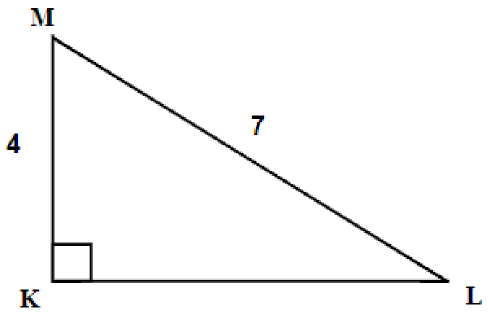
20. If reference angle in above triangle is angle T, label the triangle and find the sine, cosine, and tangent of T.

Find missing angle of a right triangle using sine, cosine, or tangent ratios.

21. Find angle M

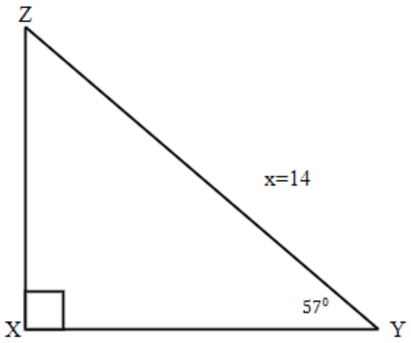


22. Find angle L.

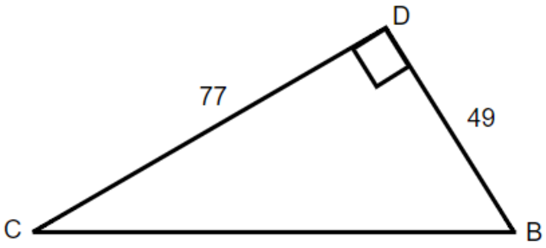


Solve the right triangle.

23. Solve the triangle.

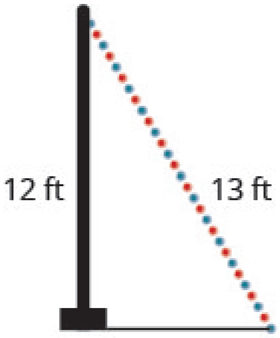


24.



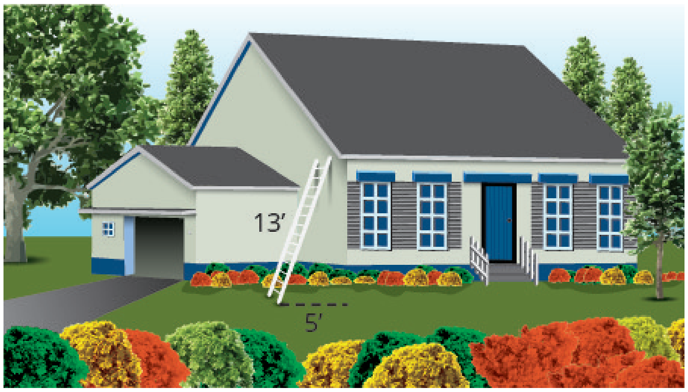
Solve applications using right angle trigonometry.

25. A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display, as shown below. What is the angle that the string of lights makes with the ground?



26. Brian borrowed a 20 foot extension ladder to use when he paints his house. If he sets the base of the ladder 6 feet from the house, as shown below, what is the angle that the ladder makes with the ground?

27. John puts the base of a 13-foot ladder five feet from the wall of his house as shown below. What is the angle between the top of the ladder and the house ?

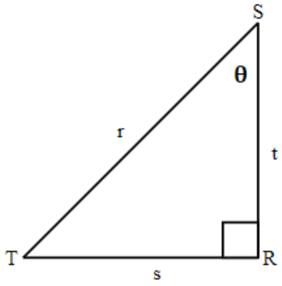


28. The sun is at an angle of elevation of 35°. If Bob casts a shadow that is 6 ft long, how tall is Bob?

29. A 27 foot guy wire to a pole makes an angle of 63.7° with the ground. How high from the ground is the wire attached to the pole?

30. A lighthouse is 20 metres tall. If the observer is looking at a boat that is 30 metres away from the base of the lighthouse, what is the angle of depression?

Review Answers

1. 132°	3. 33°, 57°	5. 73°
7. 30°, 60°, 90°	9. 15	11. 26
13. 8	15. 8.1	17. 6 feet
19.  sin θ $=\frac{s}{r}$, cos θ $=\frac{t}{r}$, tan θ $=\frac{s}{t}$	21. 55.2°	23.$\angle X$ = 90° , $\angle Y$ = 57° , $\angle Z$ = 33°x = 14, y = 11.7, z= 7.6
25. 67.4°	27. 22.6°	29. 24

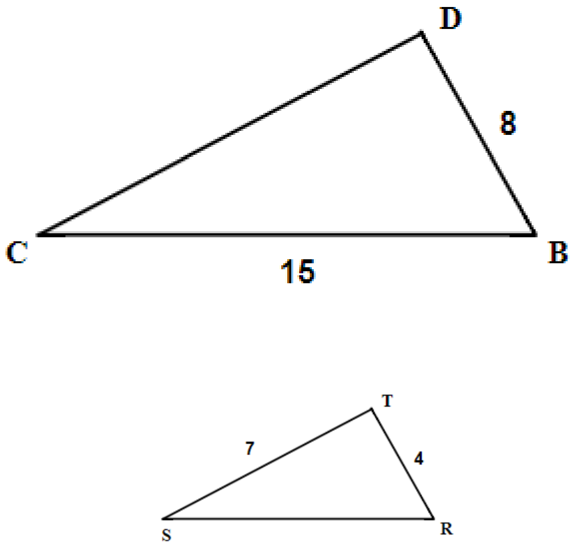
Practice Test

1. What is the supplement of a $\angle 57°$ angle?

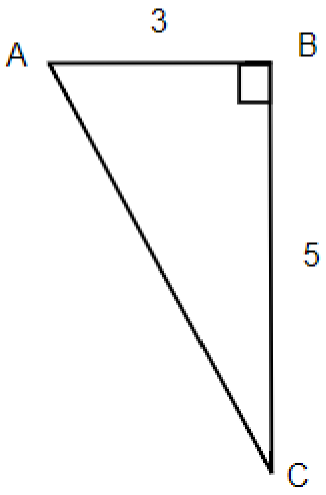
2. Two angles are complementary. The smaller angle is $16°$ less than the larger angle. Find the measures of both angles.

3. The measures of two angles of a triangle are 29 and 75 degrees. Find the measure of the third angle.

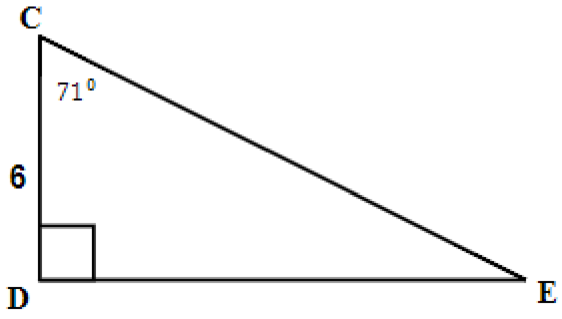
4. ΔBCD is similar to ΔSRT. Find the missing sides.

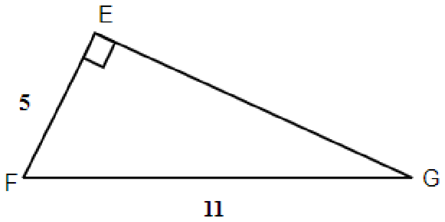
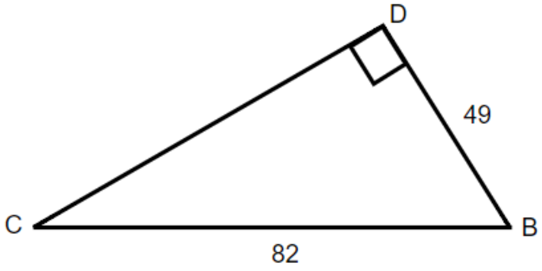


5. Use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.



6. Find the hypotenuse.



7. Find angle G. 	8. Solve the triangle. 
9. The sun is at an angle 28°. If Adam casts a shadow that is 7 ft long, how tall is Adam?	10. The road rises 6 metres per every 100 horizontal metres. What is the angle of elevation.

Answers

1. 123°	2. 53°, 37°	3. 76°
4. b = 14, t = 7.5	5. b = 15.3	6. d = 18.4
7. $\angle G$ = 27°	8.$\angle C$ = 36.7°, $\angle B$ = 53.3°, $\angle D$ = 90°, c = 49, b = 65.7, d= 82	9. 5.5 ft
10. 3.4°		

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Versioning History

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.01. If the edits involve substantial updates, the version number increases to the next full number.

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Version	**Date**	**Change**	**Details**
1.01	March 26, 2021	Book published	
1.02	May 19, 2021	Acknowledgements section added to the front matter.	
1.03	November 17, 2021	Corrections made to Chapter 9.2	Solutions for TRY IT 11.1 altered, questions for Exercise 16 and 22 altered.